CHAPTER 4

Jordan Elements

Throughout this chapter, L will be a Lie algebra over a ring of scalars Φ (some restrictions on the torsion of Φ will be required). By a Jordan element of L we mean an element $a \in L$ such that $\operatorname{ad}_a^3 L = 0$. Although without given them this name, Jordan elements were used by G. Benkart in the characterization of those finite-dimensional simple Lie algebras over a field of characteristic $p > 5$ which are classical. The reason why we choose this terminology is twofold. On the one hand, the quadratic operator $A^2 = \operatorname{ad}_a^2$ satisfies the identity $\operatorname{ad}^2_{A^2x} = A^2X^2A^2$, $x \in L$, which resembles the fundamental Jordan identity $U_{U_ax} = U_aU_xU_a$, and on the other hand, as will be seen in Chapter 8, it is possible to attach a Jordan algebra L_a to any Jordan element a of a Lie algebra L (becoming in this way an important Jordan tool in the solution of the Kurosh Problem). We list in Section 4.1 the main identities involving Jordan elements, among them the Jordan identity referred to above. In Section 4.2 we study the relationship between Jordan elements and abelian inner ideals. Section 4.3 is a fairly detailed development of the main consequences of the Jordan identity in nondegenerate Lie algebras, proving among other results a quadratic characterization of the annihilator of an ideal. Minimal abelian inner ideals in nondegenerate Lie algebras are studied in Section 4.4. The Kostrikin Descent Lemma and the refinement of this result given by M. Gómez and E. García are the aims of Section 4.5. In Section 4.6 we determine the structure of the Jordan elements of the Lie algebra R^- of a semiprime ring R. Finally, in Section 4.7, we study the Jordan elements of the Lie algebra $K = \operatorname{Skew}(R, *)$, where R is a centrally closed prime ring with involution.

4.1. Identities involving Jordan elements

Except for slight modifications, the results of this section as well as those of the next one are taken from [**Ben77**].

Definition 4.1. By a *Jordan element* of a Lie algebra L we mean an ad-nilpotent element $a \in L$ of index at most 3.

Example 4.2. Let R be a ring and let $a, z \in R$ be such that $a^2 = 0$ and $z \in Z(R)$. Then $a + z$ is a Jordan element of the Lie algebra R^-:

$$\operatorname{ad}^3_{a+z} R = [a, aRa] = 0.$$

Example 4.3. Let $L = L_{-n} \oplus \cdots \oplus L_n$ be a Lie algebra with a finite $\mathbb{Z}$-grading. Then any $x \in L_n \cup L_{-n}$ is Jordan element of L.

LEMMA 4.4. *Let L be a 3-torsion free Lie algebra and let a be a Jordan element of L. For any $x, y \in L$, we have:*

(1) $A^2XA = AXA^2$,
(2) $A^2XA^2 = 0$,
(3) $\mathrm{ad}^2_{A^2x} = A^2X^2A^2$,
(4) A^2x *is a Jordan element,*
(5) $A^2X^2AXA^2 = A^2XAX^2A^2$,
(6) $[A^2x, Ay] = [A^2y, Ax] = -A^2XAy$,
(7) $[A^2x, A^2y] = 0$,
(8) $\mathrm{ad}_{A^2x}\,\mathrm{ad}_{A^2y} = A^2XYA^2 = A^2YXA^2$,

where capital letters denote the adjoint operators with respect to those elements: $X = \mathrm{ad}_x$, $A = \mathrm{ad}_a$.

PROOF. Because $A^3x = 0$ for all $x \in L$,

$$0 = [A, [A, [A, X]]] = \mathbf{ad}^3_A(X) = (l_A - r_A)^3 X = -3A^2XA + 3AXA^2 = 0,$$

which proves (1) since L is 3-torsion free. Multiplying (1) on the right by A gives (2). Similarly,

$$\mathrm{ad}^2_{A^2x} = (A^2X - 2AXA + XA^2)^2 = A^2X^2A^2,$$

where we have used (1) and (2) to eliminate the remaining terms, so proving (3). Item (4) follows from (2) and (3):

$$\mathrm{ad}^3_{A^2x} = \mathrm{ad}_{A^2x}\,\mathrm{ad}^2_{A^2x} = (A^2X - 2AXA + XA^2)A^2X^2A^2 = 0.$$

To prove (5), write $B := \mathrm{ad}_{[x,[x,[x,a]]]} = \mathbf{ad}^3_X(A) = (l_X - r_X)^3 A$. Using (2) we get

$$\begin{aligned} 0 = A^2BA^2 = A^2(l_X - r_X)^3(A)A^2 &= A^2(X^3A - 3X^2AX + 3XAX^2 - AX^3)A^2 \\ &= 3A^2XAX^2A^2 - 3A^2X^2AXA^2, \end{aligned}$$

which implies $A^2X^2AXA^2 = A^2XAX^2A^2$, since L is 3-torsion free. (6) and (7) are particular cases of (2) and (1) of Lemma 2.17 (the last equality in (6) follows from the Jacobi identity). By (1) and (2), together with $A^3 = 0$, we have

$$\mathrm{ad}_{A^2x}\,\mathrm{ad}_{A^2y} = (A^2X - 2AXA + XA^2)(A^2Y - 2AYA + YA^2) = A^2XYA^2,$$

while the equality $A^2XYA^2 = A^2YXA^2$ is a direct consequence of (2). This proves (8) and completes the proof. □

4.2. Jordan elements and abelian inner ideals

Let L be a Lie algebra over a ring of scalars Φ. Recall that a Φ-submodule B of L is an inner ideal if $[B[B, L]] \subset B$, and an abelian inner ideal is an inner ideal which is also an abelian subalgebra.

LEMMA 4.5. *Let B be an abelian inner ideal of L. Then every $b \in B$ is a Jordan element of L.*

PROOF. $\mathrm{ad}^3_b L = \mathrm{ad}_b(\mathrm{ad}^2_b L) \subset [b, B] = 0.$ □

Note that Example 4.3 is a particular case of this lemma since both L_n and L_{-n} are abelian inner ideals. The converse is also true if L is 3-torsion free. In fact, we have the following more general result.

PROPOSITION 4.6. *Let L be a 3-torsion free Lie algebra and let $a \in L$ be a Jordan element. If V is an inner ideal of L, then $\mathrm{ad}_a^2 V$ is an abelian inner ideal. In particular,* $[a]_L := \mathrm{ad}_a^2 L$ (the principal inner ideal determined by a) *and* $(a)_L := \Phi a + [a]_L$ *are abelian inner ideals.*

PROOF. Let V be an inner ideal of L. It follows from 4.4(7) that $\mathrm{ad}_a^2 V$ is an abelian subalgebra. Thus it suffices to show that $\mathrm{ad}_x \mathrm{ad}_y L \subset \mathrm{ad}_a^2 V$ for $x = \mathrm{ad}_a^2 b$ and $y = \mathrm{ad}_a^2 c$, $b, c \in V$. By 4.4(8)we have

$$\mathrm{ad}_x \mathrm{ad}_y = \mathrm{ad}_{A^2 b} \mathrm{ad}_{A^2 c} = A^2 BCA^2.$$

Consequently,

$$\mathrm{ad}_x \mathrm{ad}_y L = \mathrm{ad}_a^2 \mathrm{ad}_b \mathrm{ad}_c \mathrm{ad}_a^2 L \subset \mathrm{ad}_a^2 \mathrm{ad}_b \mathrm{ad}_c L \subset \mathrm{ad}_a^2 [V, [V, L]] \subset \mathrm{ad}_a^2 V,$$

which proves that $\mathrm{ad}_a^2 V$ is an abelian inner ideal. In particular, $[a] = \mathrm{ad}_a^2 L$ is an abelian inner ideal, and the same is true for $(a) = \Phi a + [a]$:

$$[a, [a]] = \mathrm{ad}_a^3 L = 0 \quad \text{and} \quad [(a), [(a), L]] = [[a], [[a], L]] \subset [a] \subset (a).$$

□

4.3. Jordan elements in nondegenerate Lie algebras

Recall that a Lie algebra L is nondegenerate if $\mathrm{ad}_x^2 L = 0$ implies $x = 0$.

