CHAPTER 10

Solvability in $W^2_{p,\mathrm{loc}}$ of fully nonlinear elliptic equations

1. Main results

In this chapter we consider elliptic equations

$$H[v](x) := H\big(v(x), Dv(x), D^2v(x), x\big) = 0 \tag{1}$$

in subdomains of $\mathbb{R}^d$, where $H(\mathsf{u}, x)$ is a function given for $x \in \mathbb{R}^d$ and $\mathsf{u} = (\mathsf{u}', \mathsf{u}'')$,

$$\mathsf{u}' = (\mathsf{u}'_0, \mathsf{u}'_1, ..., \mathsf{u}'_d) \in \mathbb{R}^{d+1}, \quad \mathsf{u}'' \in \mathbb{S}.$$

Recall that $\mathbb{S}$ is the set of symmetric $d \times d$ matrices and, for $\delta \in (0, 1]$,

$$\mathbb{S}_\delta = \big\{a \in \mathbb{S} : \delta|\xi|^2 \le a^{ij}\xi^i\xi^j \le \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d\big\}.$$

Let Ω be an open bounded subset of $\mathbb{R}^d$ satisfying the exterior ball condition. Fix

$$p > d \quad \text{and a measurable function} \quad \bar{G} \ge 0 \quad \text{on} \quad \mathbb{R}^d.$$

Also fix some constants $K_0, K_F \in [0, \infty)$, $\delta \in (0, 1]$.

Quite often we deal with

$$\Omega^\rho = \big\{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \rho\big\},$$

where $\rho > 0$ is a given number. Obviously, $\Omega^\rho \ne \emptyset$ only if $\rho < \rho_{\mathrm{int}}(\Omega)$. However, mentioning this restriction many times looks awkward and quite often we avoid doing that by defining the norms of functions in function spaces over empty domains as being zero.

1. Example. The reader will see that, if $d = 3$ and $f, \bar{G} \in L_p(\Omega)$, the equation $\big(a \wedge b = \min(a, b)\big)$

$$H(D^2u, x) := \bar{G}(x) \wedge |D_{12}u| + \bar{G}(x) \wedge |D_{23}u| + \bar{G}(x) \wedge |D_{31}u|$$
$$+\Delta u - f(x) = 0 \tag{2}$$

satisfies all assumptions listed below. Observe that H in (2) is neither convex nor concave with respect to D^2u. Also note that we can replace Δu with $a^{ij}(x)D_{ij}u$ if $a(x) = (a^{ij}(x))$ is an $\mathbb{S}_\delta$-valued VMO-function such that $a(x) \ge (\delta^{ij})$.

2. Example. For $\tau > 0$ take

$$H(\mathsf{u}) = \Big(1 + \tau \cos\sqrt{\big|\ln|\mathsf{u}''|\big|}\Big)\mathrm{tr}\,\mathsf{u}'',$$

where for $\mathsf{u}'' \in \mathbb{S}$ by $|\mathsf{u}''|$ we mean $\mathrm{tr}^{1/2}(\mathsf{u}''\mathsf{u}'')$, and choose τ so small that $D_{\mathsf{u}''}H \in \mathbb{S}_\delta$ for a $\delta \in (0, 1]$. Then again H is neither convex nor concave with respect to u'' and our assumptions are satisfied perhaps with a further reduced τ and $\bar{F}(\mathsf{u}'') = \mathrm{tr}\,\mathsf{u}''$.

3. Assumption. The function $H(\mathsf{u},x)$ is Borel measurable with respect to (u',x).

The following assumptions contain parameters $\hat\theta,\theta\in(0,1]$ which are specified later in our results.

4. Assumption. There are Borel measurable functions $F(\mathsf{u},x)=F\big(\mathsf{u}'_0,\mathsf{u}'',x\big)$ and $G(\mathsf{u},x)$ such that

$$H=F+G.$$

Furthermore, for all $\mathsf{u}''\in\mathbb{S},\mathsf{u}'\in\mathbb{R}^{d+1}$, and $x\in\mathbb{R}^d$ we have

$$\big|G(\mathsf{u},x)\big|\le\hat\theta|\mathsf{u}''|+K_0|\mathsf{u}'|+\bar G(x),\quad F(0,x)\equiv 0. \tag{3}$$

5. Example. One can take $F\big(\mathsf{u}'_0,\mathsf{u}'',x\big)=H(0,\mathsf{u}'',x)$ and $G=H-F$ to satisfy the first part of Assumption 4. Since we require that $F(0,x)=0$, one can then take $F\big(\mathsf{u}'_0,\mathsf{u}'',x\big)=H(0,\mathsf{u}'',x)-H(0,x)$ and $G=H-F$. However, we are not bound by this option.

Observe that Assumption 4 is satisfied in Example 1 with $F(u,D^2u)=\Delta u$ and $G=H-F$.

Recall that

$$B_r(x)=\big\{y\in\mathbb{R}^d:|x-y|<r\big\},\quad B_r=B_r(0)$$

and that for measurable $\Gamma\subset\mathbb{R}^d$ we denote by $|\Gamma|$ the volume of Γ.

The combination of Assumptions 4 and 6 we call "a relaxed convexity assumption on H and an almost VMO condition on F". Recall that Lipschitz continuous functions are almost everywhere differentiable.

6. Assumption. (i) The function F is Lipschitz continuous with respect to u'' with Lipschitz constant K_F.

Moreover, there exist $R_0\in(0,1]$ and $\tau_0\in[0,\infty)$ such that, if $r\in(0,R_0]$, $z\in\Omega$, $B_r(z)\subset\Omega$, and $\mathsf{u}'_0\in\mathbb{R}$, then one can find a *convex* function $\bar F(\mathsf{u}'')=\bar F_{z,r,\mathsf{u}'_0}(\mathsf{u}'')$ (independent of x) for which

(ii) We have $\bar F(0)=0$ and $D_{\mathsf{u}''}\bar F\in\mathbb{S}_\delta$ at all points of differentiability of $\bar F$;

(iii) For any $\mathsf{u}''\in\mathbb{S}$ with $|\mathsf{u}''|=1$, we have

$$\int_{B_r(z)}\sup_{\tau>\tau_0}\tau^{-1}\big|F\big(\mathsf{u}'_0,\tau\mathsf{u}'',x\big)-\bar F(\tau\mathsf{u}'')\big|\,dx\le\theta\big|B_r(z)\big|; \tag{4}$$

(iv) There exists a continuous increasing function $\omega_F(\tau)$, $\tau\ge 0$, such that $\omega_F(0)=0$ and for any $\mathsf{u}'_0,\mathsf{v}'_0\in\mathbb{R}$, $x\in\Omega$, and $\mathsf{u}''\in\mathbb{S}$ we have

$$\big|F\big(\mathsf{u}'_0,\mathsf{u}'',x\big)-F\big(\mathsf{v}'_0,\mathsf{u}'',x\big)\big|\le\omega_F\big(|\mathsf{u}'_0-\mathsf{v}'_0|\big)|\mathsf{u}''|.$$

7. Remark. It is useful to note that Assumptions 4 and 6 (iv) imply that $F(\mathsf{u}'_0,0,x)=0$ for any $\mathsf{u}'_0\in\mathbb{R}$ and $x\in\Omega$.

8. Remark. Assumptions 4 and 6 (iv) imply

$$\big|H(\mathsf{u}',0,x)\big|\le K_0|\mathsf{u}'|+\bar G(x)\quad\forall\mathsf{u}',x. \tag{5}$$

Note that Assumption 6 is obviously satisfied in Examples 1 and 2 if we take $F(u,D^2u,x)=\Delta u$ and choose τ appropriately in Example 2.

Next, we introduce the following definition.

9. Definition. For a function $u \in C(\bar{\Omega})$ set

$$\omega_u(\Omega,\rho)=\sup\big\{\big|u(x_1)-u(x_2)\big| : x_1,x_2\in\Omega, |x_1-x_2|\le\rho\big\},$$

$$\omega_{F,u,\Omega}(\rho)=\omega_F\big(\omega_u(\Omega,\rho)\big),$$

and in the formulations of a theorem, lemma,... let us say that a certain constant depends only on A,B,..., and the function $\omega_{F,u,\Omega}$ if it depends only on A,B,..., and on the maximal solution of an inequality like $N_0\omega_{F,u,\Omega}(\rho)\le 1/2$, where the range of ρ and the value of N_0 depending only on A,B,... could be always traced down in our arguments.

We now come to the main a priori estimate in Sobolev spaces. Observe that in Theorem 10 we do not even assume that equation (1) is elliptic. Note also that the standing assumption that Ω satisfies the exterior ball condition, actually, is not needed in Theorem 10.

10. Theorem. *Under the above assumptions there exist constants $\hat{\theta},\theta\in(0,1]$, depending only on d, p, δ, and K_F, such that, if Assumptions* 6 *and* 4 *are satisfied with these θ and $\hat{\theta}$, respectively, then, for any $u\in W^2_{p,\text{loc}}(\Omega)\cap C(\bar{\Omega})$ that satisfies* (1) *in Ω (a.e.) and $\rho>0$ (such that $\Omega^\rho\ne\emptyset$), we have*

$$\|u\|_{W^2_p(\Omega^\rho)}\le N\|\bar{G}\|_{L_p(\Omega)}+N\rho^{-2}\|u\|_{C(\Omega)}+N\tau_0, \tag{6}$$

where the constants N depend only on K_0, K_F, d, p, δ, R_0, diam(Ω), *and the function $\omega_{F,u,\Omega}$.*

We prove this theorem in Section 5.

Generally, having a priori estimates does not guarantee that there are existence results. Say, if $d=1$ and the equation is

$$u''-(2u'')I_{|u''|\le 5}-10u=1 \quad\text{in}\quad (-1,1),$$

a priori W^2_p-estimates are trivial since $\big|(2u'')I_{|u''|\le 5}\big|\le 10$. However, the solvability is quite questionable.

To have the solvability we need ellipticity and more regularity of H.

11. Assumption. The function $H(\mathsf{u},x)$ is continuous in u for any x, is Lipschitz continuous with respect to u'', $D_{\mathsf{u}''}H\in\mathbb{S}_\delta$ at all points of differentiability of H with respect to u''.

12. Assumption. For all values of the arguments,

$$H(\mathsf{u}',0,x)\operatorname{sign}\mathsf{u}'_0\le K_0\big|[\mathsf{u}']\big|+\bar{G}(x)\quad(\operatorname{sign}0:=\pm 1) \tag{7}$$

(regarding the notation $[\mathsf{u}']$ see (4.1.15)).

Assumption 11 is satisfied in Example 1 because $H_{\mathsf{u}''_{ii}}=1$, $|H_{\mathsf{u}''_{13}}|=|H_{\mathsf{u}''_{31}}|\le 1/2$, $\big|H_{\mathsf{u}''_{12}}\big|=\big|H_{\mathsf{u}''_{21}}\big|\le 1/2$, $\big|H_{\mathsf{u}''_{23}}\big|=\big|H_{\mathsf{u}''_{32}}\big|\le 1/2$ (see Remark 4.1.10).

13. Remark. We draw the reader's attention to the fact that we do not assume that H is a nonincreasing function of u'_0 (cf. Remark 6.6.8).

Here is one of the main results of the chapter concerning the solvability of (1) in Sobolev spaces. We fix a function

$$g\in C(\partial\Omega).$$

14. THEOREM. *There exist constants $\hat\theta,\theta\in(0,1]$, depending only on d, p, δ, and K_F, which are, generally, smaller than $\hat\theta,\theta$ from Theorem* 10 *and such that, if Assumptions* 6 *and* 4 *are satisfied with these θ and $\hat\theta$, respectively, and Assumptions* 11 *and* 12 *are also satisfied and $\bar G\in L_p(\Omega)$, then there exists $u\in W^2_{p,\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$ satisfying* (1) *in Ω (a.e.) and such that $u=g$ on $\partial\Omega$.*

15. REMARK. Since none of characteristics of Ω, apart from $\rho_{\mathrm{int}}(\Omega)$ and $\mathrm{diam}(\Omega)$ enters Theorem 10 and Lemma 6.6.10, one can use Theorem 14 to prove the solvability in much worst domains than those satisfying the exterior ball condition. Usually one does it by approximating from inside a given domain, say with smooth ones. For instance, it would suffice to have

$$\lim_{\rho\downarrow 0}\inf_{x\in\partial\Omega}\frac{\big|B_\rho(x)\cap\Omega^c\big|}{\rho^d}>0, \tag{8}$$

see Theorem 3.1 of [**103**].

