Introduction

In 2013–2014, the Simons Center for Geometry and Physics hosted a one-year program to discuss various problems related to the foundations of that part of symplectic geometry which concerns the theory of moduli spaces of pseudo-holomorphic curves. An important part of this program were lecture courses on the 'Virtual fundamental chain and cycle' determined by these moduli spaces, a technique of central importance in symplectic geometry. This volume consists of written and expanded versions of these lecture courses.

Our goal in bringing together experts in these techniques to give the lecture courses and in producing this follow-up volume is to facilitate a wider understanding of the various approaches to the basics of the virtual fundamental chain and cycle theory in symplectic geometry and the assumptions that each approach makes. We believe that future major advances in symplectic geometry will require use of these techniques without restrictive topological or geometric assumptions on the symplectic manifold. It is our hope that this volume will contribute to the wider application of these methods.

The Foundational Work of Gromov and Floer. The method of using pseudo-holomorphic curves in the study of global symplectic geometry was introduced in a ground-breaking paper by M. Gromov, [**3**]. From the appearance of this paper until today, this approach has been the most important tool in global symplectic geometry. The method is based on the following facts: (i) Symplectic manifolds have compatible almost complex structures (though the manifold often does not have any complex structure). (ii) The equation for a map from an almost complex structure on a smooth 2-manifold to an almost complex structure on a higher dimensional manifold is an elliptic equation so the usual elliptic theory applies. (iii) Since almost complex structures on smooth 2-manifolds are automatically integrable, pseudo-holomorphic maps from smooth 2-manifolds to almost complex manifolds are in fact holomorphic maps from complex curves[1]. As a result, the rich and highly developed holomorphic curve theory applies in the context of pseudo-holomorphic maps from a complex one-dimensional manifold to a symplectic manifold.

Even though there are many almost complex structures compatible with a given symplectic structure, the space of all such is contractible. Together with Gromov's compactness theorem for these pseudo-holomorphic maps, which relies essentially on the fact that we are mapping to a symplectic manifold not just an almost complex manifold, this implies that the cobordism class of the moduli space of pseudo-holomorphic maps from a complex one-dimensional manifolds to a given symplectic manifold is independent of the choice of the compatible almost complex structure. Such a cobordism class is actually an invariant of the symplectic structure, not just of the almost complex structure or its homotopy class.

Combined with physicist's view of the topological sigma model, see [**7**], this discovery by Gromov produces celebrated invariants of symplectic manifolds, which are now called *Gromov-Witten invariants*. There is also the quantum cup product on cohomology defined using these invariants. Some of the basic properties,

[1]Typically there is no holomorphic map from a complex manifold of dimension > 1 to an almost complex manifold.

especially the associativity of the quantum cup product, were first observed by physicists working on String Theory.

Another important application of Gromov's theory of pseudo-holomorphic curves is Floer homology. A. Floer observed that the gradient line of the classical action functional of periodic Hamiltonian system can be identified with a pseudo-holomorphic curve (with an order zero perturbation term). This makes it possible to apply the theory of pseudo-holomorphic curves to the problems of studying periodic solutions of periodic Hamiltonian systems (which is one of the most important problems in symplectic geometry), see [**1**]. Using similar techniques he also studied the problem of giving a lower bound for the number of intersection points of two Lagrangian submanifolds.

In symplectic geometry nowadays there are various 'Floer-type' homology theories defined using moduli spaces of pseudo-holomorphic curves and their variants.

Early applications of the theory. In early days of the development of these theories, the moduli spaces of pseudo-holomorphic curves were first studied under various additional simplifying assumptions. A typical assumption is monotonicity of the symplectic manifold (X, ω), meaning the existence of a positive number c such that the equality

$$\langle [\omega], \alpha \rangle = c \langle c_1(TX), \alpha \rangle \tag{1}$$

holds for any $\alpha \in \pi_2(X)$, or semi-positivity, meaning that

$$\langle c_1(TX), \alpha \rangle \geq 0$$

for any $\alpha \in \pi_2(X)$ with $\langle [\omega], \alpha \rangle > 0$. In the case of Floer homology associated with Lagrangian submanifolds $L_i \subset X,\ i = 1, 2$ typical assumptions are monotonicity (a similar equality to (1) above) or exactness, meaning the existence of one form θ on X such that $d\theta = \omega$ and $[\theta] = 0$ in the de-Rham cohomology of the L_i.)

These assumptions are used to establish the transversality of the moduli space of pseudo-holomorphic curves. Here, transversality means that the linearization of the Cauchy-Riemann equation is a surjective linear operator. Under this condition the relevant moduli space of pseudo-holomorphic curves is a smooth manifold and its dimension is calculated by the topological data using the Atiyah-Singer index theorem or the Riemann-Roch theorem.