PROPOSITION 4.7. *Let I be an ideal of a 3-torsion free Lie algebra L. If L is nondegenerate, then I is a nondegenerate Lie algebra.*

PROOF. Let a be an absolute zero divisor of I. Then a is a Jordan element of L. Indeed, $\mathrm{ad}_a^3 L = \mathrm{ad}_a^2 [a, L] \subset \mathrm{ad}_a^2 I = 0$. So, by 4.4(3), for every $x \in L$, we have

$$\mathrm{ad}_{A^2 x}^2 L = \mathrm{ad}_a^2 \mathrm{ad}_x^2 \mathrm{ad}_a^2 L \subset \mathrm{ad}_a^2 I = 0.$$

Applying the nondegeneracy of L twice, we get $a = 0$. □

REMARK 4.8. Inheritance of nondegeneracy by ideals was obtained in [**Zel83a**, Lemma 4] as a consequence of the inheritance of the Kostrikin radical, which will be studied later.

Recall that a Lie algebra is said to be strongly prime if it is prime and nondegenerate.

COROLLARY 4.9. *Let I be an ideal of a 3-torsion free Lie algebra L. If L is strongly prime, then I is a strongly prime Lie algebra.*

PROOF. By Proposition 4.7, I is nondegenerate and therefore semiprime, so we can apply Proposition 2.46 to get that I is also prime. □

The annihilator of an ideal I of a semiprime associative algebra R can be described in quadratic terms: $xIx = 0 \Rightarrow (xI)^2 = 0 \Rightarrow xI = 0 \Rightarrow x \in \mathrm{Ann}_R(I)$. A similar result holds for nondegenerate Lie algebras.

PROPOSITION 4.10. *Let I be an ideal of a 6-torsion free Lie algebra L. If I is a nondegenerate Lie algebra, then* $\mathrm{Ann}_L(I) = \{a \in L : \mathrm{ad}_a^2 I = 0\}$.

PROOF. Let $a \in L$. Clearly $a \in \mathrm{Ann}_L(I)$ implies $\mathrm{ad}_a^2 I = 0$. Suppose conversely that $\mathrm{ad}_a^2 I = 0$. Since I is an ideal of L, the uppercase notation for adjoint representations is well defined when their arguments are restricted to I. We will assume that all the implicit arguments are in the ideal I. According with this convention, $A^2 = 0$ and the identities of Lemma 3.3 remain valid. Given $x \in I$, set $b = [a, x]$. By 3.3(4), b is a Jordan element of I, with $B^2 = -AX^2A$ by 3.3(3). Then, by 4.4(3),

$$\mathrm{ad}_{B^2x}^2 = B^2X^2B^2 = AX^2AX^2AX^2A = 0$$

since $AX^2AX^2A = 0$ by 3.3(6). Hence $B^2x = 0$ by nondegeneracy of I. Note that x was fixed and related to b, so we are not finished yet. Linearizing

$$B^2x = -AX^2Ax = 0,\ x \in I,$$

we get for any $y \in I$ and $n \in \mathbb{Z}$,

$$0 = A(X + nY)^2A(x + ny) = 3nAX^2Ay + 3n^2AY^2Ax,$$

where we have eliminated the terms AX^2Ax and AY^2Ay and used the identities (2) and (5) of Lemma 3.3. Take $n = 1$ and $n = 2$ to get the system of linear equations

$$3AX^2Ay + 3AY^2Ax = 0, \quad 6AX^2Ay + 12AY^2Ax = 0,$$

which yields the solution $6AX^2Ay = 0$. Since L is 6-torsion free, this implies

$$\mathrm{ad}_{Ax}^2 y = B^2y = -AX^2Ay = 0$$

for all $y \in I$, and hence that $Ax = 0$ because I is nondegenerate. Since $x \in I$ is arbitrary, this proves that $a \in \mathrm{Ann}_L(I)$. [1] □

COROLLARY 4.11. *Let I be an ideal of a 6-torsion free Lie algebra L. If I is nondegenerate, then the Lie algebra $L/\mathrm{Ann}_L(I)$ is nondegenerate.*

PROOF. Denote by $x \mapsto \overline{x}$ the homomorphism of L onto $\overline{L} = L/\mathrm{Ann}_L(I)$, and let $\overline{a}$ be an absolute zero divisor of $\overline{L}$. Then $\mathrm{ad}_a^2 I \subset I \cap \mathrm{Ann}_L(I) = 0$. Hence $a \in \mathrm{Ann}_L(I)$ by Proposition 4.10, i.e. $\overline{a} = 0$. □

PROPOSITION 4.12. *Let E be an essential ideal of a 6-torsion free Lie algebra L. If E is nondegenerate, then L is nondegenerate.*

PROOF. Let a be an absolute zero divisor of L. Then $\mathrm{ad}_a^2 E = 0$, and hence, Proposition by 4.10, $a \in \mathrm{Ann}_L(E)$. But $\mathrm{Ann}_L(E) = 0$ by Proposition 1.6(1). □

PROPOSITION 4.13. [**FLGGL09**, Proposition 1.6] *Let L be a 6-torsion free Lie algebra and let I be an ideal of L. If I is nondegenerate, then every Jordan element of I is a Jordan element of L.*

PROOF. Let a be a Jordan element of I. Then $\mathrm{ad}_a^4 L = \mathrm{ad}_a^3[a, L] \subset \mathrm{ad}_a^3 I = 0$. So for any $x \in L,\ y \in I$ we have

$$\begin{aligned} 0 &= \mathrm{ad}_{A^4x}\, y = (A^4X - 4A^3XA + 6A^2XA^2 - 4AXA^3 + XA^4)y \\ &= -4A^3XAy + 6A^2XA^2y - 4AXA^3y = 6A^2XA^2y. \end{aligned} \tag{4.1}$$

[1] A different proof of Proposition 4.10 is given in [**Zel83a**, Corollary 1, page 543].

By 4.4(4), $A^2[a,x]$ is a Jordan element of I. Hence we have by (4.1)

$$\begin{aligned}\operatorname{ad}^2_{A^2[a,x]} y &= \operatorname{ad}^2_a \operatorname{ad}^2_{[a,x]} \operatorname{ad}^2_a y = (A^2(AX - XA)^2 A^2)y \\ &= (-A^2XA^2XA^2 + A^2XAXA^3)y = -A^2XA^2XA^2y = 0.\end{aligned}$$

Then, by Proposition 4.10, $A^3x = A^2[a,x] \in I \cap \operatorname{Ann}(I) = 0$, so a is a Jordan element of L. □

PROPOSITION 4.14. *Let L be a 6-torsion free Lie algebra and let E be an essential ideal of L. If E is nondegenerate, then L contains nonzero Jordan elements if and only if so does E.*

PROOF. By Proposition 4.13, every Jordan element of E is a Jordan element of L. Let $a \in L$ be a nonzero Jordan element of L. Since E is an essential ideal, $\operatorname{Ann}_L(E) = 0$ and hence, by Proposition 4.10, $\operatorname{ad}^2_a E \neq 0$. Then it follows from Lemma 4.5 and Proposition 4.6 that E contains a nonzero Jordan element. □

LEMMA 4.15. *Let E be an essential ideal of a Lie algebra L and let $\alpha \in \Phi$. If E is α-torsion free, then so is L.*

PROOF. Let $\operatorname{Ann}_L(\alpha) = \{x \in L : \alpha x = 0\}$. Then $\operatorname{Ann}_L(\alpha)$ is an ideal of L, and since E is essential and α-torsion free, $\operatorname{Ann}_L(\alpha) = 0$, which proves that L is α-torsion free. □

COROLLARY 4.16. *Let L be a a 6-torsion free Lie algebra. If L is nondegenerate, then the Lie algebra* $\operatorname{Der}(L)$ *is also nondegenerate.*

PROOF. As L is nondegenerate, $Z(L) = 0$. Hence, by Proposition 2.14, L is isomorphic to the essential ideal $\operatorname{ad}(L)$ of $\operatorname{Der}(L)$. Since $\operatorname{Der}(L)$ is also 6-torsion free by the above lemma, it is nondegenerate by Proposition 4.12. □

PROPOSITION 4.17. *Let L be a Lie algebra over a ring of scalars Φ such that $6 \in \Phi^*$ and let $z \in L$ be an absolute zero divisor. We have:*

(1) *For any Jordan element $a \in L$, $[a,z] = w_1 + w_2 + w_3$ is a sum of absolute zero divisors.*
(2) *The span C_1 of the absolute zero divisors of L is an inner ideal.*

PROOF. (1) By Lemma 1.2, $\exp(A) \in \operatorname{Aut}(L)$, and hence

$$\exp(A)z = z + Az + \frac{1}{2}A^2z$$

is an absolute zero divisor. Furthermore, by 4.4(3)

$$\operatorname{ad}^2_{A^2z} = A^2Z^2A^2 = 0,$$

so A^2z is also an absolute zero divisor. Thus $[a,z] = \exp(A)z - z - \frac{1}{2}A^2z$ is a sum of three absolute zero divisors.