We are not pursuing this path and leave it to the interested reader.

We give the proof of Theorem 14 in Section 5 after some preparations. As a simple corollary of this theorem we obtain the following in which Assumption 6 is not used.

16. THEOREM. *Suppose that Assumptions* 3, 11, *and* 12 *are satisfied, $\bar G\in L_p(\Omega)$, and* (5) *holds true. Let $P(\mathsf{u}'')$ be a convex function on $\mathbb{S}$ such that $D_{\mathsf{u}''}P\in\mathbb{S}_{\delta'}$ at each point of its differentiability, where $\delta'\in(0,\delta]$. Also assume that for any $a\in\mathbb{S}_\delta$ and $\mathsf{u}''\in\mathbb{S}$ we have*

$$a^{ij}\mathsf{u}''_{ij}\le P(\mathsf{u}'')+K,$$

where K is a constant. Then the equation

$$\max\big(H[u],P[u]\big)=0$$

in Ω (a.e.) with boundary condition $u=g$ on $\partial\Omega$ has a solution $u\in W^2_{p,\mathrm{loc}}(\Omega)\cap C(\bar\Omega)$.

Proof. Introduce

$$\hat H(\mathsf{u},x)=\max\big(H(\mathsf{u},x),P(\mathsf{u}'')\big),\quad \hat F(\mathsf{u}'',x)=P(\mathsf{u}'')-P(0),\quad \hat G=\hat H-\hat F.$$

Obviously Assumptions 3, 6, 11, and 12 are satisfied for $\hat H$, $\hat F$, and $\hat F$ in place of H, F, and $\bar F$, respectively, with a K_F, $\tau_0=\theta=0$, the same K_0, δ' in place of δ, and $\bar G+\big|P(0)\big|$ in place of $\bar G$. Finally, for any u,x,

$$\hat G(\mathsf{u},x)=\max\big(H(\mathsf{u},x)-P(\mathsf{u}'')+P(0),P(0)\big)\ge P(0),$$

where for an $a\in\mathbb{S}_\delta$

$$H(\mathsf{u},x)-P(\mathsf{u}'')=H(\mathsf{u},x)-H(\mathsf{u}',0,x)-P(\mathsf{u}'')+H(\mathsf{u}',0,x)$$
$$=a^{ij}\mathsf{u}''_{ij}-P(\mathsf{u}'')+H(\mathsf{u}',0,x)\le K+H(\mathsf{u}',0,x),$$

which shows that Assumption 4 is also satisfied with $\hat\theta=0$ and $\bar G+K+\big|P(0)\big|$ in place of $\bar G$.

Hence, Theorem 14 is applicable and our theorem is proved. □

One more solvability result, in which there is no monotonicity assumption on the dependence of H on u'_0, is the following.

17. THEOREM. *There exist constants $\hat{\theta},\theta \in (0,1]$, depending only on d, p, δ, and K_F, such that, if Assumptions* 6 *and* 4 *are satisfied with these θ and $\hat{\theta}$, respectively, and Assumptions* 11 *and* 6.6.4 *are also satisfied with $M_0 \geq 0, \varepsilon > 0$, and $|g| \leq M_0$, and $\bar{G} \in L_p(\Omega)$, then there exists $u \in W^2_{p,\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfying* (1) *in Ω (a.e.) and such that $u = g$ on $\partial\Omega$.*

We prove this theorem in Section 5.

18. REMARK. Similarly to Remark 6.6.14 we observe that generally there is no uniqueness in Theorems 14 and 17. For instance, in the one-dimensional case the (quasilinear) equation

$$D^2 u + \sqrt{12|Du|} = 0$$

for $x \in (-1,1)$ with zero boundary data has two solutions: one is identically equal to zero and the other one is $1 - |x|^3$.

Another example is given by the (semilinear) equation

$$D^2 u + 2u(1 + \sin^2 x + u^2)^{-1} = 0$$

on $(-\pi/2, \pi/2)$ with zero boundary condition. Again there are two solutions: one is $\cos x$ and the other one is identically equal to zero.

To have uniqueness we need additional assumptions (see Section 4.1:2).

19. REMARK. The parameter θ in Theorems 14 and 17 depends on p and we cannot guarantee that it stays bounded away from zero for all $p > d$. Our arguments are only valid if we take θ sufficiently small and, as $p \to \infty$, θ should go to zero.

20. REMARK. For measurable $\Gamma \subset \mathbb{R}^d$ with nonzero Lebesgue measure and locally summable f denote

$$\fint_\Gamma f(x)\,dx = \frac{1}{|\Gamma|}\int_\Gamma f(x)\,dx$$

(recall that $|\Gamma|$ is the volume of Γ). Then for $z \in \Omega$, $r > 0$, and $\mathsf{u}_0' \in \mathbb{R}$ introduce

$$\hat{F}(\mathsf{u}'') = \hat{F}_{z,r,\mathsf{u}_0'}(\mathsf{u}'') = \fint_{B_r(z)} F\big(\mathsf{u}_0', \mathsf{u}'', x\big)\,dx.$$

Observe that

$$\big|F\big(\mathsf{u}_0', \tau\mathsf{u}'', x\big) - \hat{F}(\tau\mathsf{u}'')\big| \leq \big|F\big(\mathsf{u}_0', \tau\mathsf{u}'', x\big) - \bar{F}(\tau\mathsf{u}'')\big|$$

$$+ \fint_{B_r(z)} \big|F\big(\mathsf{u}_0', \tau\mathsf{u}'', y\big) - \bar{F}(\tau\mathsf{u}'')\big|\,dy,$$

which implies that

$$\fint_{B_r(z)} \sup_{\tau > \tau_0} \tau^{-1}\big|F\big(\mathsf{u}_0', \tau\mathsf{u}'', x\big) - \hat{F}(\tau\mathsf{u}'')\big|\,dx \leq 2\theta, \tag{9}$$

if $|u''| = 1$, $B_r(z) \subset \Omega$ and $r \leq R_0$. Thus, one can be tempted to always take $\hat{F}$ as $\bar{F}$. However, there is no guarantee that $\hat{F}(\mathsf{u}'')$ is convex in u''.

21. REMARK. Under the above assumptions the function

$$F(\mathsf{u}'', x) = H(0, \mathsf{u}'', x) - H(0, x)$$

does not necessarily satisfy Assumption 6, so that this choice of F and G in Example 5 may not be optimal. The simplest example in case $d = 2$ is given by

$$H(0, \mathsf{u}'', x) = G(x) \wedge |\mathsf{u}''_{11}| + 2\mathsf{u}''_{11} + \mathsf{u}''_{22},$$

where $G(x) = (x^1)^{-\alpha}$ for $x^1 > 0$ and $G(x) = 0$ for $x^1 \leq 0$ with a small $\alpha > 0$, so that G is summable to a high power.

Indeed, assume that $0 \in \Omega$. Then for $z = 0$, small $r > 0$, and $\mathsf{u}''_{11} = 1$ the left-hand side of (9) (with $H(0, \mathsf{u}'', x) - H(0, x)$ in place of $F(\mathsf{u}'', x)$) becomes

$$\fint_{B_r} \sup_{\tau > \tau_0} \Big| 1 \wedge \big(G(x)/\tau\big) - \fint_{B_r} 1 \wedge \big(G(y)/\tau\big)\, dy \Big|\, dx$$

$$\geq \fint_{B_r} \Big| 1 \wedge \big(G(x)/\tau_0\big) - \fint_{B_r} 1 \wedge \big(G(y)/\tau_0\big)\, dy \Big|\, dx,$$

which for $r \leq \tau_0^{1/\alpha}$ equals

$$\fint_{B_r} \Big| I_{x^1 > 0} - \fint_{B_r} I_{y^1 > 0}\, dy \Big|\, dx = \fint_{B_r} |I_{x^1 > 0} - 1/2|\, dx = 1/2.$$

Hence, (4) cannot be satisfied with small θ and the natural choice for F in this example is, of course, $2\mathsf{u}''_{11} + \mathsf{u}''_{22}$.

22. Remark. In condition (4) no restriction is imposed on $F\big(\mathsf{u}'_0, \mathsf{u}'', x\big)$ for $|\mathsf{u}''| \leq \tau_0$. But even for large $|\mathsf{u}''|$ the function F satisfying Assumption 6 need not be even locally convex. For instance

$$F(D^2u, x) = \Big(1 + \varepsilon \sin\big(\ln |D^2 u|\big)\Big)\, a^{ij}(x) D_{ij} u$$

satisfies Assumption 6 with $\tau_0 = 0$ and any $\theta > 0$ given in advance if $a^{ij} \in VMO$, $(a^{ij}) \in \mathbb{S}_\delta$, and $\varepsilon > 0$ is small enough.

Indeed, it suffices to take sufficiently small R_0 and set $\bar{F}_{z,r}(\mathsf{u}'') = \mathsf{u}''_{ij} \bar{a}^{ij}_{z,r}$, where $\bar{a}^{ij}_{z,r}$ are the averages of a^{ij} over $B_r(z)$. Moreover, since $\theta > 0$, a^{ij} need not be exactly in VMO but, as we say, almost in VMO.

23. Remark. A somewhat more transparent condition than Assumption 6 can be stated in terms of

$$\hat{F}\big(\mathsf{u}'_0, \mathsf{u}'', x\big) := \lim_{\tau \to \infty} \frac{1}{\tau} F\big(\mathsf{u}'_0, \tau \mathsf{u}'', x\big)$$

provided that the limit exists. Then requiring that $\hat{F}$ be convex in u'', $D_{\mathsf{u}''} \hat{F} \in \mathbb{S}_\delta$, and $\hat{F}(\mathsf{u}'_0, \mathsf{u}'', \cdot) \in VMO$ for any $\mathsf{u}'_0, \mathsf{u}''$ seems to be enough to satisfy Assumption 6 (perhaps, modulo some kind of uniformity).

However, observe that $\hat{F}$ does not exist for F from Remark 22. One way to remedy the situation is to absorb $\varepsilon \sin\big(\ln |D^2 u|\big) a^{ij}(x) D_{ij} u$ into G. But, generally, requiring the above limit to exist is an additional, rather strong, and unnecessary assumption.

Yet it is worth mentioning [**97**], which shows that for $H = F(\mathsf{u}'') - f(x)$, any viscosity solution of $H[u] = 0$ is in $W^2_{p,\mathrm{loc}}$ with a corresponding estimate provided that $\hat{F}$ exists, is convex, and $f \in L_p$, $p > d$. We show that in such situations there exists a unique $W^2_{p,\mathrm{loc}}$-solution admitting such estimates.

In addition to the examples presented above we give two more examples related to the stochastic differential games and the Isaacs equations.

24. EXAMPLE. Let A and B be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, $\mathsf{u}'_0 \in \mathbb{R}$, and $x \in \mathbb{R}^d$ we are given functions

$$a^{\alpha}\big(\mathsf{u}'_0, x\big), \quad b^{\alpha\beta}\big(\mathsf{u}'_0, x\big), \quad c^{\alpha\beta}\big(\mathsf{u}'_0, x\big), \quad f^{\alpha\beta}\big(\mathsf{u}'_0, x\big)$$

with values in

$$\mathbb{S}_\delta, \quad \mathbb{R}^d, \quad [0, \infty), \quad \mathbb{R},$$

respectively. Assume that these functions are measurable in x, a^{α}, $b^{\alpha\beta}$, and $c^{\alpha\beta}$ are bounded, they are continuous with respect to u'_0 uniformly with respect to α, β, x, and

$$\bar{G} := \sup_{\mathsf{u}'_0, \alpha, \beta} \big|f^{\alpha\beta}\big(\mathsf{u}'_0, \cdot\big)\big| \in L_p(\Omega).$$

Consider the following Isaacs equation

$$H(v, Dv, D^2 v, x) = 0, \tag{10}$$

where

$$H(\mathsf{u}, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \Big[\sum_{i,j=1}^{d} a^{\alpha}_{ij}\big(\mathsf{u}'_0, x\big)\mathsf{u}''_{ij} + \sum_{i=1}^{d} b^{\alpha\beta}_i\big(\mathsf{u}'_0, x\big)\mathsf{u}'_i$$

$$- c^{\alpha\beta}\big(\mathsf{u}'_0, x\big)\mathsf{u}'_0 + f^{\alpha\beta}\big(\mathsf{u}'_0, x\big)\Big].$$

Our measurability, boundedness, and countability assumptions guarantee that H is measurable in x and Lipschitz continuous in u''. One can also easily check that at all points of differentiability $D_{\mathsf{u}''}H \in \mathbb{S}_\delta$ (see Corollary 4.1.4). Next assume that there is an $R_0 \in (0, \infty)$ such that for any $z \in \Omega$, $r \in (0, R_0]$, and $\mathsf{u}'_0 \in \mathbb{R}$ one can find $\bar{a}^{\alpha} \in \mathbb{S}_\delta$ (independent of x) such that

$$\fint_{B_r(z)} \sup_{\alpha \in A} \big|a^{\alpha}\big(\mathsf{u}'_0, x\big) - \bar{a}^{\alpha}\big|\, dx \le \theta,$$

where θ is taken from Theorem 14 (with $K_F = \delta^{-1}\sqrt{d}$).