In the case of monotone or semi-positive symplectic manifolds (or monotone or exact Lagrangian submanifolds), a standard perturbation argument shows that transversality holds for generic compatible almost complex structure.

Beyond these cases, an obstruction to being able to achieve the transversality arises from ramified multiple covers of transversal solutions. The problem is that the formal dimension of these covers can be negative even though the formal dimension of the underlying simple curve is non-negative. The main motivation for introducing the virtual fundamental chain and cycle technique (the main theme of this book) is to resolve this problem.

An Example. Let us give an explicit example of this phenomenon. We consider the moduli space of pseudo-holomorphic maps $u\colon S^2 \to X$ from a two-sphere with three marked points, taken to be $0, 1, \infty$, to a symplectic manifold X of real dimension $2n$, and denote by α the homology class $u_*[S^2]$. We assume that $\langle c_1(X), \alpha \rangle > 0$.

The dimension of this moduli space is

$$2\langle c_1(TX), \alpha\rangle + 2n.$$

We denote this moduli space by $\mathcal{M}_{0,3}(\alpha)$. (Here the first subscript, 0, is the genus of the curve and the second subscript, 3, is the number of marked points on the curve.) The map $u \mapsto (u(0), u(1), u(\infty)) \in X \times X \times X$ is called the *evaluation map*. We take cycles $Q_1, Q_2, Q_3 \subset X$ such that

$$\sum_{i=1}^{3}(2n - \dim Q_i) = 2\langle c_1(TX), \alpha\rangle + 2n. \tag{2}$$

Then, the formal dimension of the moduli space

$$\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3) = \{u \in \mathcal{M}_{0,3}(\alpha) \mid (u(0), u(1), u(\infty)) \in Q_1 \times Q_2 \times Q_3\}$$

is 0. If this moduli space and its compactification are cut out transversally, then the sum over the (finite set of) points of the moduli space of signs coming from orientations is a typical example of the Gromov-Witten invariant. Indeed, because of the assumption $\langle c_1(TX), \alpha\rangle > 0$, for a generic compatible almost complex structure the moduli space $\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3)$ is cut out transversally, so that it is a smooth manifold of dimension 0. However, it is not in general true that this moduli space is compact for such a generic choice of almost complex structure.

Gromov's compactness theorem holds and implies that if we consider a sequence u_i of elements of $\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3)$ we may choose a subsequence so that its limit becomes a stable map of a holomorphic curve to X.

An example of stable curve which appears as a limit is drawn in Figure 1. It

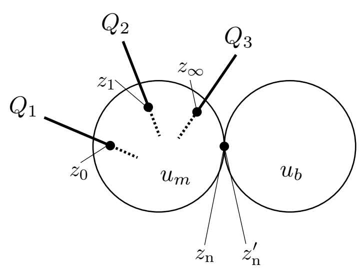

FIGURE 1. Limiting Configuration

is a union of curves $u_m : S^2 \to X$, $u_b : S^2 \to X$. The domain S^2 of u_m comes with four special points $(z_0, z_1, z_\infty, z_{\rm n})$ where z_0, z_1, z_∞ are limits of $0, 1, \infty$ in the domain of u_i, respectively. The domain of u_b comes with one special point $z'_{\rm n}$, and we require $u_m(z_{\rm n}) = u_b(z'_{\rm n})$. So we may regard the source of our map u_∞ as a union of two spheres jointed at $z_{\rm n} = z'_{\rm n}$. We put $\alpha_m = (u_m)_*([S^2])$ and $\alpha_b = (u_b)_*([S^2])$, giving $\alpha_m + \alpha_b = \alpha$.

Then $(u_m; z_0, z_1, z_\infty, z_{\rm n})$ and $(u_b; z'_{\rm n})$ represent elements of the moduli spaces $\mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3)$ and $\mathcal{M}_{0,1}(\alpha_b)$, respectively.

The 'virtual' dimensions of the moduli spaces are given by

$$\begin{aligned} \dim \mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3) &= 2\langle c_1(TX), \alpha_m\rangle + 2n + 2 - \sum_{i=1}^{3}(2n - \dim Q_i), \\ \dim \mathcal{M}_{0,1}(\alpha_b) &= 2\langle c_1(TX), \alpha_b\rangle + 2n - 4. \end{aligned} \tag{3}$$

The numbers +2 and −4 in the above dimension formulae are explained as follows. The source curve $(S^2; z_0, z_1, z_\infty, z_{\rm n})$ of u_m has a 2-dimensional moduli space: The source curve $(S^2; z'_{\rm n})$ of u_b has an automorphism group of dimension 4. The condition $u_m(z_{\rm n}) = u_b(z'_{\rm n})$ adds $2n$ constraints. On the other hand, by Equations (2) and (3)

$$\dim \mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3) + \dim \mathcal{M}_{0,1}(\alpha_b) - 2n = -2.$$

This implies that if all the moduli spaces involved are transversal then this configuration does not appear. In order to prove the compactness of $\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3)$ we need these limiting moduli spaces to be cut out transversally (so that they are empty).