(2) It suffices to show that for every $x \in L$ and z_1, z_2 absolute zero divisors, $[[x,z_1],z_2] \in C_1$. Since $[x,z_1]$ is a Jordan element by 3.3(4)), we have by (1) that $[[x,z_1],z_2] \in C_1$. □

REMARK 4.18. Suppose that L is a simple Lie algebra over a field $\mathbb{F}$ of characteristic $p > 3$. It follows from (1) of the above proposition that if L is generated by Jordan elements, then the linear span of its absolute zero divisors is an ideal of L. This result had been already obtained in Proposition 3.25 using the Vandermonde argument.

4.4. Minimal abelian inner ideals

Let B be a minimal abelian inner ideal of a Lie algebra L. Since every inner ideal contained in B is abelian, it is cleat that B is a minimal inner ideal of L. Conversely, every minimal inner ideal of L which is also abelian is a minimal abelian inner ideal of L. So we can interchange the adjectives abelian and minimal without changing the meaning.

LEMMA 4.19. *Let L be a 3-torsion free nondegenerate Lie algebra and let B be an abelian subalgebra of L. Then B is a minimal abelian inner ideal of L if and only if $B = \mathrm{ad}_b^2 L$ for every nonzero element $b \in B$.*

PROOF. If B is a minimal abelian inner ideal of L, then it follows from Lemma 4.5 and Proposition 4.6 that $B = \mathrm{ad}_b^2 L$ for every $0 \neq b \in B$. The converse is trivial. □

COROLLARY 4.20. *Suppose that L is 3-torsion free and nondegenerate. Let B be a minimal abelian inner ideal of L and let $a \in L$ be a Jordan element. Then either $\mathrm{ad}_a^2 B = 0$ or it is a minimal abelian inner ideal.*

PROOF. By Proposition 4.6, $\mathrm{ad}_a^2 B$ is an abelian inner ideal of L. Since L is nondegenerate, for any nonzero element $\mathrm{ad}_a^2 b \in \mathrm{ad}_a^2 B$, we have (again by 4.6) that $\mathrm{ad}_{A^2b}^2 L$ is a nonzero abelian inner ideal of L. Then using 4.4(3) and the minimality of B, we get

$$\mathrm{ad}_{A^2b}^2 L = \mathrm{ad}_a^2(\mathrm{ad}_b^2 \mathrm{ad}_a^2 L) = \mathrm{ad}_a^2 B,$$

which proves that $\mathrm{ad}_a^2 B$ is minimal by Lemma 4.19. □

COROLLARY 4.21. *Let L be a 6-torsion free Lie algebra, let I be an ideal of L, and let B be an abelian subalgebra of I. If I is nondegenerate, then B is a minimal inner ideal of L if and only if it is a minimal inner ideal of I.*

PROOF. Let B be an minimal abelian inner ideal of I. It follows from Proposition 4.13 that every $b \in B$ is a Jordan element of L. Hence, for every nonzero $b \in B$, $B = \mathrm{ad}_b^2 I$ is an abelian inner ideal of L by Proposition 4.6. Since B is minimal as an inner ideal of I, it is also minimal as an inner ideal of L. Conversely, suppose that B is a minimal abelian inner ideal of L contained in I. As I is nondegenerate, it follows from Proposition 4.6 that $\mathrm{ad}_b^2 I$ is a nonzero inner ideal of L for any nonzero element $b \in B$,. Hence $B = \mathrm{ad}_b^2 I$ by minimality of B, which proves that B is a minimal abelian inner ideal of I. □

4.5. On the existence of Jordan elements

In this section we study conditions guaranteeing the existence of nonzero Jordan elements in Lie algebras. We keep using capital letter to denote adjoint operators.

The Kostrikin Descent Lemma.

LEMMA 4.22. [**Zel92b**, III.2.10] *Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic 0 or $p \geq 5$ and let $a \in L$ be such that $\mathrm{ad}_a^n = 0$ for $4 \leq n \leq p-1$. Then for every $x \in L$ we have $\mathrm{ad}_{A^{n-1}x}^{n-1} = 0$.*

PROOF. From $A^n = 0$ we get, for any $x \in L$, $\mathbf{ad}_A^n X = \mathrm{ad}_{A^n x} = 0$, i.e.

$$-nA^{n-1}XA + \binom{n}{2}A^{n-2}XA^2 + \cdots + (-1)^{n-1}nAXA^{n-1} = 0. \tag{4.2}$$

Multiplying (4.2) on the right by A^{n-2} we get

$$A^{n-1}XA^{n-1}=0, \tag{4.3}$$

since n is invertible in $\mathbb{F}$. Now, multiplying (4.2), first on the right and then on the left by A^{n-3}, we get the system of linear equations

$$-nA^{n-1}XA^{n-2}+\binom{n}{2}A^{n-2}XA^{n-1}=0$$
$$(-1)^{n-2}\binom{n}{2}A^{n-1}XA^{n-2}+(-1)^{n-1}nA^{n-2}XA^{n-1}=0,$$

whose determinant is

$$\Delta=(-1)^{n-1}\frac{n^2(n-3)(n+1)}{4}.$$

Since $4\le n\le p-1$, $\Delta=0$ if and only if $n<p-1$. In this case we have

$$A^{n-1}XA^{n-2}=0. \tag{4.4}$$

Return to the statement of the lemma. We must prove that $\mathrm{ad}^{n-1}_{A^{n-1}x}=0$ for any $x\in L$. Suppose on the contrary that

$$\mathrm{ad}^{n-1}_{A^{n-1}x}=(\mathbf{ad}^{n-1}_A X)^{n-1}=\sum m_{\mathbf{i}}A^{i_1}XA^{i_2}X\cdots XA^{i_n}\neq 0, \tag{4.5}$$

where the integers m_i are nonzero in $\mathbb{F}$, $0\le i_k<n$ for all $1\le k\le n$, and

$$i_1+i_2+\cdots i_n=(n-1)^2. \tag{4.6}$$

Let

$$A^{i_1}XA^{i_2}X\cdots XA^{i_n}\neq 0 \tag{4.7}$$

be the summand in (4.5) such that the sequence $(i_1,\cdots,i_n)$ is lexicographically maximal. We claim that $i_k>0$ for all $1\le k\le n$. If $i_k=0$ for some k, then it follows from (4.6) that all i_j, $j\neq k$, are equal to $n-1$. Hence, since $n\ge 4$ by assumption, the product (4.7) contains a factor $A^{n-1}XA^{n-1}$, which is equal to zero by (4.3), a contradiction.

Our next claim is that $i_k\le n-2$ for all $2\le k\le n$. Suppose on the contrary that $i_k=n-1$ for some k such that $2\le k\le n$. Since $i_{k-1}\neq 0$, we have by (4.2)

$$A^{i_{k-1}}XA^{i_k}=A^{i_{k-1}-1}(AXA^{n-1})=A^{i_{k-1}-1}(\sum_{i\ge 2}\alpha_iA^iXA^{n-i}),\ \alpha_i\in\mathbb{F},$$

which contradicts the lexicographically maximality of $(i_1,\cdots,i_n)$ in (4.4). Now it follows from (4.6) that

$$i_1=n-1,\ i_2=i_3=\cdots=i_n=n-2.$$

If $n\neq p-1$, then we have by (4.4) that $A^{i_1}XA^{i_2}=0$, a contradiction. The proof will be complete by proving that the assumption $n=p-1$ implies

$$A^{n-1}XA^{n-2}XA^{n-2}=0,$$

which is again a contradiction. Multiplying (4.2) on the right by A^{n-4} we get

$$A^{n-2}XA^{n-2}=\alpha_1A^{n-1}XA^{n-3}+\alpha_2A^{n-3}XA^{n-1}$$

with $\alpha_i\in\mathbb{F}$, $i=1,2$. Hence, by (4.3),

$$A^{n-1}XA^{n-2}XA^{n-2}=\alpha_2A^{n-1}XA^{n-3}XA^{n-1}.$$

Since the prime number p is odd, $n = p-1$ is even, and $n-3$ is again odd, we have

$$[X, \mathbf{ad}_A^{n-3} X] = 2XA^{n-3}X + \sum_{i+j>0} \beta_{ij} A^i X A^{n-3-i-j} X A^j.$$

Hence, by (4.3),

$$A^{n-1} X A^{n-3} X A^{n-1} = \frac{1}{2} A^{n-1} [X, \mathbf{ad}_A^{n-3} X] A^{n-1} = 0.$$

This completes the proof of the lemma. □

Corollary 4.23. *Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic 0 or $p \geq 5$, and let $a \in L$ be an ad-nilpotent element of index n, $4 \leq n \leq p-1$. Then L contains a nonzero Jordan element.*

Proof. By Lemma 4.22, $\mathrm{ad}_{A^{n-1}(L)}^{n-1} = 0$. Let b be a nonzero element in $A^{n-1}(L)$. If $n = 4$, then b is a Jordan element and we have finished. Otherwise we can repeat the process to get finally the desired Jordan element, since b is a nonzero ad-nilpotent element of index less or equal to $n-1$. □

For $n = 4$, Corollary 4.23 remains true for any Lie algebra over a ring of scalars just assuming 6-torsion free.