Then we claim that there exists $u \in W^2_{p,\text{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfying (10) in Ω (a.e.) and such that $u = g$ on $\partial\Omega$. Furthermore, (6) holds with $\tau_0 = 0$.

To prove the claim introduce

$$F(\mathsf{u}'_0, \mathsf{u}'', x) = \sup_{\alpha \in A} \sum_{i,j=1}^{d} a^{\alpha}_{ij}\big(\mathsf{u}'_0, x\big)\mathsf{u}''_{ij}, \quad G = H - F.$$

Notice that Assumption 6 is satisfied with $\tau_0 = 0$ and

$$\bar{F}(\mathsf{u}'') := \sup_{\alpha \in A} \sum_{i,j=1}^{d} \bar{a}^{\alpha}_{ij}\, \mathsf{u}''_{ij}$$

because these functions are convex, positive homogeneous of degree one with respect to u'' and, for $|\mathsf{u}''| = 1$,

$$\fint_{B_r(z)} \big|F\big(\mathsf{u}_0', \mathsf{u}'', x\big) - \bar F(\mathsf{u}'')\big|\,dx \le \fint_{B_r(z)} \sup_{\alpha\in A}\Big|\sum_{i,j=1}^d \big[a^\alpha_{ij}\big(\mathsf{u}_0', x\big) - \bar a^\alpha\big]\mathsf{u}''_{ij}\Big|\,dx$$

$$\le \fint_{B_r(z)} \sup_{\alpha\in A}\big|a^\alpha\big(\mathsf{u}_0', x\big) - \bar a^\alpha\big|\,dx \le \theta.$$

On can easily check that the remaining item (iv) in Assumption 6 and Assumptions 4 (with $\hat\theta = 0$) are satisfied as well. Assumption 12 is satisfied because $c \ge 0$.

As a result we have a solvability theorem for (10), which covers (apart from the restriction on p), as A and B are singletons, the first result about solvability of linear elliptic equations with VMO coefficients obtained by Chiarenza, Frasca, and Longo in their pioneering paper [**20**]. In this singleton case we in addition treat semilinear equations.

25. Example. Let A and B be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, $x \in \mathbb{R}^d$, and $\mathsf{u}' \in \mathbb{R}^{d+1}$ we are given an $\mathbb{S}_\delta$-valued function $a^\alpha(\mathsf{u}_0', x)$ and a real-valued function $b^{\alpha\beta}(\mathsf{u}', x)$. Assume that these functions are measurable in x, a^α and $b^{\alpha\beta}$ are continuous with respect to u' uniformly with respect to α, β, x, and

$$\big|b^{\alpha\beta}(\mathsf{u}', x)\big| \le K_0\big|\big(\mathsf{u}_1', ..., \mathsf{u}_d'\big)\big| + \bar G(x),$$

where $\bar G \in L_p(\Omega)$. Also assume that a^α satisfies the condition from Example 24.

Consider equation (10), where

$$H(\mathsf{u}, x) := \inf_{\beta\in B}\sup_{\alpha\in A}\Big[\sum_{i,j=1}^d a^\alpha_{ij}\big(\mathsf{u}_0', x\big)\mathsf{u}''_{ij} + b^{\alpha\beta}(\mathsf{u}', x)\Big].$$

As in Example 24 one easily sees that Theorem 14 is applicable.

26. Remark. In the literature, the interior W^2_p, $p > d$, estimates like (6) with $\tau_0 = 0$ and $\Omega = B_1$ for viscosity solutions of a class of fully nonlinear uniformly elliptic equations of the form

$$H(D^2u, x) = f(x)$$

were first obtained by Caffarelli in [**12**] (see also [**14**]). His proof is based on an ingenious application of the Aleksandrov–Bakel'man–Pucci a priori estimate, the Krylov–Safonov Harnack inequality, and a covering result a version of which can be found in [**84**] and [**100**]. Our results are based on technically less sophisticated ideas and results from [**75**], which uses the Evans-Krylov, Fang-Hua Lin, and Fefferman–Stein theorems. By exploiting a weak reverse Hölder's inequality, the result of [**12**] was sharpened by Escauriaza in [**32**], who obtained the interior W^2_p-estimate for the same equations allowing $p > d - \varepsilon$, with a small constant $\varepsilon > 0$ depending only on the ellipticity constant and d.

The above cited works [**12**], [**14**], and [**32**] are quite remarkable in one respect–they do not suppose that H is convex or concave in D^2u and relate to any viscosity solution. The assumptions in [**12**] and [**14**] are quite different from ours. One of these assumptions is that the equations $H(D^2u, x_0) = 0$ admit $C^2_{\mathrm{loc}}\big(B_r(x_0)\big)$-solutions for any $B_r(x_0) \subset B_1$ and any continuous boundary data. Until now we only know that, generally, this assumption is satisfied if H is convex or concave with respect to u''. Paper [**109**] and the references there present a few exceptions. Another condition is stated in terms of

$$\beta(x,x_0)=\sup_{\mathsf{u}''\in\mathbb{S}}\frac{\big|H(\mathsf{u}'',x)-H(\mathsf{u}'',x_0)\big|}{|\mathsf{u}''|}.$$

It is assumed that for all $B_r(x_0)\subset B_1$

$$⨍_{B_r(x_0)}\beta^d(x,x_0)\,dx\le\theta^d, \tag{11}$$

where θ is sufficiently small.

Since condition (11) is repeated in very many papers, we want to compare (11) and (4) taking $d=3$, $p>3$, $G\in L_p(\mathbb{R}^d)$, $G\ge 0$, and

$$H(\mathsf{u}'',x)=G(x)\wedge\big|\mathsf{u}''_{11}\big|+2\mathsf{u}''_{11}+\mathsf{u}''_{22}+\mathsf{u}''_{33}.$$

If we take $F(\mathsf{u}'',x)=2\mathsf{u}''_{11}+\mathsf{u}''_{22}+\mathsf{u}''_{33}$, then *all* our conditions are automatically satisfied and we have a priori estimates and *solvability*.

At the same time, note that $H(\mathsf{u}'',x_0)$ is neither concave nor convex with respect to u'' and it is not clear if the solvability assumption in [**14**] is satisfied.

Also, as is easy to see, condition (11) implies that, if $B_r(x_0)\subset B_1$ and, say $G\le 1$, then (take $|\mathsf{u}''|=1$ in the definition of $\beta(x,x_0)$)

$$\Big|⨍_{B_r(x_0)}G(x)\,dx-G(x_0)\Big|\le\theta. \tag{12}$$

In particular, G should lie between two continuous functions, which collapse as $\theta\downarrow 0$. This condition is reminiscent of the usual condition in the L_p-theory of *linear* elliptic equations when one requires the leading coefficients be not necessarily continuous but be sufficiently uniformly close to continuous ones. We just repeat that our conditions, one of which may look like (11), are automatically satisfied even without anything like (12).

A number of existence results of $W^2_{p,\mathrm{loc}}$-solutions and a priory estimates in $W^2_{p,\mathrm{loc}}$ obtained by means of the theory of viscosity solutions can be found in [**13**], [**14**], and [**15**]. In all of them H is supposed to be Lipschitz continuous in u uniformly with respect to x and satisfy a continuity assumption with respect to x similar to (11). Note that these assumptions exclude, for instance, Example 25. On the other hand, our results do not cover those from [**13**], [**14**], and [**15**] either, in particular, just because $p<d$ is allowed there.

Note that in Theorem 4.2 of [**116**] one more interior estimate of type (6) is obtained under the assumptions that H is convex in u'', Lipschitz continuous in u and satisfies a condition similar to (11). On the other hand, again some values of $p<d$ are allowed. Finally, in [**15**] and [**116**] the function H is assumed to be nonincreasing with respect to u'_0 unlike in our Theorem 14.

2. A priori estimates in $C^{2+\alpha}_{\mathrm{loc}}$ for equations $F(D^2u)=0$

If $\alpha\in(0,1)$ and Q is a domain in $\mathbb{R}^d$ and u is a function on Q, we define

$$[u]_{C^\alpha(Q)}=\sup_{\substack{x,y\in Q\\ x\neq y}}\frac{\big|u(x)-u(y)\big|}{|x-y|^\alpha}$$

and by $C^{2+\alpha}(Q)$ we mean the subset of $C^2(Q)$ consisting of functions u for which

$$\|u\|_{C^{2+\alpha}(Q)}=\|u\|_{C^2(Q)}+[D^2u]_{C^\alpha(Q)}<\infty.$$

One knows that, if Q is regular enough, the norm $\|u\|_{C^{2+\alpha}(Q)}$ is equivalent to

$$\|u\|_{C^{2+\alpha}(Q)}=\|u\|_{C(Q)}+[D^2u]_{C^\alpha(Q)}<\infty.$$

As usual, by $C^{2+\alpha}_{\mathrm{loc}}(Q)$ we mean the set of functions which belong to $C^{2+\alpha}(Q')$ for any bounded domain $Q'\subset\bar{Q}'\subset Q$.

The following fundamental result is usually attributed to L.C. Evans and the author. Let $F(\mathsf{u}'')$ be a *convex* function defined for $\mathsf{u}''\in\mathbb{S}$ such that at all points of its differentiability we have

$$D_{\mathsf{u}''}F(\mathsf{u}'')\in\mathbb{S}_\delta,$$

where $\delta\in(0,1]$ is a fixed number.

1. THEOREM. *There exists a constant*

$$\alpha_0=\alpha_0(\delta,d)\in(0,1)$$

such that for any $R\in(0,\infty)$ and $u\in C^{1,1}(B_R)$, satisfying $F(D^2u)=0$ (a.e.) in B_R, we have $u\in C^{2+\alpha_0}_{\mathrm{loc}}(B_R)$. Furthermore, for any $r\in(0,R)$ and $\alpha\in(0,\alpha_0]$,

$$[D^2u]_{C^\alpha(B_r)}\le N\frac{1}{(R-r)^{2+\alpha}}\sup_{\partial B_R}|u-l|, \tag{1}$$

where $l=l(x)$ is an arbitrary quadratic *function and $N=N(\delta,d,\alpha)<\infty$.*

To prove the theorem we need a few auxiliary results, but first we observe that we may take $l\equiv0$ and assume that

$$F(0)=0. \tag{2}$$

Indeed, take an arbitrary quadratic function $l(x)$ and introduce

$$\hat{F}(\mathsf{u}'')=F\big(\mathsf{u}''+D^2l+tD^2|x|^2\big),$$

where t is chosen so that

$$\hat{F}(0)=F\big(D^2l+tD^2|x|^2\big)=0.$$

The latter is possible since the derivative with respect to t of the middle term is $2\operatorname{tr}D_{\mathsf{u}''}F$ which is between $2\delta d$ and $2\delta^{-1}d$. Then, under the assumption that the theorem is true if $F(0)=0$ and $l\equiv0$, we apply it to $v(x)=u(x)-t\big(|x|^2-R^2\big)-l(x)$ and $\hat{F}$ in place of u and F, respectively, to get

$$[D^2u]_{C^\alpha(B_r)}=[D^2v]_{C^\alpha(B_r)}\le N\frac{1}{(R-r)^{2+\alpha}}\sup_{\partial B_R}|v|$$

$$=N\frac{1}{(R-r)^{2+\alpha}}\sup_{\partial B_R}|u-l|.$$

Therefore, below we always take $l \equiv 0$, assume that $u \in C^{1,1}(B_R)$, $F(D^2u) = 0$ (a.e.) in B_R, and (2) holds. However, in the first auxiliary result the fact that $F(D^2u) = 0$ (a.e.) in B_R is not used.

Observe that for almost any $x_0 \in B_R$, as $x \to x_0$,

$$u(x) = u(x_0) + D_i u(x_0)(x^i - x_0^i)$$

$$+ (1/2)D_{ij}u(x_0)(x^i - x_0^i)(x^j - x_0^j) + o\big(|x - x_0|^2\big). \tag{3}$$

Let U be a Borel subset of these points having full measure.