However it is not in general true that $\mathcal{M}_{0,1}(\alpha_b)$ is transversal for generic almost complex structure. We consider the case when $\alpha_b = k\alpha'_b$ with

$$2\langle c_1(TX), \alpha'_b\rangle < 0.$$

Then, it can happen that there is a pseudo-holomorphic curve with

$$\begin{aligned} &2\langle c_1(TX), \alpha'_b\rangle + 2n - 4 \geq 0 \quad \text{and} \\ &2\langle c_1(TX), \alpha_b\rangle + 2n - 4 = 2\langle c_1(TX), k\alpha'_b\rangle + 2n - 4 < 0. \end{aligned}$$

One explicit numerical example is $n = 3$, $\langle c_1(TX), \alpha'_b\rangle = -1$, and $k = 2$.

Because of the first inequality, we can not show $\mathcal{M}_{0,1}(\alpha'_b) = \emptyset$ by dimension counting, and indeed it may be non-empty for every almost complex structure. But if $(u', 0)$, $u' \colon S^2 \to X$ represents an element of $\mathcal{M}_{0,1}(\alpha'_b)$ then $(u'', 0)$ with $u''(z) = u'(z^k)$ represents an element of $\mathcal{M}_{0,1}(\alpha_b)$. In particular $\mathcal{M}_{0,1}(\alpha_b)$ is non-empty. Thus, the space $\mathcal{M}_{0,1}(\alpha_b)$ with negative 'virtual' dimension is non-empty for every almost complex structure and so is not cut out transversally for any such structure.

This is a typical example of the problem of multiple covers of negative formal dimension. Avoiding this problem is the reason one often makes certain (sometimes restrictive) assumptions in the applications of pseudo-holomorphic curves in symplectic geometry.

The Virtual Fundamental Chain and Cycle Approach. Because it is not possible to use perturbation of the almost complex structure to produce the fundamental class of the moduli space when there are multiple covers of formally negative dimension, one is led to introduce the virtual fundamental chain or cycle technique. This approach resolves this problem in practically all the situations that appear in the study of pseudo-holomorphic curves in symplectic geometry. However, the method comes with the drawback that one is forced to use the rational numbers as a coefficient ring. Namely, Gromov-Witten invariants obtained in this way are necessarily rational numbers rather than integers and coefficients for the Floer homology is the field of rational numbers. (The reason why this approach does not work over integers is similar to the reason that Euler number of an orbifold is, in general, a rational number rather than an integer.)

Let us elaborate this point a bit more below. An orbifold is a space that looks like U/Γ locally, where U is an open ball centered at the origin of $\mathbb{R}^n$ and Γ is a finite group acting orthogonally on $\mathbb{R}^n$. We consider an orbifold X which is a union of $\mathbb{C}/\mathbb{Z}_2$ and $\mathbb{C}/\mathbb{Z}_3$. We identify $[z^2] \in \mathbb{C}/\mathbb{Z}_2$ with $[w^3] = [1/z^2] \in \mathbb{C}/\mathbb{Z}_3$. This orbifold X has two singular points which correspond to $z = 0 \in \mathbb{C}/\mathbb{Z}_2$ and

$w = 0 \in \mathbb{C}/\mathbb{Z}_3$, respectively. There exists a vector field on X which vanishes only at these two singular points. Because of the presence of non-trivial isotropy group we need to count these zeros with multiplicity 1/2 and 1/3, respectively. Therefore the orbifold Euler number of X is

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

There are various versions of virtual fundamental chain or cycle technique. Some versions work with infinite dimensional spaces directly, other versions are based on local, finite dimensional reductions. All three articles in this book work in the context of finite dimensional reductions. In these versions the 'virtual fundamental chain or cycle technique' consists of the following three steps.

Step 1: Represent a moduli space locally as a zero set of certain section s of an orbibundle E over an orbifold U.

Step 2: Regard such a local description of the moduli space as a 'coordinate chart' of certain geometric object and introduce an appropriate notion of 'coordinate change'.