Lemma 4.24. *Let L be a 6-torsion free Lie Φ-algebra, and let $0 \neq a \in L$ be such that $\mathrm{A}^4 = 0$. Then L contains a nonzero Jordan element.*

Proof. If $A^3 = 0$, then a itself is a Jordan element, so we may suppose $A^3 x \neq 0$ for some $x \in L$. We claim that $A^3 x$ is a Jordan element, i.e.

$$\mathrm{ad}_{A^3 x}^3 = \sum_i m_i A^{i_1} X A^{i_2} X A^{i_3} X A^{i_4} = 0. \quad i_1 + i_2 + i_3 + i_4 = 9,$$

If this were not the case, we would have, arguing as in the proof of Lemma 4.22,

$$A^3 X A^2 X A^2 X A^2 \neq 0. \tag{4.8}$$

But this leads to a contradiction. From

$$0 = \mathrm{ad}_{A^4 x} = -4A^3 X A + 6A^2 X A^2 - 4AXA^3 \tag{4.9}$$

and just using 2-torsion free, we get

$$A^3 X A^3 = 0. \tag{4.10}$$

Hence $0 = A^3[[X, A], X]A^3 = 2A^3 XAXA^3$. Then, by (4.9),

$$6(A^3 X) A^2 X A^2 = 4(A^3 X) A^3 X A + 4(A^3 X) AXA^3 = 0,$$

which contradicts (4.8). □

A refinement of Kostrikin's descent lemma. Let L be a Lie algebra over a ring of scalars Φ and let $a \in L$ be such that $\mathrm{ad}_a^n = 0$, $n \geq 2$. Following the paper of E. García and M. Gómez [**GGL09**], we prove in this subsection a result which extends those obtained for $n = 2$ (Proposition 3.6) and for $n = 3$ (Proposition 4.6). In fact, Kostrikin's descent lemma is sharpened in the sense that $\mathrm{ad}_a^{n-1} L$ is actually an abelian inner ideal of L and hence any $x \in \mathrm{ad}_a^{n-1} L$ is a Jordan element.

To illustrate the process of the proof of this result, which is quite technical and involves long calculations, we begin by analyzing the case that $n = 6$ (assuming $7! \in \Phi^*$). Note first that, by Lemma 2.17, the Φ-submodule $\mathrm{ad}_a^5 l$ is an abelian subalgebra, so, by Lemma 2.35, it is enough to show that $\mathrm{ad}_{A^5 x}^2 L \subset \mathrm{ad}_a^5 L$, $x \in L$.

Set $B = \mathrm{ad}_{A^5x} = A^5X - 5A^4XA + 10A^3XA^2 - 10A^2XA^3 + 5AXA^4 - XA^5$. We will prove that $B^2 \in A^5 \operatorname{Ad}(L)$, i.e. $B^2 \equiv 0 \pmod{A^5 \operatorname{Ad}(L)}$.

From $A^6 = 0$, we get

$$-6A^5XA + 15A^4XA^2 - 20A^3XA^3 + 15A^2XA^4 - 6AXA^5 = 0. \tag{4.11}$$

Multiply (4.11) by A^2, first on the left and then on the right. Then multiply (4.11) by A on both sides. We get the system of linear equations

$$\begin{aligned} -6A^5XA^3 + 15A^4XA^4 - 20A^3XA^5 &= 0\\ -20A^5XA^3 + 15A^4XA^4 - 6A^3XA^5 &= 0\\ 15A^5XA^3 - 20A^4XA^4 + 15A^3XA^5 &= 0. \end{aligned}$$

As $\begin{vmatrix} -6 & 15 & -20\\ -20 & 15 & -6\\ 15 & -20 & 15 \end{vmatrix} = -980 = -2^2.5.7^2 \in \Phi^*$, the system has a unique solution, so

$$A^5XA^3 = A^3XA^5 = A^4XA^4 = 0. \tag{4.12}$$

Hence we have

$$A^iXAXA^j = A^i[[X, A], X]A^j = 0,\ i + j \geq 0. \tag{4.13}$$

Using (4.12) and (4.13) we get that B^2 is congruent with

$$50A^4XA^3XA^3 - 25A^4XA^2XA^4 - 100A^3XA^4XA^3 + 50A^3XA^3XA^4 - 10A^3XA^2XA^5$$

We deal with each summand of the above expression separately. Multiplying (4.11) on the left by A and on the right by XA^3 we get

$$-20A^4XA^3XA^3 + 15A^3XA^4XA^3 = -15A^5XA^2XA^3.$$

Again, multiplying (4.11) on the right by AXA^3 we obtain

$$15A^4XA^3XA^3 - 20A^3XA^4XA^3 = -A^5XA^2XA^3$$

The determinant of the system formed by these two equations is 7.5^2, so this system can be solved obtaining

$$A^4XA^3XA^3,\ A^3XA^4XA^3 \in A^5 \operatorname{Ad}(L). \tag{4.14}$$

Multiplying (4.11) on the left by A^4X and using (4.12) and (4.13), we get

$$15A^4XA^2XA^4 = 20A^4XA^3XA^3.$$

Hence, by (4.14),

$$A^4XA^2XA^4 \in A^5 \operatorname{Ad}(L). \tag{4.15}$$

Multiplying now (4.11) on the right by XA^4 we obtain

$$15A^4XA^2XA^4 = 20A^3XA^3XA^4.$$

Hence, by (4.15),

$$A^3XA^3XA^4 \in A^5 \operatorname{Ad}(L). \tag{4.16}$$

Finally, multiplying (4.11) on the left by A^3XA, we get

$$-20A^3XA^4XA^3 + 15A^3XA^3XA^4 - 6A^3XA^2XA^5 = 0.$$

Hence, by (4.14) and (4.16), $A^3XA^2XA^5 \in A^5 \operatorname{Ad}(L)$, as desired.

LEMMA 4.25. *Let L be a Lie Φ-algebra, and let $a \in L$ be such that $A^n = 0$, $n > 2$. For each $0 < m < n$ and any $x \in L$, put*

$$y_1 = A^m X A^{n-1},\ y_2 = A^{m+1} X A^{n-2},\ \ldots,\ y_{n-m} = A^{n-1} X A^m.$$

(1) *If $2m \geq n$ and $(2n - m - 1)! \in \Phi^*$, then all $y_1, \ldots, y_{n-m}$ are zero.*
(2) *If $2m < n$ and $(n + m - 1)! \in \Phi^*$, then for any choice of m indexes*

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n - m,$$

we can express the elements $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$ as a linear combination of the other $n - 2m$ elements of $\{y_1, \ldots, y_{n-m}\}$.

PROOF. Let $a \in L$ be such that $A^n = 0$. For any $x \in L$, we have

$$\sum_{k=1}^{n-1} (-1)^k \binom{n}{k} A^{n-k} X A^k = 0. \tag{4.17}$$

Denote by ♣ the left hand side of (4.17)and consider the equalities

$$A^{m-1}\clubsuit = 0, \quad A^{m-2}\clubsuit A = 0, \quad \ldots, \quad A\clubsuit A^{m-2} = 0, \quad \clubsuit A^{m-1} = 0, \tag{4.18}$$

regarded as a linear system of m equations in the unknowns $y_1, y_2, \ldots, y_{n-m}$.

(1) If $2m \geq n$, the first $n - m$ equations form an homogeneous linear system whose matrix is invertible (see [**Kos90**, Theorem 3.1, p. 40]), since we are assuming that $(2n - m - 1)! \in \Phi^*$, so the only solution of the system is the trivial one. Therefore, $A^i X A^j = 0$ for $i + j = n + m - 1$.