Recall that for $\lambda \in \mathbb{R}^d$ and $h > 0$

$$\Delta_{h,\lambda} u(x) := \frac{u(x + h\lambda) - 2u(x) + u(x - h\lambda)}{h^2}.$$

2. LEMMA. *For any unit $\lambda \in \mathbb{R}^d$ we have $\Delta_{h,\lambda} u \to D_{(\lambda)(\lambda)} u$ on U as $h \downarrow 0$ and for any ball $B \subset B_R$ we have*

$$\sup_{B \cap U} D_{(\lambda)(\lambda)} u = \lim_{h \downarrow 0} \sup_{B^h} \Delta_{h,\lambda} u, \tag{4}$$

$$\inf_{B \cap U} D_{(\lambda)(\lambda)} u = \lim_{h \downarrow 0} \inf_{B^h} \Delta_{h,\lambda} u. \tag{5}$$

Proof. Owing to (3) we have $\Delta_{h,\lambda} u = D_{(\lambda)(\lambda)} u + o(1)$ at any given point of U as $h \downarrow 0$. This proves the first assertion and also shows that (4) holds with $\leq$ in place of $=$.

To prove the converse inequality, fix a unit λ and for $x_0 \in B_R$ denote by $\ell(x_0)$ the intersection of the line $\{x_0 + t\lambda : t \in \mathbb{R}\}$ with B_R. We call a point $x_0 \in B_R$ admissible if $\ell(x_0) \cap U$ has full (one-dimensional) measure.

By Fubini's theorem the set of admissible points has full measure. Furthermore, on each line $\ell(x_0)$ the function u is of class $C^{1,1}$ and hence it is twice differentiable in the sense of Taylor's formula almost everywhere on the line. If x_0 is admissible, then by comparing (3) with the Taylor's expansion mentioned above we conclude that

$$u_{tt}(x_0 + t\lambda) := \frac{\partial^2 u(x_0 + t\lambda)}{(\partial t)^2} = D_{(\lambda)(\lambda)} u(x_0 + t\lambda)$$

on $\ell(x_0)$ almost everywhere. Since, for $x_0 \in B^h$, it holds that

$$u(x_0 + h\lambda) - 2u(x_0) + u(x_0 - h\lambda) = \int_{-h}^{h} \big(h - |r|\big) u_{tt}(x_0 + t\lambda)\, dt,$$

it follows that, for admissible $x_0 \in B^h$, $\Delta_{h,\lambda} u(x_0)$ is dominated by the left-hand side of (4). To finish proving (4), it only remains to add that $\Delta_{h,\lambda} u$ is a continuous function. Equation (5) is proved by replacing u with $-u$ in (4). The lemma is proved. □

3. LEMMA. *Let U, V, W be $d \times d$ symmetric matrices. Then there exist $a \in \mathbb{S}_\delta$ such that*

$$F(U) - 2F(V) + F(W) \geq a^{ij}(U - 2V + W)_{ij}. \tag{6}$$

Proof. Since F is differentiable almost everywhere we may assume that it is differentiable at V. Then observe that owing to the convexity of F

$$F(U) - F(V) = \int_0^1 \frac{\partial}{\partial t} F(tU + (1-t)V)\,dt$$

$$\geq \int_0^1 \frac{\partial}{\partial t} F\big(tU + (1-t)V\big)\big|_{t=0}\,dt = D_{\mathsf{u}''_{ij}} F(V)(U_{ij} - V_{ij}).$$

Similarly,

$$F(W) - F(V) \geq D_{\mathsf{u}''_{ij}} F(V)(W_{ij} - V_{ij}),$$

and the lemma is proved. □

4. Lemma. *For any constant* $\nu \in (0,1)$ *there exists a constant*

$$\beta = \beta(\nu, d, \delta) \in (0,1)$$

such that if λ *is a unit vector in* $\mathbb{R}^d$, $2r \leq R$, $\omega > 0$, *and*

$$\big|\big\{x \in B_{2r} : D_{(\lambda)(\lambda)}u - \inf_{U \cap B_{2r}} D_{(\lambda)(\lambda)}u > \omega\big\}\big| \geq \nu |B_{2r}|, \tag{7}$$

then

$$\inf_{U \cap B_r} D_{(\lambda)(\lambda)}u - \inf_{U \cap B_{2r}} D_{(\lambda)(\lambda)}u \geq \beta\omega. \tag{8}$$

Proof. By (7) and Lemma 2 for all sufficiently small $h > 0$ we have

$$\big|\big\{x \in B^h_{2r} : \Delta_{h,\lambda}u - \inf_{B^h_{2r}} \Delta_{h,\lambda}u > \omega'\big\}\big| \geq \nu' |B^h_{2r}|, \tag{9}$$

where $\omega' = \omega/2$, $\nu' = \nu/2$. Furthermore, Lemma 3 (combined with Lemma 4.1.5) implies that for an $\mathbb{S}_\delta$-valued measurable function a (and $h \leq r/4$)

$$a_{ij} D_{ij} \Delta_{h,\lambda} u \leq 0$$

almost everywhere in B^h_{2r}. The same inequality is true if we replace $\Delta_{h,\lambda}u$ with

$$\Delta_{h,\lambda}u - \inf_{B^h_{2r}} \Delta_{h,\lambda}u \quad (\geq 0).$$

It follows by Theorem 9.5.4 (a) and Theorem 9.5.1 that there exists a constant $\beta = \beta(\nu, d, \delta) \in (0,1)$ such that

$$\inf_{B_r} \Delta_{h,\lambda}u - \inf_{B^h_{2r}} \Delta_{h,\lambda}u \geq \beta\omega.$$

By letting $h \downarrow 0$ we get (8) and the lemma is proved. □

5. Remark. Lemma 4 with the proof not using finite differences can be found in [**104**]. Below we follow even closer the exposition in [**104**], that is shorter than the original arguments of Evans and Krylov, which have their own advantage of being more intuitively clear. They reveal themselves in the original form only in the proof of Theorem 13.7.1. It is also worth saying that another approach to proving Theorem 1 is given in [**16**] and extended to integro-differential operators in [**18**].

Define

$$\omega(r, x) = \max_{|\lambda|=1} \operatorname*{osc}_{U \cap B_r(x)} D_{(\lambda)(\lambda)}u, \quad \omega(r) = \omega(r, 0).$$

6. Lemma. *For unit* $\lambda\in\mathbb{R}^d$, $\mu,\nu\in(0,1)$, *and* $r\in(0,R]$ *introduce*

$$\Gamma_{\lambda,\mu,r}=\big\{x\in U\cap B_r: D_{(\lambda)(\lambda)}u(x)-\inf_{U\cap B_r}D_{(\lambda)(\lambda)}u>\mu\omega(r)\big\},$$

$$\Sigma_{\mu,\nu,r}=\big\{\lambda\in\partial B_1:|\Gamma_{\lambda,\mu,r}|>\nu|B_r|\big\}.$$

Then there exist $\mu=\mu(d,\delta),\nu=\nu(d,\delta)\in(0,1)$ *such that* $|\Sigma_{\mu,\nu,r}|\geq\nu|\partial B_1|$ *(where this time* $|\cdot|$ *stands for the surface measure).*

Proof. For any $x\in U\cap B_r$ and unit λ, if

$$D_{(\lambda)(\lambda)}u(x)\geq(1/2)\Big[\sup_{U\cap B_r}D_{(\lambda)(\lambda)}u+\inf_{U\cap B_r}D_{(\lambda)(\lambda)}u\Big],$$

then

$$D_{(\lambda)(\lambda)}u(x)-\inf_{U\cap B_r}D_{(\lambda)(\lambda)}u\geq(1/2)\operatorname*{osc}_{U\cap B_r}D_{(\lambda)(\lambda)}u.$$

In the remaining case

$$\sup_{U\cap B_r}D_{(\lambda)(\lambda)}u-D_{(\lambda)(\lambda)}u(x)>(1/2)\operatorname*{osc}_{U\cap B_r}D_{(\lambda)(\lambda)}u.$$

Thus,

$$\sup_{y\in U\cap B_r}\big|D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big|\geq(1/2)\operatorname*{osc}_{U\cap B_r}D_{(\lambda)(\lambda)}u.$$

By taking the supremums over $|\lambda|=1$, we conclude that for each $x\in U\cap B_r$ there is a $y\in U\cap B_r$ such that

$$\max_{|\lambda|=1}\big|D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big|\geq(1/3)\omega(r). \tag{10}$$

Next, there are two cases: either there exists $\lambda\in\partial B_1$ such that

$$D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\geq(1/3)\omega(r),$$

or the opposite inequality holds for all $\lambda\in\partial B_1$, but

$$-\min_{|\lambda|=1}\big[D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big]\geq(1/3)\omega(r).$$

In the latter case we observe that there exists an $a\in\mathbb{S}_\delta$ such that $a_{ij}\big[D_{ij}u(x)-D_{ij}u(y)\big]=0$, which shows that

$$0\leq\frac{d-1}{\delta}\max_{|\lambda|=1}\big[D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big]+\delta\min_{|\lambda|=1}\big[D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big]$$

$$\leq\frac{d-1}{\delta}\max_{|\lambda|=1}\big[D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\big]-(\delta/3)\mu(r),$$

and along with the former case leads to the conclusion that there is a $\mu=\mu(d,\delta)\in(0,1)$ and a unit $\lambda=\lambda(x,y)$ such that

$$D_{(\lambda)(\lambda)}u(x)-D_{(\lambda)(\lambda)}u(y)\geq2\mu\omega(r). \tag{11}$$

Moreover, for $\lambda \in \partial B_1$ the absolute value of the ratio

$$\frac{1}{\omega(r)}\big[D_{(\lambda)(\lambda)}u(x) - D_{(\lambda)(\lambda)}u(y)\big]$$

is less than one and therefore has a modulus of continuity with respect to λ depending only on d. It follows that (11) with μ in place of 2μ also holds in a neighborhood of $\lambda(x,y)$ of a fixed size.

Since

$$D_{(\lambda)(\lambda)}u(y) \geq \inf_{U\cap B_r} D_{(\lambda)(\lambda)}u$$

it follows that, for any $x \in U \cap B_r$, we have

$$\int_{\partial B_1} I_{\Gamma_{\lambda,\mu,r}}(x)\, d\sigma_\lambda \geq 2\nu|\partial B_1|, \tag{12}$$

where σ_λ is the surface measure on ∂B_1 and $\nu = \nu(d,\delta) \in (0,1)$. By integrating through (12) with respect to x and using Fubini's theorem, we obtain

$$\int_{\partial B_1} |\Gamma_{\lambda,\mu,r}|\, d\sigma_\lambda \geq 2\nu|\partial B_1|\cdot|B_r|.$$

Due to the fact that $|\Gamma_{\lambda,\mu,r}| \leq |B_r|$, the left-hand side is less than

$$|B_r|\cdot|\Sigma_{\mu,\nu,r}| + \nu|\partial B_1|\cdot|B_r|,$$

which yields the desired result. The lemma is proved. □

7. LEMMA. *Define*

$$\omega^*(r) = \int_{\partial B_1} \operatorname*{osc}_{U\cap B_r} D_{(\lambda)(\lambda)}u\, d\sigma_\lambda.$$

Then there is a constant $N = N(d)$ such that $\omega \leq N\omega^*$ and $\omega^* \leq N\omega$ for $r \leq R$.

This result follows easily from the observation that

$$\frac{\operatorname{osc}_{U\cap B_r} D_{(\lambda)(\lambda)}u}{\omega(r)}$$

is a Lipschitz continuous function of $\lambda \in \partial B_1$ with Lipschitz constant depending only on d.

8. LEMMA. *There are constants $\alpha_0 = \alpha_0(\delta,d) \in (0,1)$ and $N = N(d,\delta) \in (0,\infty)$ such that for $r \leq R$ and $\alpha \in (0,\alpha_0]$*

$$\omega(r) \leq N\Big(\frac{r}{R}\Big)^{\alpha}\omega(R). \tag{13}$$

Proof. By Lemma 7 it suffices to prove (13) with ω^* in place of ω. Furthermore, since $r/R \leq 1$, it suffices to prove (13) with $\alpha = \alpha_0$.