Step 3: Either (a) 'perturb' the section s in a way compatible with the coordinate changes so that it becomes transversal to 0 and then glue the zero sets of perturbed sections in 'various coordinate charts' by the coordinate change (this is the method used in the first two articles) or (b) develop and use general techniques from 'derived topology' to produce a fundamental chain or cycle from the geometric object in Step 2. An approach along these 'derived' lines is described in the third article. The application of the material in that article to produce the fundamental cycle and chain will appear in an article in preparation by Joyce.

It is worth pointing out that the arguments for Step 3 use only the formal properties of the geometric objects constructed in Step 2.

Brief descriptions of the articles in this volume. As we indicated, the goal of all the articles in this volume is to produce a fundamental cycle and chain associated to the moduli space of pseudo-holomorphic maps of curves to a symplectic manifold X equipped with a compatible almost complex structure. The starting point is a local description of the moduli space as the zeros of a smooth section of a smooth finite dimensional vector bundle over a smooth finite dimensional base modulo a finite group action – the *Kuranishi chart.* If the sections were transverse, the moduli space would inherit the structure of a finite dimensional smooth orbifold, which is oriented and thus has a (rational) fundamental class. But one must deal with non-transverse sections, especially in the case when the finite groups are non-trivial.

The article by Dusa McDuff describes her joint work with Katrin Wehrheim, [**6**]. Their goal is to produce the virtual fundamental cycle for the moduli space of pseudo-holomorphic maps. As they say in the introduction, "However, in practice, [the moduli space] usually has a more complicated structure since the operator is not transverse to zero. Intuitively the fundamental class is therefore the zero set of a suitable perturbation of the Fredholm operator: all the difficulty in constructing it lies in finding a suitable framework in which to build this perturbation." The framework described in this article is for the transitions between Kurannishi charts, formulated in what they call a *weak SS Kuranishi atlas.* In this article they show that any moduli space of genus zero pseudo-holomorphic curves in a compact symplectic manifold with a compatible almost complex structure has such an atlas and any two are compatible. From such an atlas, they produce a weighted branched topological manifold and a virtual fundamental class in the Cech homology of this branched manifold, unique up to cobordism. This then gives the virtual fundamental class for such moduli spaces. The case of moduli spaces of higher genus curves is covered by papers referenced in this article.

The second article is written by Mohammad Farajzadeh Tehrani partially using materials taken from a series of talks by Kenji Fukaya at the Simons Center for Geometry and Physics and also using material from papers written by Fukaya-Y.-G. Oh, H. Ohta and K. Ono [**2**]. The first half of the article is on the abstract theory of Kuranishi structures. After the definition of a Kuranishi structure is explained, the author provides a detailed account of the construction of the virtual fundamental class on a space with a Kuranishi structure. The first part of this process is a construction of a system with a finite number of charts (called a dimensionally graded system, DGS). (The initial Kuranishi structure can, and usually does, consist of an infinite number of charts.) DGS is a special case of a so called 'good coordinate system' appearing in [**2**]. The construction of a compatible system of multi-sections (multi-valued perturbations) on the those finitely many charts (transversal to zero) then are described. The triangulation of the zero set of such multi-section is given and as a result one is able to define a singular homology class, that is the virtual fundamental class. In the second half of this article, the construction of a Kuranishi structure on the moduli space of pseudo holomorphic curves is discussed. The author starts by explaining such moduli spaces and their topologies. Examples are given to show how the construction works. This article provides a comprehensive introduction to the construction of the Gromov-Witten invariant for general symplectic manifold.

The third article is a survey by Joyce of his paper [**5**] and his in-progress multivolume book "Kuranishi spaces and Symplectic Geometry." Preliminary versions of volumes I, II of this book are available at:

http://people.maths.ox.ac.uk/ joyce/Kuranishi.html.

This article proposes a new definition of Kuranishi space that has the nice property that the resulting objects form a 2-category, denoted **Kur**.

Any Fukaya–Oh–Ohta–Ono Kuranishi space **X**, as in the second article, can be made into a Kuranishi space **X′** in this sense, uniquely up to equivalence in **Kur**. The same holds for McDuff and Wehrheim's 'Kuranishi atlases' described in the first article in this volume and also holds for the Hofer, Wysocki and Zehnder's 'Polyfold Fredholm structures', [**4**].

The Kuranishi spaces as defined in this article, i.e., the objects of **Kur**, are objects in derived differential geometry, a concept introduced by the author in earlier work. Derived differential geometry is the study of classes of derived manifolds and derived orbifolds that the author calls 'd-manifolds' and 'd-orbifolds', respectively. There is an equivalence of 2-categories **Kur** $\sim$ **dOrb**, where **dOrb** is the 2-category of d-orbifolds. So Kuranishi spaces are really a form of derived orbifolds. As such they have their own differential geometry, with notions of orientation, immersions, submersions, transverse fibre products, and the other standard properties of good geometric categories.

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