(2) Suppose that $2m < n$ and let $1 \leq i_1 < i_2 < \cdots < i_m \leq n - m$. Then the m columns $(i_1, \ldots, i_m)^t$ of the coefficients of the linear systems (4.18) form a matrix which is invertible. Thus we can express the unknowns $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$ as a linear combination of the other $n - 2m$ elements in $\{y_1, \ldots, y_{n-m}\}$. □

LEMMA 4.26. *Let L be a Lie Φ-algebra and let $a \in L$ be such that $A^n = 0$, with $n > 2$ and $(n + [\frac{n}{2}] - 1)! \in \Phi^*$. Then for any $x \in L$ and any triple (i, j, k) of nonnegative integers such that $i + j + k = 2n - 2$, we have that $A^i X A^j X A^k$ belongs to $A^{n-1} \operatorname{Ad}(L)$.*

PROOF. If $j = 0$, then $i + k = 2n - 2$, and hence either $A^i X^2 A^k = 0$ (when $i \geq n$ or $k \geq n$) or both i, k are equal to $n - 1$. In the two cases,

$$A^i X A^j X A^k \in A^{n-1} \operatorname{Ad}(L).$$

We will assume from now on that n is odd, $n = 2t + 1$. For n even the proof is similar.

If $k < t$ or $i < t$, then $A^i X A^j X A^k = 0$. Since both cases are similar, let us suppose that $k < t$. Taking $m = t+1$ in (4.18), we have by 4.25(1) that $A^i X A^j = 0$ whenever $i + j = n + m - 1 = (2t + 1) + (t + 1) - 1 = 3t + 1$. But $k < t$, together with $i + j + k = 2n - 2 = 4t$, implies $i + j > 3t$. Therefore $A^i X A^j X A^k = 0$ as desired.

If $k = t$ or $i = t$ then $A^i X A^j X A^k \in A^{n-1} \operatorname{Ad}(L)$. Consider first the case of a term $A^i X A^j X A^t$ ending in A^t. Then $i + j + t = 4t$ implies $i + j = 3t$. Take $m = t$ in (4.18). Then $n - 2m = (2t + 1) - 2t = 1$ and hence it follows from 4.25(2) that $A^i X A^j$ is a scalar multiple of $A^{2t} X A^t$, so $(A^i X A^j) X A^t \in A^{n-1} X A^m$. Consider now a term of the form $A^tXA^jXA^k$. Then $j+k=3t$ and hence, by 4.25(2) again, $A^jXA^k=\lambda A^{2t}XA^t$ for some $\lambda\in\Phi$. Then

$$A^tX(A^jXA^k)=\lambda A^tX(A^{2t}XA^t)\in A^{n-1}\operatorname{Ad}(L)$$

by we have just proved.

To finish the proof we will use induction. Suppose that given $0\le r<[\frac{t}{2}]$, every term $A^iXA^jXA^k$ such that $i\le t+r$ or $k\le t+r$ belongs to $A^{n-1}\operatorname{Ad}(L)$. We will prove that the same holds for every term beginning or ending with A^{t+r+1}. Let us first consider the case $k=t+r+1$. Then $i+j+t+r+1=4t$ implies $i+j=3t-r-1$. Taking $m=t-r-1$ in (4.18), it follows from Lemma 4.25(2) that A^iXA^j can be expressed as a linear combination of any set of $n-2m=(2t+1)-2(t-r-1)=2r+3$ elements taken among the terms

$$A^{2t}XA^{t-r-1},\ A^{2t-1}XA^{t-r},\ \ldots,\ A^{t+r}XA^{t-2r-1},\ldots,\ A^{t-r-1}XA^{2t},$$

so we can express $(A^iXA^j)XA^{t+r+1}$ as a linear combination of the $2r+3$ terms

$$A^{2t}XA^{t-r-1}XA^{t+r+1},\quad A^{t-r-1}XA^{2t}XA^{t+r+1},$$
$$A^{t-r}XA^{2t+1}XA^{t+r+1},\ldots,A^{t+r}XA^{t-2r-1}XA^{t+r+1}.$$

It is clear that $A^{2t}XA^{t-r-1}XA^{t+r+1}\in A^{n-1}\operatorname{Ad}(L)$, since $n=2t+1$. The other $2r+2$ terms also belong to $A^{n-1}\operatorname{Ad}(L)$ by the induction hypothesis. In the same way, any term beginning with A^{t+r+1} can be written as a linear combination of the $2r+3$ terms ending in A^k, $t-r-1\le k\le t+r+1$, which belong to $A^{n-1}\operatorname{Ad}(L)$ by we have just proved. □

Theorem 4.27. [**GGL09**, Theorem 2.3] *Let L be a Lie Φ-algebra and let $a\in L$ be such that $\mathrm{A}^n=0$. If $n>2$ and $(n+[\frac{n}{2}]-1)!\in\Phi^*$, then $\operatorname{ad}_a^{n-1}L$ is an abelian inner ideal of L. Hence any $x\in\operatorname{ad}_a^{n-1}L$ is a Jordan element.*

Proof. By Lemma 2.17, $\operatorname{ad}_a^{n-1}L$ is an abelian subalgebra. Thus, by Lemma 2.35, it suffices to prove that for each $x\in L$, $\operatorname{ad}^2_{A^{n-1}x}\in A^{n-1}\operatorname{Ad}(L)$. Since

$$\operatorname{ad}_{A^{n-1}x}=\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}A^{n-1-k}XA^k,$$

$\operatorname{ad}^2_{A^{n-1}x}$ is a linear combination of monomials of the form $A^iXA^jXA^k$, where $i,j,k\ge0$ and $i+j+k=2n-2$. Thus all we need to see is that each one of these monomials is equal to $A^{n-1}M$ for some $M\in\operatorname{Ad}(L)$. But this is just what proves Lemma 4.26. □

Remark 4.28. Let A be a nonassociative algebra over a field of characteristic 0 and let d be a derivation of A such that $d^n=0$ for some $n\ge2$. Then the subspace $D=d^{n-1}(A)$ satisfies $D^2=0$ and $(DA)D\subset D$. As observed in [**BFL10**, Theorem 2.2], this assertion can be proved by an adaptation of the arguments used in the proof of the (apparently very special case) [**GGL09**, Theorem 2.3].

Minimal inner ideals revisited. The structure theorem on minimal inner ideals (see Theorem 2.43) can be sharpened as follows.

Theorem 4.29. [**Ben77**, Lemma 1.12] *Let L be a 6-torsion free Lie algebra and let B be a minimal inner ideal of L. Then either:*

(1) *$B=\Phi b$, where b is an absolute zero divisor of L and Φ is a field,*

(2) $B = \mathrm{ad}_b^2 L$ *for all* $b \neq 0$ *in* B *and* $[B, B] = 0$, *or*
(3) B *is an ideal of* L *which is innerly simple as a Lie algebra.*

Proof. We may assume that B has no nonzero absolute zero divisors (since otherwise we would be in case (1) of Theorem 2.43). If B is as in case (2) of 2.43, then it follows from Proposition 4.6 that $B = \mathrm{ad}_b^2 L$ for all $b \neq 0$ in B. Thus we may assume that B is an ideal of L which is simple as a Lie algebra and every proper inner ideal V of B is abelian. For every $v \in V$, $\mathrm{ad}_v^4 L \subset [v, [v, [v, B]]] \subset [v, V] = 0$, so v is ad-nilpotent of index less than or equal to 4. If the index is 4, then it follows from Lemma 4.24 that there exist $x \in L$ such that $u = \mathrm{ad}_v^3 x$ is a nonzero Jordan element of L. Since $u \in B$, we have by 4.6 that $\mathrm{ad}_u^2 L$ is an abelian inner ideal of L contained in B, and hence either $\mathrm{ad}_u^2 L = B$ or $\mathrm{ad}_u^2 L = 0$. In the first case, $[B, B] = 0$, which is a contradiction since B is simple as an algebra. The case $\mathrm{ad}_u^2 L = 0$ gives $u = 0$ (since otherwise $B = \Phi u$, which has been discarded), which is again a contradiction. Therefore, $\mathrm{ad}_v^3 L = 0$. By the same reasoning applied to v we obtain $V = 0$, so B does not contain proper nonzero inner ideals. □

Example 4.30. The following Lie algebras are innerly simple and therefore they are themselves minimal inner ideals of type (3) in the theorem above.