In light of Lemmas 6 and 4 for

$$\lambda \in \Sigma_{\mu,\nu,2r}$$

inequality (8) holds with $\omega = \mu\omega(2r)$, provided that $r \leq R/2$. Observe that

$\beta\mu\omega(2r) \geq \mu'\omega^*(2r)$, where $\mu' = \mu'(d,\delta) \in (0,1)$. In particular, for those λ

$$\operatorname*{osc}_{U\cap B_{2r}} D_{(\lambda)(\lambda)}u - \operatorname*{osc}_{U\cap B_r} D_{(\lambda)(\lambda)}u$$

$$\geq \inf_{U\cap B_r} D_{(\lambda)(\lambda)}u - \inf_{U\cap B_{2r}} D_{(\lambda)(\lambda)}u \geq \mu'\omega^*(2r).$$

It follows that

$$\omega^*(2r) - \omega^*(r) = \int_{\partial B_1} \big[\operatorname*{osc}_{U\cap B_{2r}} D_{(\lambda)(\lambda)}u - \operatorname*{osc}_{U\cap B_r} D_{(\lambda)(\lambda)}u\big]\, d\sigma_\lambda$$

$$\geq |\Sigma_{\mu,\nu,2r}|\mu'\omega^*(2r) \geq \gamma\omega^*(2r),$$

where $\gamma = \gamma(d,\delta) \in (0,1)$. Hence, $\omega^*(r) \leq (1-\gamma)\omega^*(2r)$. Now by repeating an iterative argument on page 190 we conclude that (13) holds with ω^* in place of ω and with $\alpha_0 = \log_2(1+\gamma)$ if $r \leq R/2$. For the remaining range of r and R it holds because of the monotonicity of ω^*. The lemma is proved. □

Proof of Theorem 1. Lemma 8 is applicable not only to the balls centered at the origin but at any other point in B_R as well, as long as these balls belong to B_R. Then we see, in particular, that $D_{(\lambda)(\lambda)}u$ are uniformly continuous on $U\cap B_{R'}$ with any $R' < R$. By calculus they admit a unique continuation on $B_{R'}$ and these continuations are second-order derivative of u. Furthermore, by enclosing $x, y \in B_r$ into a ball of radius $|x-y|$, finding the largest concentric ball still lying in B_R, and using Lemma 8 we easily see that, for $r < R$,

$$M(r) := [D^2u]_{C^\alpha(B_r)} \leq N\frac{1}{(R-r)^\alpha}\omega(R) \leq N\frac{1}{(R-r)^\alpha}\sup_{B_R,|\lambda|=1}|D_{(\lambda)(\lambda)}u|. \quad (14)$$

In particular, $M(r) < \infty$ for $r < R$.

To convert the right-hand side of (14) to the one in (1), observe that, for any $r < R$, unit $\lambda \in \mathbb{R}^d$, $h < r/2$, $x \in B_r$, $x' \in B_{r-h}$, such that x' lies on the same diameter as x and has the distance to x equal to h, we have

$$\big|D_{(\lambda)(\lambda)}u(x)\big| \leq \big|D_{(\lambda)(\lambda)}u(x) - \Delta_{h,\lambda}u(x')\big| + \frac{4}{h^2}\sup_{B_R}|u|$$

$$\leq (2h)^\alpha[D_{(\lambda)(\lambda)}u]_{C^\alpha(B_r)} + \frac{4}{h^2}\sup_{B_R}|u|.$$

It follows that if $r_n < r_{n+1} < R$ and $h < r_{n+1}/2$, then

$$M(r_n) \leq N\frac{1}{(r_{n+1}-r_n)^\alpha}\sup_{B_{r_{n+1}},|\lambda|=1}|D_{(\lambda)(\lambda)}u|$$

$$\leq N_0^\alpha\frac{h^\alpha}{(r_{n+1}-r_n)^\alpha}M(r_{n+1}) + N\frac{1}{(r_{n+1}-r_n)^\alpha h^2}\sup_{B_R}|u|. \quad (15)$$

Without losing generality, we assume that $N_0 \geq 2$ and we take here $h = (r_{n+1}-r_n)/N_0$ $(< r_{n+1}/2)$. Then we get

$$M(r_n) \leq 2^{-\alpha}M(r_{n+1}) + N\frac{1}{(r_{n+1}-r_n)^{2+\alpha}}\sup_{B_R}|u|.$$

Now we choose $r_1 = r$ and for $n \geq 1$ define

$$r_{n+1} = r + (R-r)\kappa \sum_{k=1}^{n} \frac{1}{k^2},$$

where κ^{-1} is the value of the above series with ∞ in place of n, so that $r_n \uparrow R$. Then we conclude

$$\sum_{n=1}^{\infty} 2^{-n\alpha} M\big(r(n)\big) \leq \sum_{n=2}^{\infty} 2^{-n\alpha} M\big(r(n)\big) + N\frac{1}{(R-r)^{2+\alpha}} \sup_{B_R} |u| \sum_{n=1}^{\infty} n^2 2^{-n\alpha}.$$

To ensure that the above series converge we replace R with a smaller number R' and use the fact that $M(R') < \infty$. Then we cancel like terms and let $R' \uparrow R$ to get (1) with $l \equiv 0$ and B_R in place of ∂B_R. After that we observe that (recall (2))

$$0 = F(D^2u) - F(0) = a_{ij}D_{ij}u$$

in B_R (a.e.) with a measurable $\mathbb{S}_\delta$-valued matrix (a_{ij}). Then the maximum principle shows that

$$\sup_{B_R} |u| = \sup_{\partial B_R} |u|.$$

The theorem is proved. □

9. Corollary. *Let $u \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy $F(D^2u) = 0$ in $\mathbb{R}^d$ (a.e.). Assume that*

$$\varliminf_{|x|\to\infty} \frac{u(x)}{|x|^{2+\alpha_0}} = 0.$$

Then u is a quadratic function.

3. Solvability in $C^{2+\alpha}_{\mathrm{loc}}$ for equations $F(D^2v) = 0$ and estimates of sharp functions of D^2u

Let $F(\mathsf{u}'')$ be a *convex* function defined for $\mathsf{u}'' \in \mathbb{S}$ such that at all points of its differentiability we have

$$D_{\mathsf{u}''}F(\mathsf{u}'') \in \mathbb{S}_\delta,$$

where $\delta \in (0,1]$ is a fixed number. Let $\alpha_0 = \alpha_0(\delta, d) \in (0,1)$ be the constant from Theorem 2.1.

1. Theorem. *For any $g \in C(\partial B_2)$ there exists a unique $v \in C(\bar{B}_2) \cap C^{2+\alpha_0}_{\mathrm{loc}}(B_2)$ satisfying*

$$F(D^2v) = 0 \quad in \quad B_2, \quad v = g \quad on \quad \partial B_2. \tag{1}$$

Furthermore,

$$\big|D^2v(x) - D^2v(y)\big| \leq N|x-y|^{\alpha_0} \sup_{\partial B_2} |g - p|$$

as long as $x, y \in B_1$, where p is an arbitrary polynomial of degree 2 on $\mathbb{R}^d$ and N depends only on δ and d.

Proof. As usual, uniqueness follows from the maximum principle. Furthermore, replacing $F(\mathsf{u}'')$ with $F(\mathsf{u}''+D^2p)$ and g with $g-p$ shows that it suffices to concentrate on $p\equiv 0$. Also, in view of Theorem 2.1, it suffices to prove the existence of $v\in C(\bar B_2)\cap C^{1,1}_{\text{loc}}(B_2)$ satisfying (1).

Recall that $\bar\delta$ is introduced in Remark 6.2.2, and set

$$\bar\alpha=\bar\alpha(\delta,d)=\alpha_0\big(\bar\delta(d,\delta),d\big)\quad(\in(0,1)). \tag{2}$$

First assume that $g\in C^2(\mathbb{R}^d)$. Take P from (6.2.4), introduce $P[u](x)=P\big(D^2u(x)\big)$, and for $K>0$ consider the equation

$$\max\big(F[u],P[u]-K\big)=0$$

in B_2 (a.e.) with boundary condition $u=g$ on ∂B_2. Observe that $\big|F(0)\big|\leq \bar H+0|\mathsf{u}'|$, where $\bar H=\big|F(0)\big|$. Hence, by Theorem 6.6.20 and Remark 6.6.18, this problem has a unique solution $u_K\in C(\bar B_2)\cap C^{1,1}_{\text{loc}}(B_2)$.

By Theorem 2.1 $u_K\in C^{2+\bar\alpha}_{\text{loc}}(B_2)$ and for any $r\in(0,2)$,

$$[D^2u_K]_{C^{\bar\alpha}(B_r)}\leq N(\delta,d)\frac{1}{(2-r)^{2+\bar\alpha}}\sup_{\partial B_2}|g|. \tag{3}$$

In light of Remark 6.6.19 and the Arzelà-Ascoli theorem, we see that the set $\{u_K:K>0\}$ is precompact in $C(\bar B_2)$. Interpolation theorems and (3), tell us that the set $\{u_K:K>0\}$ is precompact in $C^2(B_r)$ with any $r<R$. Therefore, there is a sequence $K_n\to\infty$ such that u_{K_n},Du_{K_n}, and $D^2u_{K_n}$ converge uniformly in $\bar B_2$, $C(B_r)$, and $C(B_r)$ for any $r<2$, respectively, to a function v, Dv, and D^2v, respectively. Obviously, $v\in C(\bar B_2)\cap C^2_{\text{loc}}(B_2)$ and v satisfies (1). Also (3) holds with v in place of u_K. This proves the theorem if $g\in C^2(\mathbb{R}^d)$.

In the case of general $g\in C(\partial B_2)$ we approximate it in $C(\partial B_2)$-norm by $g_n\in C^3(\mathbb{R}^d)$. Then we obtain the functions v_n corresponding to g_n. By the maximum principle

$$\sup_{\bar B_2}|v_n-v_m|\leq\sup_{\partial B_2}|g_n-g_m|.$$

Hence, there is $v\in C(\bar B_2)$ such that $v_n\to v$ uniformly in $\bar B_2$. Of course, $v=g$ on ∂B_2. Furthermore, estimate (3) with v_n and g_n in place of u_K and g yields a similar estimate for v, which shows that $v\in C^{2+\alpha}_{\text{loc}}(B_2)$ and proves the theorem. □

Below in this section we fix

$$\alpha\in(0,\alpha_0].$$

2. LEMMA. *Let $r\in(0,\infty)$, $\nu\geq 2$ and let $\phi\in C(\partial B_{\nu r})$. Then there exists a unique $v\in C(\bar B_{\nu r})\cap C^{2+\alpha}_{\text{loc}}(B_{\nu r})$ such that*

$$F(D^2v)=0\quad in\quad B_{\nu r},\qquad v=\phi\quad on\quad \partial B_{\nu r}.$$

Furthermore,

$$⨍_{B_r}⨍_{B_r}\big|D^2v(x)-D^2v(y)\big|\,dxdy\leq N(d,\alpha,\delta)\nu^{-2-\alpha}r^{-2}\sup_{\partial B_{\nu r}}|\phi|.$$

Proof. Scalings show that it suffices to concentrate on $r=2/\nu$. In that case the existence of solution follows from Theorem 1, which also implies that for $x,y\in B_{2/\nu}\subset B_1$

$$\big|D^2v(x)-D^2v(y)\big|\leq N\nu^{-\alpha}\sup_{\partial B_2}|\phi|.$$

It only remains to observe that

$$\fint_{B_{2/\nu}} \fint_{B_{2/\nu}} \big|D^2 v(x) - D^2 v(y)\big|\, dxdy \leq \sup_{x,y\in B_{2/\nu}} \big|D^2 v(x) - D^2 v(y)\big|.$$

The lemma is proved. □

Recall that $\gamma_0 = \gamma_0(d,\delta,K)$ is introduced in Lemma 9.4.2 and set

$$\bar\gamma = \bar\gamma(d,\delta) := \gamma_0(d,\delta,0) \quad (\in (0,1)). \tag{4}$$

3. LEMMA. *Let $r \in (0,\infty)$ and $\nu \in [2,\infty)$. Then for any $u \in W^2_d(B_{\nu r})$ and $\gamma \in (0,\bar\gamma]$ we have*

$$\Big(\fint_{B_r}\fint_{B_r} \big|D^2u(x) - D^2u(z)\big|^\gamma\, dxdy\Big)^{1/\gamma}$$

$$\leq N\nu^{d/\gamma}\Big(\fint_{B_{\nu r}} \big|F[u]\big|^d\, dx\Big)^{1/d} + N\nu^{-\alpha}\Big(\fint_{B_{\nu r}} |D^2u|^d\, dx\Big)^{1/d}, \tag{5}$$

where N depends only on d and δ.