(i) $[\Delta, \Delta]/Z(\Delta) \cap [\Delta, \Delta]$, where Δ is a division associative algebra such that $[[\Delta, \Delta], \Delta] \neq 0$, [**Ben76**, Corollary 3.15].

(ii) The finitary orthogonal Lie algebra $\mathfrak{fo}(X, \langle\,,\rangle)$ (Section 2.2) where the bilinear form $\langle\,,\rangle$ is anisotropic and X has dimension (possibly infinite) greater than 4.

Finite gradings and Jordan elements. Let Λ be a torsion free abelian group and let L be a Lie algebra. Recall that a Λ-grading $L = \bigoplus_{\lambda \in \Lambda} L_\lambda$ of L is said to be finite if the set $\Lambda^* = \{\lambda \in \Lambda : L_\lambda \neq 0\}$ is finite, and nontrivial if Λ^* contains a nonzero element. Note that if a Λ-grading is finite and nontrivial, then the subgroup $G = G(\Lambda^*)$ of Λ generated by Λ^* is free of finite rank, and therefore is isomorphic to $\mathbb{Z}^r$ for some positive integer r. In fact, as pointed out by E. Zelmanov in [**Zel83a**, Proof of Lemma 14], there is no loss of generality in assuming $\Lambda = \mathbb{Z}$ and that the finite Λ-grading is actually a finite $\mathbb{Z}$-grading.

Proposition 4.31. *Let* L *be a Lie algebra with a nontrivial finite* Λ*-grading, where* Λ *is a torsion free abelian group. Then* L *contains a nonzero Jordan element.*[2]

Proof. Take a basis $\{\lambda_1, \dots, \lambda_r\}$ of G such that for some $\alpha \in \Lambda^*$, $\alpha = n_{\alpha_1}\lambda_1 + \dots + n_{\alpha_r}\lambda_r$ with $n_{\alpha_1} \neq 0$, and let $\pi : G \to \mathbb{Z}$ be the homomorphism defined by putting $\pi(\lambda_1) = 1$ and $\pi(\lambda_i) = 0$ for $1 < i \leq r$. We may also assume that $|\pi(\beta)| \leq |\pi(\alpha)|$ $(\beta \in \Lambda^*)$. Then any $x \in L_\alpha$ is a Jordan element: for any $\beta \in \Lambda^*$, $\mathrm{ad}_x^3 L_\beta \subset L_{3\alpha+\beta} = 0$ since $|\pi(3\alpha + \beta)| > |\pi(\alpha)|$. □

Corollary 4.32. *Let* L *be a Lie algebra over an algebraically closed field* $\mathbb{F}$ *of characteristic* 0. *If* L *has a nonzero ad-algebraic element, then* L *contains a nonzero Jordan element.*

[2]The existence of nonzero Jordan elements in Lie algebras with a nontrivial finite Λ-grading is mentioned in [**Zel83a**, p. 548].

PROOF. Let a be a nonzero element of L such that ad_a is algebraic. If a is not ad-nilpotent, then we have by Example 1.4 that ad_a produces a nontrivial finite $(\mathbb{F},+)$-grading on L given by

$$L_\lambda = \{x \in L : (\mathrm{ad}_a - \lambda 1_L)^m x = 0 \text{ for some } m \geq 1\},$$

with $L_\lambda = 0$ if $\lambda \in \mathbb{F}$ is not an eigenvalue of ad_a. As $\mathrm{char}(\mathbb{F}) = 0$, the additive group of $\mathbb{F}$ is torsion free, so Proposition 4.31 applies to prove that L contains a nonzero Jordan element. If a is ad-nilpotent, then Kostrikin's descent lemma yields a nonzero Jordan element. □

4.6. Jordan elements in the Lie algebra of a ring

Let R be a semiprime ring. Following Sections 1.3 and 1.5, we denote by $\mathcal{C}(R)$ the extended centroid of R and by $\tilde{R} = \mathcal{C}(R)R$ its central closure. Sometimes we will need to work in the *unital central closure* $\hat{\tilde{R}} = \tilde{R} + \mathcal{C}(R)$, a subring of the symmetric Martindale ring of quotients $Q_s(R)$ of R. In this section we describe the Jordan elements of the Lie algebra R^- and prove a necessary and sufficient condition for the existence of minimal abelian inner ideals in the Lie algebra $\overline{R}' = [R,R]/([R,R] \cap Z(R))$.

THEOREM 4.33. [**BMM96**, Theorem 2.3.3] *Let R be a semiprime ring and let $a_1, a_2, \ldots, a_n \in R$. If $a_1 \notin \sum_{i=2}^n C(R)a_i$ in $\tilde{R}$, then there exist $r_j, s_j \in R$, $j = 1, 2, \ldots, m$, such that $\sum_{j=1}^m r_j a_1 s_j \neq 0$ and $\sum_{j=1}^m r_j a_k s_j = 0$, $k = 2, \ldots, n$.*

PROPOSITION 4.34. *A semiprime 6-torsion free ring R contains a nonzero nilpotent element if and only if the Lie algebra $\overline{R} = R/Z(R)$ (equivalently, $\overline{R}'$) contains a nonzero Jordan element.*

PROOF. Let $0 \neq a \in R$ be a nilpotent element, we may assume that $a^2 = 0$. Then, as seen in Example 4.2, a is a Jordan element of R^-. Moreover, since R is semiprime, a is not a central element, so $\bar{a}$ is a nonzero Jordan element of $\overline{R}$. Suppose conversely that $a \in R$ is such that $\bar{a}$ is a nonzero Jordan element of $\overline{R}$. Since R has no 3-torsion, $(\mathrm{ad}_a^2 R)^2 = 0$ by Proposition 3.35(1), with $\mathrm{ad}_a^2 R \neq 0$ since $\overline{R}$ is nondegenerate by Proposition 3.35(2). Note finally that $\overline{R}' \cong [\overline{R}, \overline{R}]$ is an essential ideal of $\overline{R}$ and hence, by Proposition 4.14, $\overline{R}$ contains nonzero Jordan elements if and only if so does $\overline{R}'$. □

We have just seen that if $\bar{a}$ is a Jordan element of the Lie algebra $\overline{R}$, then any element of the principal inner ideal $\mathrm{ad}_a^2 R$ of R^- has square 0 in R. It would be interesting to know whether a itself has square 0 modulo $Z(R)$. The answer to this question is affirmative if R coincides with its unital central closure, or if R is a simple ring.

THEOREM 4.35. *Let R be a semiprime 6-torsion free ring and let a be a Jordan element of the Lie algebra R^-. Then there exists $\lambda \in Z(\hat{\tilde{R}})$ such that $(a-\lambda)^2 = 0$. If R is a centrally closed prime ring, then $\lambda \in Z(R)$.*

PROOF. By Proposition 3.35(1), for $x, y \in R$ we have $(\mathrm{ad}_a^2 x)(\mathrm{ad}_a^2 y) = 0$, i.e.

$$\begin{aligned} 0 = (\mathrm{ad}_a^2 x)(\mathrm{ad}_a^2 y) &= a^2x(a^2y + ya^2 - 2aya) \\ &\quad + ax(-2a^3y - 2aya^2 + 4a^2ya) \\ &\quad + x(a^4y - 2a^3ya + a^2ya^2). \end{aligned} \tag{4.19}$$

We claim that $a^2 = \mu a + \tau$ for some $\mu, \tau \in \mathcal{C}(R)$. If this were not the case, by Theorem 4.33 applied to the elements $a^2, a, 1$ in $\hat{R}$, there would exist $r_j, s_j \in R$, $j = 1, \ldots, m$, such that $\sum_{j=1}^m r_j a^2 s_j \neq 0$ and $\sum_{j=1}^m r_j a^k s_j = 0$ for $k = 0, 1$. In (4.19), for each $j = 1, \ldots m$, replace x by $s_j x$ and multiply on the left by r_j. We obtain

$$0 = \sum_{j=1}^m r_j (a^2 s_j x(a^2 y + ya^2 - 2aya) + as_j x(-2a^3 y - 2aya^2 + 4a^2 ya)$$

$$+ s_j x(a^4 y - 2a^3 ya + a^2 ya^2)) = (\sum_{j=1}^m r_j a^2 s_j) x(a^2 y + ya^2 - 2aya)$$

which tells us that for each $y \in R$

$$a^2 y + ya^2 - 2aya \in \operatorname{Ann}(I), \tag{4.20}$$

where I denotes the ideal generated by $\sum_{j=1}^m r_j a^2 s_j$. For each $j = 1, \ldots m$, replace y by s_j in formula (4.20) and multiply on the left by r_j. Then, by semiprimeness of R, we obtain

$$\sum_{j=1}^m r_j a^2 s_j = \sum_{j=1}^m r_j (a^2 s_j + s_j a^2 - 2as_j a) \in I \cap \operatorname{Ann}(I) = 0,$$

which is a contradiction. Therefore there exist $\mu, \tau \in \mathcal{C}(R)$ such that

$$a^2 = \mu a + \tau \text{ and } a^3 = (\mu^2 + \tau)a + \mu\tau. \tag{4.21}$$

Substituting these expressions of a^2 and a^3 in the equation

$$0 = \operatorname{ad}_a^3 x = a^3 x - 3a^2 xa + 3axa^2 - xa^3,$$

we obtain

$$0 = (\mu^2 + 4\tau)ax - (\mu^2 + 4\tau)xa = [(\mu^2 + 4\tau)a, x]$$

for every $x \in R$, which proves that $(\mu^2 + 4\tau)a$ is a central element of $\tilde{R}$.