Proof. Define v to be a unique $C(\bar B_{\nu r}) \cap C^{2+\alpha}_{\mathrm{loc}}(B_{\nu r})$-solution of the equation $F[v] = 0$ in $B_{\nu r}$ with boundary condition $v = u$ on $\partial B_{\nu r}$. Such a function exists by Lemma 2. Furthermore, $v(x) - b^i x^i - c$ satisfies the same equation for any constants b^i, c. Hence by Lemma 2 and Hölder's inequality

$$I_r := \Big(\fint_{B_r}\fint_{B_r} \big|D^2v(x) - D^2v(y)\big|^\gamma\, dxdy\Big)^{1/\gamma}$$

$$\leq N\nu^{-2-\alpha}r^{-2} \sup_{x\in\partial B_{\nu r}} \big|u(x) - (D_i u)_{B_{\nu r}} x^i - u_{B_{\nu r}}\big|.$$

By Poincaré's inequality (see, for instance, Lemma 2.1 in [**70**]) the last supremum is dominated by a constant times

$$\nu^2 r^2 \Big(\fint_{B_{\nu r}} |D^2u|^d\, dx\Big)^{1/d}.$$

It follows that

$$I_r \leq N\nu^{-\alpha}\Big(\fint_{B_{\nu r}} |D^2u|^d\, dx\Big)^{1/d}. \tag{6}$$

Next, the function $w = u - v$ is of class $C(\bar B_{\nu r}) \cap W^2_{d,\mathrm{loc}}(B_{\nu r})$ and for an operator $\mathcal{L} \in \mathbb{L}_{\delta,0}$ we have

$$F[u] - F[v] = \mathcal{L}(u - v), \quad \mathcal{L}w = F[u]$$

in $B_{\nu r}$ (a.e.). Moreover, $w = 0$ on $\partial B_{\nu r}$. Therefore, by Theorem 9.4.6 there exists $N = N(d,\delta) < \infty$ such that

$$\fint_{B_r} |D^2w|^\gamma\, dx \leq \nu^d \fint_{B_{\nu r}} |D^2w|^\gamma\, dx \leq N\nu^d\Big(\fint_{B_{\nu r}} \big|F[u]\big|^d\, dx\Big)^{\gamma/d}.$$

In fact, Theorem 9.4.6 provides the above estimate only for $\nu r = 1$. The general case is taken care of by scalings. Upon combining this result with (6) we come to (5) and the lemma is proved. □

4. A priori estimates in $W^2_{p,\mathrm{loc}}$

Here we suppose that Assumptions 1.4 and 1.6 are satisfied. Thus, we suppose that all assumptions stated before Theorem 1.10 are satisfied.

First we derive the following.

1. Lemma. *For any $q \in [1,\infty)$ and $\mu > 0$ there is a $\theta = \theta(d,\delta,K_F,\mu,q) > 0$ such that, if Assumption 1.6 is satisfied with this θ, then for any $\mathsf{u}'_0 \in \mathbb{R}$, $r \in (0,R_0]$ and $z \in \Omega$ such that $B_r(z) \subset \Omega$ we have*

$$I(r,q,z) := \fint_{B_r(z)} \sup_{\substack{\mathsf{u}''\in\mathbb{S},\\ |\mathsf{u}''|>\tau_0}} \frac{\big|F\big(\mathsf{u}'_0,\mathsf{u}'',x\big) - \bar F(\mathsf{u}'')\big|^q}{|\mathsf{u}''|^q}\,dx \le \mu^q,$$

where $\bar F = \bar F_{z,r,\mathsf{u}'_0}$.

Proof. The variable u'_0 in this statement is just a parameter and we drop it. Then observe that $\big|F(\mathsf{u}'',x)\big| = \big|F(\mathsf{u}'',x) - F(0,x)\big| \le K_F|\mathsf{u}''|$ and $\big|\bar F(\mathsf{u}'')\big| \le \delta^{-1}|\mathsf{u}''|$, so that $I(r,q,z) \le N(\delta,K_F,q)I(r,1,z)$ and we may assume that $q = 1$.

Next, the functions $\tau^{-1}F(\tau\mathsf{u}'',x)$ and $\tau^{-1}\bar F(\tau\mathsf{u}'')$ are Lipschitz continuous with respect to u'' with constants depending only on δ and K_F. Therefore, there exist points $\mathsf{u}''(1),...,\mathsf{u}''(n)$ with $n = n(\mu,d,\delta,K_F)$, such that $\big|\mathsf{u}''(k)\big| = 1$ and for any $\mathsf{u}'' \in \mathbb{S}$ with $|\mathsf{u}''| = 1$ there exists a k such that

$$\big|\tau^{-1}F(\tau\mathsf{u}'',x) - \tau^{-1}F\big(\tau\mathsf{u}''(k),x\big)\big| \le \mu/4,\ \big|\tau^{-1}\bar F(\tau\mathsf{u}'') - \tau^{-1}\bar F\big(\tau\mathsf{u}''(k)\big)\big| \le \mu/4$$

for all $\tau > 0$. Also note that setting $\tau = |\mathsf{u}''|$ we get

$$\sup_{\mathsf{u}''\in\mathbb{S},|\mathsf{u}''|>\tau_0} \frac{\big|F(\mathsf{u}'',x) - \bar F(\mathsf{u}'')\big|}{|\mathsf{u}''|} = \sup_{\mathsf{u}'':|\mathsf{u}''|=1}\sup_{\tau>\tau_0} \tau^{-1}\big|F(\tau\mathsf{u}'',x) - \bar F(\tau\mathsf{u}'')\big|$$

$$\le \sum_{k=1}^n \sup_{\tau>\tau_0} \tau^{-1}\big|F\big(\tau\mathsf{u}''(k),x\big) - \bar F\big(\tau\mathsf{u}''(k)\big)\big| + \mu/2.$$

After that it is seen that our assertion is true with $q = 1$ for $\theta(d,\delta,K_F,\mu,1) = \mu/(2n)$. The lemma is proved. □

Recall that α_0 and $\bar\gamma$ are introduced in Theorem 2.1 and (3.4), respectively.

2. Lemma. *Let $r \in (0,\infty)$ and $\nu \ge 2$ be such that $\nu r \le R_0$ and $\Omega^{\nu r} \ne \emptyset$. Take*

$$\mu \in (0,\infty), \quad \beta \in (1,\infty),$$

and suppose that Assumption 1.6 is satisfied with $\theta = \theta(d,\delta,K_F,\mu,\beta d)$ (see Lemma 1). Take $\gamma \in (0,\bar\gamma]$, $\alpha \in (0,\alpha_0]$, a function $u \in W^2_d(\Omega)$, and for $x_0 \in \Omega^{\nu r}$ denote

$$I_r(x_0) = \Big(\fint_{B_r(x_0)}\fint_{B_r(x_0)} \big|D^2u(x_1) - D^2u(x_2)\big|^\gamma\,dx_1dx_2\Big)^{1/\gamma}.$$

Then for $x_0 \in \Omega^{\nu r}$

$$I_r(x_0) \le N\nu^{d/\gamma}\Big(\fint_{B_{\nu r}(x_0)} \big|F[u]\big|^d\,dx\Big)^{1/d} + N\tau_0\nu^{d/\gamma}$$

$$+ N\Big[\big(\mu + \omega_{F,u,\Omega}(\nu r)\big)\nu^{d/\gamma} + \nu^{-\alpha}\Big]\Big(\fint_{B_{\nu r}(x_0)} |D^2u|^{\beta' d}\,dx\Big)^{1/(\beta' d)}, \quad (1)$$

where $\beta' = \beta/(\beta-1)$ *and* N *depends only on* d, K_F, α, *and* δ.

Proof. Set $\rho := \nu r$. Since $\rho \le R_0$, $\bar F = \bar F_{x_0,\rho,u(x_0)}$ is well defined and by Lemma 3.3

$$I_r(x_0) \le N\nu^{d/\gamma}\Big(\fint_{B_\rho(x_0)} \big|\bar F[u]\big|^d\,dx\Big)^{1/d} + N\nu^{-\alpha}\Big(\fint_{B_\rho(x_0)} |D^2u|^d\,dx\Big)^{1/d}. \quad (2)$$

By setting $\hat F[u](x) = F\big(u(x_0), D^2u(x), x\big)$ we find

$$\fint_{B_\rho(x_0)} \big|\bar F[u]\big|^d\,dx \le N \fint_{B_\rho(x_0)} \big|F[u]\big|^d\,dy + NJ_1 + NJ_2,$$

where

$$J_1 = \fint_{B_\rho(x_0)} \big|\hat F[u] - \bar F[u]\big|^d\,dx$$

is dominated by (cf. Remark 1.7)

$$\fint_{B_\rho(x_0)} I_{|D^2u|>\tau_0} \frac{\big|\hat F[u] - \bar F[u]\big|^d}{|D^2u|^d} |D^2u|^d\,dx + N\tau_0^d,$$

which in turn owing to Lemma 1 and Hölder's inequality is less than

$$\mu^d\Big(\fint_{B_\rho(x_0)} |D^2u|^{\beta' d}\,dx\Big)^{1/\beta'} + N\tau_0^d,$$

and

$$J_2 = \fint_{B_\rho(x_0)} \big|\hat F[u] - F[u]\big|^d\,dx \le \omega_F^d\big(\operatorname*{osc}_{B_\rho(x_0)} u\big) \fint_{B_\rho(x_0)} |D^2u|^d\,dx.$$

It follows that

$$\Big(\fint_{B_\rho(x_0)} \big|\bar F[u]\big|^d\,dy\Big)^{1/d} \le N\Big(\fint_{B_\rho(x_0)} \big|F[u]\big|^d\,dy\Big)^{1/d}$$

$$+N\mu\Big(\fint_{B_\rho(x_0)} |D^2u|^{\beta' d}\,dy\Big)^{1/(\beta' d)} + N\tau_0 + N\omega_{F,u,\Omega}(\rho)\Big(\fint_{B_\rho(x_0)} |D^2u|^d\,dx\Big)^{1/d}.$$

This and (2) yield (1) since

$$\Big(\fint_{B_\rho(x_0)} |D^2u|^d\,dx\Big)^{1/d} \le \Big(\fint_{B_\rho(x_0)} |D^2u|^{\beta' d}\,dx\Big)^{1/\beta' d}$$

by Hölder's inequality. The lemma is proved. □

3. LEMMA. *Set $\gamma = \bar\gamma$, recall that $p > d$, and take $R \in (0,1]$ and $u \in W^2_p(B_{2R})$. Assume that $B_{2R} \subset \Omega$. Then there exist constants $\hat\theta, \theta \in (0,1]$, depending only on d, p, δ, and K_F, such that, if Assumptions* 1.4 *and* 1.6 *are satisfied with these $\hat\theta$ and θ, respectively, then there is a constant N, depending only on R_0, d, p, K_0, K_F, δ, and the function $\omega_{F,u,B_{2R}}$, such that*

$$\|D^2u\|_{L_p(B_R)} \le N\big\|H[u]\big\|_{L_p(B_{2R})} + N\|\bar G\|_{L_p(B_{2R})} + N\tau_0 R^{d/p}$$

$$+ NR^{d/p-d/\gamma}\big\| |D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})} + N\|u\|_{L_p(B_{2R})}, \tag{3}$$

$$\|D^2u\|_{L_p(B_R)} \le N\tau_0 R^{d/p} + NR^{d/p-2}\sup_{B_{2R}}|u|$$

$$+ N\big(\big\|H[u]\big\|_{L_p(B_{2R})} + \|\bar G\|_{L_p(B_{2R})}\big). \tag{4}$$

Proof. For $\rho > 0$ and $x \in \mathbb{R}^d$ introduce

$$h^{\#}_{\gamma,\rho}(x) = \sup_{\substack{r\in(0,\rho],\\ B_r(x_0)\ni x}} \Big(\fint_{B_r(x_0)} \fint_{B_r(x_0)} \big|h(x_1) - h(x_2)\big|^\gamma \, dx_1 dx_2\Big)^{1/\gamma},$$

$$\mathbb{M}h(x) = \sup_{\substack{r>0,\\ B_r(x_0)\ni x}} \fint_{B_r(x_0)} |h(y)|\, dy, \tag{5}$$

whenever these definitions make sense.

Take $\varepsilon \in (0,1]$ to be specified later and take $R_1 < R_2 \le 2R$ such that

$$R_2 - R_1 \le \varepsilon R_0, \quad R_2 \le 2R_1. \tag{6}$$

Then take $\nu \ge 2$ and set

$$r_0 = (R_2 - R_1)/(\nu + 1).$$

Next, take x, x_0, and $r > 0$ such that

$$r \le r_0, \quad x \in B_{R_1}, \quad x \in B_r(x_0)$$

and observe that, since $R_2 - \nu r_0 = R_1 + r_0$, we have $x_0 \in B_{R_2-\nu r_0}$ and $B_{\nu r}(x_0) \subset B_{R_2}$. Also $\nu r \le \nu r_0 \le R_0$. Therefore, by Lemma 2 applied to $\Omega = B_{R_2}$ and $\alpha = \alpha_0$ we have (note x_0 on the left and x on the right)

$$I_r(x_0) \le N\nu^{d/\gamma}\mathbb{M}^{1/d}\big(\big|F[u]\big|^d I_{B_{R_2}}\big)(x) + N\tau_0\nu^{d/\gamma}$$

$$+ N\Big[\big(\mu + \omega_{F,u,B_{2R}}(\nu r_0)\big)\nu^{d/\gamma} + \nu^{-\alpha}\Big]\mathbb{M}^{1/(\beta' d)}\big(|D^2u|^{\beta' d} I_{B_{R_2}}\big)(x)$$

with N depending only on d, K_F, and δ. It follows that in B_{R_1}

$$(D^2u)^{\#}_{\gamma,r_0} \le N\nu^{d/\gamma}\mathbb{M}^{1/d}\big(\big|F[u]\big|^d I_{B_{R_2}}\big) + N\tau_0\nu^{d/\gamma}$$

$$+ N\Big[\big(\mu + \omega_{F,u,B_{2R}}(\varepsilon R_0)\big)\nu^{d/\gamma} + \nu^{-\alpha}\Big]\mathbb{M}^{1/(\beta' d)}\big(|D^2u|^{\beta' d} I_{B_{R_2}}\big).$$

By Theorem C.2.6 with

$$\kappa = r_0/R_1 \le 1/3, \quad \chi_1 = (d+2)/\gamma, \quad \chi_2 = d/\gamma - d/p$$

and the Hardy-Littlewood maximal function theorem, by taking β so that $p > \beta' d$, we obtain

$$\|D^2u\|_{L_p(B_{R_1})} \le N\nu^{d/\gamma}\big\|F[u]\big\|_{L_p(B_{R_2})} + N\tau_0\nu^{d/\gamma}R_1^{d/p}$$
$$+ \Big[N\big(\mu + \omega_{F,u,B_{2R}}(\varepsilon R_0)\big)\nu^{d/\gamma} + N_0\nu^{-\alpha}\Big]\|D^2u\|_{L_p(B_{R_2})}$$
$$+ N\nu^{\chi_1}(R_2 - R_1)^{-\chi_1}R_1^{-\chi_2+\chi_1}\big\|\,|D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})}, \tag{7}$$

where and in a few lines below the constants N, N_i depend only on d, p, K_F, and δ. Somewhat later they will also depend on K_0, and the moment when they depend on the data as in the statement of the lemma will be specifically noted. Now we take and fix $\nu \ge 2$ so that

$$N_0\nu^{-\alpha} \le 1/8.$$

Then (7) becomes

$$\|D^2u\|_{L_p(B_{R_1})} \le N_1\big\|F[u]\big\|_{L_p(B_{R_2})} + N\tau_0R_1^{d/p}$$
$$+ \Big[N_2\big(\mu + \omega_{F,u,B_{2R}}(\varepsilon R_0)\big) + 1/8\Big]\|D^2u\|_{L_p(B_{R_2})}$$
$$+ N(R_2 - R_1)^{-\chi_1}R_1^{-\chi_2+\chi_1}\big\|\,|D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})}. \tag{8}$$

Next, we use the fact that

$$\big|F[u]\big| \le \big|H[u]\big| + K_0|u| + K_0|Du| + \bar G + \hat\theta|D^2u|$$

and that by interpolation inequalities

$$K_0N_1\|Du\|_{L_p(B_{R_2})} \le (1/8)\|D^2u\|_{L_p(B_{R_2})} + N\|u\|_{L_p(B_{R_2})}.$$

Then we take $\hat\theta$ and μ so small that

$$N_1\hat\theta \le 1/8, \quad N_2\mu \le 1/8,$$

and, finally, take the largest $\varepsilon \le 1$ such that

$$N_2\omega_{F,u,B_{2R}}(\varepsilon R_0) \le 1/8.$$

This ε, which depends only on d, p, K_F, R_0, the function $\omega_{F,u,B_{2R}}$, and δ, will appear later in our arguments and this is the way how the constant N in the statement of the lemma depends on $\omega_{F,u,B_{2R}}$.

We require Assumptions 1.4 and 1.6 be satisfied with the above chosen $\hat\theta$ and $\theta = \theta(d, \delta, K_F, \mu, \beta d)$ (see Lemma 1), respectively. By combining the above, we get

$$\|D^2u\|_{L_p(B_{R_1})} \le N\big\|H[u]\big\|_{L_p(B_{R_2})} + N\tau_0R^{d/p} + (5/8)\|D^2u\|_{L_p(B_{R_2})}$$
$$+N(R_2 - R_1)^{-\chi_1}R_1^{-\chi_2+\chi_1}\big\|\,|D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})} + N\|u\|_{L_p(B_{2R})} + N\|\bar G\|_{L_p(B_{2R})}.$$

Now we are going to iterate this estimate by defining $R_1 = R$ and for $k \ge 1$

$$R_{k+1} = R_k + cR(n_0 + k)^{-2},$$

where the constant $c = O(n_0)$ is chosen so that $R_k \uparrow 2R$ as $k \to \infty$, that is

$$c\sum_{k=1}^{\infty}(n_0 + k)^{-2} = 1,$$

and $n_0 > 0$ is chosen so that for $k \geq 1$

$$R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq Rcn_0^{-2} \leq R \leq R_k,$$

which is satisfied if n_0 is just an appropriate absolute constant, and

$$R_{k+1} - R_k = cR(n_0 + k)^{-2} \leq cn_0^{-2} \leq \varepsilon R_0$$

(this time we need $n_0^{-1} = o(\varepsilon R_0)$ as $\varepsilon R_0 \to 0$). Also observe that $R \leq R_k \leq 2R$ and

$$(R_{k+1} - R_k)^{-\chi_1} R_k^{-\chi_2+\chi_1} \leq N(n_0 + k)^{2\chi_1} R^{-\chi_2}.$$

Then for $k \geq 1$ we get

$$\|D^2u\|_{L_p(B_{R_k})} \leq N\big\|H[u]\big\|_{L_p(B_{R_{k+1}})} + N\tau_0 R^{d/p} + (5/8)\|D^2u\|_{L_p(B_{R_{k+1}})}$$

$$+N(n_0 + k)^{2\chi_1} R^{-\chi_2}\big\|\,|D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})} + N\|u\|_{L_p(B_{2R})} + N\|\bar{G}\|_{L_p(B_{2R})},$$

where and below the constants N are as in the statement of the lemma. We multiply both parts of this inequality by $(5/8)^k$ and sum up the results over $k = 1, 2, ...$ Then we cancel the like terms

$$\sum_{k=2}^{\infty}(5/8)^k\|D^2u\|_{L_p(B_{R_k})},$$

which are finite since $u \in W^2_p(B_{2R})$, and finally take into account that

$$\sum_{k=2}^{\infty}(5/8)^k(n_0 + k)^{2\chi_1} \leq Nn_0^{2\chi_1}\sum_{k=2}^{\infty}(5/8)^k + N\sum_{k=2}^{\infty}(5/8)^k k^{2\chi_1} \leq N.$$

Then we come to (3).

Next, by using equation (9.7.2) and performing scaling in Theorem 9.4.6 (here we need $R \leq 1$), using Hölder's inequality (to go from d to p), and denoting

$$I = \|\bar{G}\|_{L_p(B_{2R})} + \big\|H[u]\big\|_{L_p(B_{2R})}$$

we infer that in (3)

$$\big\|\,|D^2u|^\gamma\big\|^{1/\gamma}_{L_1(B_{2R})} \leq NR^{\chi_2}\Big(\|\bar{G} + K_0|u|\,\|_{L_p(B_{2R})} + \big\|H[u]\big\|_{L_p(B_{2R})}\Big)$$

$$+ NR^{\chi_3}\sup_{B_{2R}}|u| \leq NR^{\chi_2} I + NR^{\chi_3}\sup_{B_{2R}}|u|,$$

where $\chi_3 = d/\gamma - 2$. After that it suffices to roughly estimate $\|u\|_{L_p(B_{2R})}$ in (3) by the last supremum above. The lemma is proved. □

Until this moment we followed a general framework laid out in [**70**] and [**30**], where we also used the localization techniques based on partitions of unity. In particular, we considered $H[\zeta u]$ and were able to relate it (due to certain assumptions) to $\zeta H[u]$ plus unimportant terms. In our present situation this localization does not work and instead we rely on the following simple fact.

The following lemma will allow us to get interior W^2_p-estimates in domains by using Lemma 3.

4. LEMMA. *Let Ω be just a bounded domain in $\mathbb{R}^d$ and $\rho > 0$ be such that $\Omega^{3\rho} \neq \emptyset$, and let f and h be nonnegative functions given in Ω. Assume that for any $z \in \Omega^{2\rho}$ we have*

$$\int_{B_\rho(z)} f(x)\,dx \le \int_{B_{2\rho}(z)} h(x)\,dx + N_1\rho^d + N_2\rho^{d-2p}, \tag{9}$$

where N_1 and N_2 are fixed constants ≥ 0. Then

$$\int_{\Omega^{3\rho}} f(x)\,dx \le 2^d \int_\Omega h(x)\,dx + NN_1 + NN_2\rho^{-2p},$$

where the constants N depend only on d and $|\Omega|$.

Proof. We integrate (9) with respect to z over $\Omega^{2\rho}$ and observe that

$$\int_{\Omega^{2\rho}} I_{B_\rho(z)}(x)\,dz = \int_{\Omega^{2\rho}} I_{B_\rho(x)}(z)\,dz = \big|B_\rho(x)\cap\Omega^{2\rho}\big|.$$

For $x \in \Omega^{3\rho}$, $B_\rho(x) \subset \Omega^{2\rho}$, and $\big|B_\rho(x)\cap\Omega^{2\rho}\big| = \rho^d|B_1|$. On the other hand,

$$\int_{\Omega^{2\rho}} I_{B_{2\rho}(z)}(x)\,dz = \big|B_{2\rho}(x)\cap\Omega^{2\rho}\big| \le 2^d\rho^d|B_1|I_\Omega(x).$$

This easily yields the desired result. The lemma is proved. □

5. Proof of Theorems 1.10, 1.14, and 1.17

Proof of Theorem 1.10. We take the constants $\hat\theta, \theta \in (0,1]$ from Lemma 4.3. By that lemma, if $\rho \in (0,1]$ and $\Omega^{2\rho} \neq \emptyset$ and $z \in \Omega^{2\rho}$, we have $\bar B_{2\rho}(z) \subset \Omega$ and

$$\|D^2u\|^p_{L_p\left(B_\rho(z)\right)} \le N\tau_0^p|\rho|^d + N\rho^{d-2p}\sup_{B_{2\rho}(z)}|u|^p + N\|\bar G\|^p_{L_p\left(B_{2\rho}(z)\right)},$$

and by Lemma 4.4 for $0 < 3\rho < \rho_{\mathrm{int}}(\Omega)\wedge 3$

$$\int_{\Omega^{3\rho}} \big|D^2u(x)\big|^p\,dx \le N\int_\Omega \big|\bar G(x)\big|^p\,dx + N\tau_0^p + N\rho^{-2p}\sup_\Omega|u|^p. \tag{1}$$

If $3\rho \ge \rho_{\mathrm{int}}(\Omega)\wedge 3$ and still $\Omega^{3\rho} \neq \emptyset$, we have $3\rho < \rho_{\mathrm{int}}(\Omega)$, hence $\rho \ge 1$, $\rho \le \mathrm{diam}(\Omega)$, and (1) follows because $\Omega^{3\rho} \subset \Omega^3$ and $3^{-2p} \le 3^{-2p}\rho^{-2p}\mathrm{diam}^{2p}(\Omega)$. Using interpolation inequalities also allows us to estimate the $L_p(\Omega^{3\rho})$-norm of Du. The theorem is proved. □

Proof of Theorem 1.17. It is based on Theorem 6.6.5 and, as there, we take P from (6.2.4) and set $P[u](x) = P(D^2u(x))$.