Set $\alpha := \mu^2 + 4\tau$. Since $\mathcal{C}(R)$ is von Neumann regular (Theorem 1.17), $\alpha = \alpha\beta\alpha$ for some $\beta \in \mathcal{C}(R)$, so $e_\alpha := \alpha\beta$ is an idempotent of $\mathcal{C}(R)$ and

$$e_\alpha a = \beta\alpha a = \beta(\mu^2 + 4\tau)a \in Z(\tilde{R}). \tag{4.22}$$

Denote by f_α the idempotent $1 - e_\alpha$ of $\mathcal{C}(R)$ and set $\lambda := e_\alpha a + \frac{1}{2} f_\alpha \mu$. Then $\lambda \in Z(\hat{\tilde{R}})$ and satisfies

$$(a - \lambda)^2 = (a - e_\alpha a - \frac{1}{2} f_\alpha \mu)^2 = (f_\alpha a - \frac{1}{2} f_\alpha \mu)^2 = f_\alpha (a^2 - \mu a + \frac{1}{4}\mu^2)$$

$$= f_\alpha(\tau + \frac{1}{4}\mu^2) = \frac{1}{4} f_\alpha \alpha = 0.$$

Suppose now that R is a centrally closed prime ring. Then we have by Lemma 1.22 that either $Z(R) = 0$ or R is unital and therefore $Z(R) = \mathcal{C}(R)$. But $(a - \lambda)^2 = 0$ implies that $\lambda^2 \in Z(R)$, so in both cases $\lambda \in Z(R)$. □

When referred to a simple ring, Proposition 4.34 adopts the following refined form [**Ben76**, Theorem 3.2].

LEMMA 4.36. *Let R be a simple ring and let $a \in R$ be ad-nilpotent of index n. Then n is odd and a is algebraic of degree $r = (n + 1)/2$. If the characteristic of R is 0 or $p > r$. Then $a = x + z$ where x is nilpotent of index r and $z \in Z(R)$.*

We are mainly interested in the following particular case of the above lemma.

LEMMA 4.37. *Let R be as in Lemma 4.36 and let $a \in R$ be such that $\operatorname{ad}_a^4 R = 0$. Then $a = x + z$ where $x^2 = 0$ and $z \in Z(R)$.*

REMARK 4.38. Lemma 4.36 was extended by Martindale and Miers [**MM89**, Corollary 1] to prime rings, and by Grzeszczuk [**Grz92**] to semiprime rings. The proof of Theorem 4.35 given above is taken from [**BGGL16**, Theorem 3.2].

4.7. Jordan elements in Lie algebras of skew-symmetric elements

Let R be a prime ring with involution $*$, extended centroid $\mathcal{C}(R)$, and central closure $\tilde{R}$. We describe in this section the Jordan elements of the Lie algebra $K = \operatorname{Skew}(R, *)$.

PROPOSITION 4.39. [**BGGL14**, Proposition 4.1(a)] *Let R be a 2-torsion free ring with involution $*$. If a is a Jordan element of K, then a^2 is a Jordan element of R^-.*

PROOF. Note first that

$$\operatorname{ad}_{a^2}^3 = (l_{a^2} - r_{a^2})^3 = (l_a^2 - r_a^2)^3 = (l_a + r_a)^3 (l_a - r_a)^3 = (l_a + r_a)^3 \operatorname{ad}_a^3 .$$

Let $x \in R$. Then $2x = x_k + x_s$, where

$$x_k = x - x^* \in \operatorname{Skew}(R, *) \text{ and } x_s = x + x^* \in \operatorname{Sym}(R, *).$$

Since $\operatorname{ad}_a^3(x_s)a = \operatorname{ad}_a^3(x_s a)$, $a \operatorname{ad}_a^3(x_s) = \operatorname{ad}_a^3(a x_s)$ and $x_s \circ a := x_s a + a x_s \in K$, we have

$$\begin{aligned}
2 \operatorname{ad}_{a^2}^3(x) = \operatorname{ad}_{a^2}^3(2x) &= \sum_{i=0}^{3} \binom{3}{i} a^i \operatorname{ad}_a^3(x_k + x_s) a^{3-i} = \sum_{i=0}^{3} \binom{3}{i} a^i \operatorname{ad}_a^3(x_k) a^{3-i} + \\
&+ \sum_{i=0}^{3} \binom{3}{i} a^i \operatorname{ad}_a^3(x_s) a^{3-i} = \sum_{i=0}^{3} \binom{3}{i} a^i \operatorname{ad}_a^3(x_s) a^{3-i} \\
&= \operatorname{ad}_a^3(x_s) a^3 + 3a \operatorname{ad}_a^3(x_s) a^2 + 3a^2 \operatorname{ad}_a^3(x_s) a + a^3 \operatorname{ad}_a^3(x_s) \\
&= \operatorname{ad}_a^3(x_s a + a x_s) a^2 + 2a \operatorname{ad}_a^3(x_s a + a x_s) a + a^2 \operatorname{ad}_a^3(x_s a + a x_s) = 0,
\end{aligned}$$

so $\operatorname{ad}_{a^2}^3(R) = 0$, since R is 2-torsion free. □

LEMMA 4.40. *Let R be a semiprime 2-torsion free ring, $x \in R$ and $\lambda \in \mathcal{C}(R)$. If $x - \lambda$ is nilpotent, then λ is uniquely determined. If R has an involution $*$ and x is symmetric (resp. skew-symmetric), then $\lambda^* = \lambda$ (resp. $\lambda^* = -\lambda$).*

PROOF. Let $\mu \in \mathcal{C}(R)$ be such that $x - \mu$ is nilpotent. As $[x - \lambda, x - \mu] = 0$, we have that $\lambda - \mu = (x - \mu) - (x - \lambda)$ is a nilpotent element of $\mathcal{C}(R)$. Hence $\lambda = \mu$, since $\mathcal{C}(R)$ has no nonzero nilpotent elements because R is semiprime.

Suppose now that R has an involution $*$ and that $x = x^*$. Then $x - \lambda^* = (x - \lambda)^*$ is nilpotent. Hence $\lambda^* = \lambda$ by the uniqueness we have just proved. For $x = -x^*$ the proof is similar. □

Lemma 4.41. [**BFLGL16**, Propostion 6.1] *Let R be a prime ring of characteristic not 2 with involution $*$ of the first kind, and let $\langle K\rangle$ be the subring of R generated by K.*

(i) *If K is not abelian, then $\langle K\rangle$ is a prime ring and $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$.*
(ii) *If K is abelian, then $K = 0$ or $\overline{\mathcal{C}(R)} \otimes_{\mathcal{C}(R)} \tilde{R} = \mathrm{M}_2(\overline{\mathcal{C}(R)})$, where $\overline{\mathcal{C}(R)}$ denotes the algebraic closure of the field $\mathcal{C}(R)$.*

Proof. (i) Taking $U = K$ in [**BMM96**, Theorem 9.1.13(d)], we have that $[K,K] \neq 0$ implies $[K,K]^2 \neq 0$. Let I be the ideal of R generated by $[K,K]^2$. By [**BMM96**, Lemma 9.1.4], $0 \neq I \subset \langle K\rangle$ and hence it follows from [**Lam99**, Theorem 14.14 and subsequent Remark] that $\langle K\rangle$ is prime with $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$.