Step 1. Assume that $g \in C^{1,1}(\mathbb{R}^d)$ and there exist a constant N_0 and a *constant* $\bar H$ such that, for all x, u',

$$\big|H(\mathsf{u}',0,x)\big| \le N_0|\mathsf{u}'| + \bar H. \tag{2}$$

We need this assumption in addition to a weaker property (1.5) in order to be able to apply Theorem 6.6.5. By that theorem for any $K > 0$ there exists a solution $v_K \in W^2_{p,\text{loc}}(\Omega) \cap C(\bar{\Omega})$, $p > 1$, of the equation

$$\max\big(H[v_K], P[v_K] - K\big) = 0 \tag{3}$$

in Ω (a.e.) and $v_K = g$ on $\partial\Omega$, and

$$\sup_{\Omega} |v_K| \le M_0. \tag{4}$$

Next, we want to use Theorems 9.7.1 and 1.10. Set

$$H_K(\mathsf{u}, x) = \max\big(H(\mathsf{u}, x), P(\mathsf{u}'') - K\big),$$

$$F_K\big(\mathsf{u}'_0, \mathsf{u}'', x\big) = \max\big(F\big(\mathsf{u}'_0, \mathsf{u}'', x\big), P(\mathsf{u}'') - K\big), \quad G_K = H_K - F_K.$$

$$\bar{F}_{K,z,r,\mathsf{u}'_0}(\mathsf{u}'') = \max\big(\bar{F}_{z,r,\mathsf{u}'_0}(\mathsf{u}''), P(\mathsf{u}'') - K\big).$$

Then condition (1.3) is satisfied for G_K in place of G just because $|G_K| \le |G|$. Assumption 6.6.4 is also easily checked.

By Remark 6.2.2, H_K satisfies Assumption 1.11 with $\bar{\delta} = \bar{\delta}(d, \delta) \in (0, \delta]$ in place of δ. Also Assumptions 1.6 (i) and (ii) are satisfied with new K_F and δ depending only on d and the original K_F and δ. Finally, Assumptions 1.6 (iii) and (iv) are satisfied with the same θ and ω_F because, for instance,

$$\big|F_K\big(\mathsf{u}'_0, \mathsf{u}'', x\big) - \bar{F}_{K,z,r,\mathsf{u}'_0}(\mathsf{u}'')\big| \le \big|F\big(\mathsf{u}'_0, \mathsf{u}'', x\big) - \bar{F}_{z,r,\mathsf{u}'_0}(\mathsf{u}'')\big|.$$

This allows us to use the results proved in Theorem 1.10 for v_K, but first we recall Remark 1.8 and combine it with (4) and Theorem 9.7.1 to conclude that the family of the functions v_K is equicontinuous in $\bar{\Omega}$, and, in particular, the functions $\omega_{F,v_K,\Omega}$ are majorated by a continuous function independent of K and vanishing at zero.

As an important observation tacitly used later we point out that in the above argument guaranteeing the precompactness of $\{v_K : K \ge 0\}$ in $C(\bar{\Omega})$, in what concerns g, only the supremum of $|g|$ and the modulus of continuity of g on $\partial\Omega$ were involved. This is important when we will need to approximate our general $g \in C(\partial\Omega)$ with the ones of class $C^{1,1}(\mathbb{R}^d)$.

Then by Theorem 1.10 we obtain that, provided that an appropriate choice of $\hat{\theta}$ and θ was made, for any $\rho > 0$ such that $\Omega^\rho \ne \emptyset$

$$\sup_{K \ge 1} \|v_K\|_{W^2_p(\Omega^\rho)} < \infty. \tag{5}$$

We now let $K \to \infty$. In light of the precompactness of $\{v_K\}$ and estimate (5) there is a sequence $K_n \to \infty$ as $n \to \infty$ and $v \in W^2_{p,\text{loc}}(\Omega)$ such that $v_{K_n} \to v$ weakly in $W^2_p(\Omega^\rho)$ for any ρ and $v_{K_n} \to v$ uniformly in $\bar{\Omega}$. Then, of course, $v \in C(\bar{\Omega})$ and $v = g$ on $\partial\Omega$. Embedding theorems and the fact that (5) holds with $p > d$ also imply that $Dv_{K_n} \to Dv$ locally uniformly in Ω.

Next, for $m = 1, 2, \ldots$ define

$$H^m(\mathsf{u}'', x) = \sup_{n \ge m} \max\big(H\big(v_{K_n}(x), Dv_{K_n}(x), \mathsf{u}'', x\big), P(\mathsf{u}'') - K_n\big).$$

Observe that $H^m(\mathsf{u}'', x)$ are Lipschitz continuous in u'' and at all points of differentiability satisfy $D_{\mathsf{u}''}H^m \in \mathbb{S}_{\bar\delta}$. Also

$$\big|H^m(0, x)\big| \le K_0 \max_{n \ge m}\big(\big|v_{K_n}(x)\big| + \big|Dv_{K_n}(x)\big|\big) + \bar G(x),$$

which is in $L_{p,\mathrm{loc}}(\Omega)$. Therefore, the operators $H^m[u]$ fit into the scheme of Section 4.2. Furthermore, for $n \ge m$ obviously

$$H^m(v_{K_n}, x) \ge 0$$

in Ω (a.e.). By Theorems 4.2.15 and 4.2.6 we conclude that for any m

$$\sup_{n \ge m} \max\big[H\big(v_{K_n}(x), Dv_{K_n}(x), D^2v(x), x\big), P\big(D^2v(x)\big) - K_n\big] \ge 0 \tag{6}$$

in Ω (a.e.). We fix x at which (6) holds for all m (that is, we fix almost any x) and since $H(\mathsf{u}', \mathsf{u}'', x)$ is continuous in u', we have that

$$\big|H\big(v_{K_n}(x), Dv_{K_n}(x), D^2v(x), x\big) - H\big(v(x), Dv(x), D^2v(x), x\big)\big| \to 0$$

as $n \to \infty$. Then in light of (6)

$$\max\big[H\big(v(x), Dv(x), D^2v(x), x\big), P\big(D^2v(x)\big) - K_m\big] \ge o(1),$$

which for $m \to \infty$ yields

$$H\big(v(x), Dv(x), D^2v(x), x\big) = H[v](x) \ge 0.$$

The inequality $H[v] \le 0$ is proved similarly starting from the function

$$\inf_{n \ge m} \max\big[H\big(v_{K_n}(x), Dv_{K_n}(x), \mathsf{u}'', x\big), P(\mathsf{u}'') - K_n\big].$$

This proves the theorem if $g \in C^{1,1}(\mathbb{R}^d)$ and (2) holds.

Step 2. Assume that $g \in C^{1,1}(\mathbb{R}^d)$ and abandon (2). Let $\eta(t) = t$ for $|t| \le 1$ and $\eta(t) = \operatorname{sign} t$ for $|t| \ge 1$. For $\kappa > 0$ define $\eta(\kappa, t) = \kappa\eta(t/\kappa)$,

$$H^n(\mathsf{u}, x) = H(\mathsf{u}, x) - H(\mathsf{u}', 0, x) + \eta\left(n, H(\mathsf{u}', 0, x)\right),$$

$$F^n(\mathsf{u}'_0, \mathsf{u}'', x) = F(\mathsf{u}'_0, \mathsf{u}'', x), \quad G^n = H^n - F^n.$$

Observe that

$$\big|H^n(\mathsf{u}', 0, x)\big| \le n,$$

so that (2) is satisfied.

In light of (1.3)

$$\big|G^n(\mathsf{u}, x)\big| = \big|G(\mathsf{u}, x) - G(\mathsf{u}', 0, x) + \eta\left(n, G(\mathsf{u}', 0, x)\right)\big|$$

$$\le \big|G(\mathsf{u}, x)\big| + \big|G(\mathsf{u}', 0, x)\big| \le \hat\theta|\mathsf{u}''| + 2K_0|\mathsf{u}'| + 2\bar G(x), \tag{7}$$

and (1.3) is satisfied for G^n in place of G with slightly modified K_0 and $\bar G$. Thus Assumption 1.4 is satisfied for H^n.

Owing to (6.6.4), for $\mathsf{u}' = (M_0 + \varepsilon_0, \varepsilon')$ with $\varepsilon' = (\varepsilon_1, ..., \varepsilon_d)$, since $\eta(\kappa, t)$ has the sign of t, we have

$$-H^n(\mathsf{u}', 0, x) = -\eta\,(n, H(\mathsf{u}', 0, x)) \geq 0.$$

Similarly, $H^n(-\mathsf{u}', 0, x) \leq 0$, which finishes checking Assumption 6.6.4. for H^n.

Assumption 1.6 is satisfied because we did not change F and Assumption 1.11 is satisfied because $D_{\mathsf{u}''}H^n = D_{\mathsf{u}''}H$.

Hence, with the same $\hat{\theta}$ and θ as in Step 1, for any n there exists $v^n \in W^2_{p,\text{loc}}(\Omega) \cap C(\bar{\Omega})$ satisfying

$$H^n[v^n] = 0 \tag{8}$$

in Ω (a.e.) and such that $v^n = g$ on $\partial\Omega$. The derivation of (5) shows that for any $\rho > 0$ such that $\Omega^\rho \neq \emptyset$

$$\sup_n \|v^n\|_{W^2_p(\Omega^\rho)} < \infty. \tag{9}$$

Then by repeating what is said after (5) we find a subsequence $v^{n'}$ and a function $v \in W^2_{p,\text{loc}}(\Omega)$ such that $v^{n'} \to v$ weakly in $W^2_p(\Omega^\rho)$ for any ρ and $v^{n'} \to v$ uniformly in $\bar{\Omega}$. Then, of course, $v = g$ on $\partial\Omega$. Embedding theorems and the fact that (5) holds with $p > d$ also imply that $Dv^{n'} \to Dv$ locally uniformly in Ω.

Next, for large $m = 1, 2, ...$ define

$$\hat{H}^m(\mathsf{u}'', x) = \sup_{n' \geq m} H^{n'}\big(v^{n'}(x), Dv^{n'}(x), \mathsf{u}'', x\big).$$

Observe that $\hat{H}^m(\mathsf{u}'', x)$ are Lipschitz continuous in u'' and at all points of differentiability satisfy $D_{\mathsf{u}''}\hat{H}^m \in \mathbb{S}_\delta$. Also (7) shows that

$$\big|\hat{H}^m(0, x)\big| \leq 2K_0 \max_{n' \geq m} \big(\big|v^{n'}(x)\big| + \big|Dv^{n'}(x)\big|\big) + 2\bar{G}(x),$$

which is in $L_{p,\text{loc}}(\Omega)$. Therefore, the operators $\hat{H}^m[u]$ fit into the scheme of Section 4.2. Furthermore, for $n' \geq m$ obviously

$$\hat{H}^m(D^2 v^{n'}, x) \geq 0$$

in Ω (a.e.). By Theorem 4.2.15 and 4.2.6 we conclude that for any m in Ω (a.e.)

$$\hat{H}^m(D^2 v, x) \geq 0. \tag{10}$$

Next, note that, for any u'' and $x \in \Omega$, for all large n we have

$$H^n\big(v^n(x), Dv^n(x), \mathsf{u}'', x\big) = H\big(v^n(x), Dv^n(x), \mathsf{u}'', x\big)$$

because the sequences $v^n(x), Dv^n(x)$ are bounded. It follows from the continuity of $H(\mathsf{u}, x)$ with respect to u that for any u'' and $x \in \Omega$

$$\hat{H}^m(\mathsf{u}'', x) \to H\big(v(x), Dv(x), \mathsf{u}'', x\big).$$

Now we infer from (10) that $H[v] \geq 0$ in Ω (a.e.).

The fact that $H[v] \leq 0$ in Ω (a.e.) is proved similarly starting with the function

$$\inf_{n' \geq m} H^{n'}\big(v^{n'}(x), Dv^{n'}(x), \mathsf{u}'', x\big).$$

This yields the desired result if $g \in C^{1,1}(\mathbb{R}^d)$. The final step where one approximates $g \in C(\partial\Omega)$ by polynomials and passes to the limit is much simpler than the argument in Step 2 and is left to the reader as a routine exercise. The theorem is proved. □

Proof of Theorem 1.14. It suffices to literally repeat the above proof using Theorem 6.6.9 in place of Theorem 6.6.5. □