(ii) Take $U = K$ in [**BMM96**, Theorem 9.1.13(a)]. □

Lemma 4.42. [**BFLGL16**, Propostion 6.2] *Let R be a prime ring of characteristic 0 or $p > 5$ with involution $*$ of the first kind such that $[K,K] \neq 0$. If a is a Jordan element of K, then $a^3 = 0$.*

Proof. By 4.41(i), $\langle K\rangle$ is a prime ring with $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$. Since $\langle K\rangle$ is spanned by the elements of K and their squares [**BMM96**, Lemma 9.1.5], we get (using the Leibniz rule) that $\mathrm{ad}_a^3 K = 0$ implies $\mathrm{ad}_a^5\langle K\rangle = 0$. Now it follows from [**MM83**, Corollary 1] that $(a-\alpha)^3 = 0$ for some $\alpha \in \mathcal{C}(R)$ (the formula making sense in the unital hull of $\langle K\rangle$). Since the involution $*$ is of the first kind, $\alpha = 0$ by Lemma 4.40. Thus $a^3 = 0$. □

Theorem 4.43. [**BGGL14**, Corollary 4.3] *Let R be a centrally closed prime ring of characteristic not 2 or 3 with involution $*$, and let a be a nonzero Jordan element of the Lie algebra K. Then one of the following possibilities holds:*

(i) *There exists $z \in \mathrm{Skew}(Z(R), *)$ such that $(a-z)^2 = 0$.*
(ii) *$a \in Z(K)$ and $a \notin Z(R)$.*
(iii) *$a^3 = 0$ and $a^2 \neq 0$.*

Proof. Suppose first that $*$ is of the second kind and let ξ be a nonzero skew-symmetric element in $\Gamma(R)$, which is a field since R is a centrally closed prime ring. Then $R = K \oplus \xi K$ and hence a is a Jordan element of the Lie algebra R^-. Now it follows from Theorem 4.35 that there exist $z \in Z(R)$ such that $(a-z)^2 = 0$, with $z^* = -z$ by Lemma 4.40, so we are in case (i).

Suppose now that $*$ is of the first kind. Then it follows from Lemma 4.42 that either K is abelian or $a^3 = 0$. If the first, $a \in Z(K)$, with $a \notin Z(R)$ since $*$ is of the first kind and a is nonzero by hypothesis, so we are in case (ii); if the second, we may suppose that $a^2 \neq 0$, since otherwise we would be in case (ii). □

Remarks 4.44. (1) Theorem 4.43 appears in [**BGGL16**, Corollary 4.3] as a corollary of a result for centrally closed semiprime 6-torsion free rings with involution. The proof given here is an adaption of that of [**BFLGL16**, Theorem 6.3].

(2) Jordan elements of type (ii) occur when R is the ring $\mathrm{M}_2(\mathbb{F})$ with transpose involution. As will be seen in Chapter 8, case (iii) takes place when the inner ideal generated by a is Clifford (see Exercise 2.99(3)).

Jordan elements of K and $[K,K]$ when R is simple. Recall that by Lemma 1.13, if R is a simple ring, then either R is unital (and hence $Z(R) \cong \Gamma(R)$), or $Z(R) = 0$.

LEMMA 4.45. [**Ben76**, Lemma 4.23] *Let R be a simple ring of characteristic not 2 or 3 with involution of the first kind. Assume that either $Z(R) = 0$ or $\dim_{Z(R)} R > 16$, and let $a \in K$ be such that $\mathrm{ad}_a^3[K,K] = 0$. Then $a^3 = 0$ and $a^2Ka^2 = 0$.*

PROOF. From $\mathrm{ad}_a^3[K,K] = 0$ we get $\mathrm{ad}_a^4 K = 0$. By [**Her69**, Theorem 2.3], $R = \langle K \rangle$ and hence, as in the proof of Lemma 4.42, we get $\mathrm{ad}_a^7 R = 0$. Now it follows from [**Ben76**, Theorem 3.2] that $(a-z)^4 = 0$ for some $z \in Z(R)$. Since a is skew-symmetric and the involution is of the first kind, we have by Lemma 4.40 that $z = 0$.

Now let $k \in [K,K]$. Then $0 = \mathrm{ad}_a^3 k = a^3k - 3a^2kz + 3aka^2 - ka^3$ implies that $a^3[K,K] \subset Ra$. Therefore $a^3[K,K]a^3 \subset Ra^4 = 0$. But $a^3[K,K]a^3 = \mathrm{ad}_{a^3}^2[K,K]$ since $(a^3)^2 = 0$, which implies by [**Ben76**, Corollary 2.12] that $a^3 = 0$. Hence, for any $k \in K$, we have $0 = \mathrm{ad}_a^4 k = 6a^2Ka^2$, which implies $a^2Ka^2 = 0$. □

COROLLARY 4.46. *Let R be as in the previous lemma and let a be a Jordan element of the Lie algebra K. Then $a^3 = 0$ and $a^2Ka^2 = 0$.*

PROOF. $\mathrm{ad}_a^3[K,K] \subset \mathrm{ad}_a^3 K = 0$. Now Lemma 4.45 applies. □

THEOREM 4.47. *Let R be a simple ring of characteristic 0 or $p > 3$ with involution $*$. Assume that either $Z(R) = 0$ or $\dim_{Z(R)} R > 16$. If a is a Jordan element of K, then one of the following holds:*

(i) $(a-z)^2 = 0$ *for some* $z \in \mathrm{Skew}(Z(R),*)$,
(ii) $a^3 = 0$, $a^2Ka^2 = 0$, *and* $a^2 \neq 0$.

The same is true if a is a Jordan element of $[K,K]$.

PROOF. Suppose first that the involution is of the second kind. Then it follows as in the proof of Theorem 4.43 that a is as in (i). If the involution is of the first kind, we have by Lemma 4.45 that $a^3 = 0$ and $a^2Ka^2 = 0$, since a is a Jordan element of K. If $a^2 = 0$ we are in (i); otherwise we are in (ii).

Now let a be a Jordan element of $[K,K]$. Suppose first that the involution is of the second kind and let ξ be a nonzero skew-symmetric element of $\Gamma(R)$. Then $R = K \oplus \xi K$. From $\mathrm{ad}_a^3[K,K] = 0$ it follows $\mathrm{ad}_a^4 K = 0$ and hence $\mathrm{ad}_a^4 R = 0$. Now it follows from [**Ben76**, Theorem 3.2] that $\mathrm{ad}_a^3 R = 0$ and there exists $z \in Z(R)$ such that $(a-z)^2 = 0$, with $z^* = -z$ by Lemma 4.40. If the involution is of the first kind, then we have by Lemma 4.45 that $a^3 = 0$ and $a^2Ka^2 = 0$. This completes the proof of the theorem. □

4.8. Exercises

EXERCISE 4.48. Let $R = \mathrm{M}_n(\mathbb{F})$, where $\mathbb{F}$ is a field. Show that every nilpotent element $a \in R$ can be expressed as a sum of Jordan elements of the Lie algebra R'.

EXERCISE 4.49. Following Section 2.2 for notation, let X be a vector space over a field $\mathbb{F}$ of characteristic not 2, $\dim_{\mathbb{F}} X > 2$, which is endowed with a nondegenerate symmetric bilinear form $\langle\,,\rangle$, and let x, z be nonzero orthogonal vectors in X such that x is isotropic. Show that $a = x^*z - z^*x$ is a Jordan element of the Lie algebra $\mathfrak{o}(X, \langle\,,\rangle)$. Note that $a^2 = 0$ if and only if z is isotropic.

EXERCISE 4.50. Let L be a 6-torsion free Lie algebra. Show that L is strongly prime if and only if for any $x \in L$ and any $I \lhd L$, $\mathrm{ad}_x^2 I = 0 \Rightarrow x = 0$ or $I = 0$.

EXERCISE 4.51. Let L be a simple Lie algebra over a field of characteristic 0 which is generated by ad-nilpotent elements. Show that L is spanned by Jordan elements.

EXERCISE 4.52. (Brešar) Let A be a 3-torsion free nonassociative Φ-algebra and let d be a derivation of A such that $d^3 = 0$. Show that the Φ-submodule $D = d^2(A)$ satisfies $D^2 = 0$ and $(DA)D \subset D$.

EXERCISE 4.53. Let $x, a \in L$ and let m be a natural number. Show that if m is even, then $[X, \mathbf{ad}_A^m X] = \sum_{i+j>0} \beta_{ij} A^i X A^{m-i-j} X A^j$. So the argument used in Lemma 4.22 for $n = p - 1$ doesn't work in the general case.

EXERCISE 4.54. Carry out all computations in Lemma 4.25 for the cases: (i) $n = 5,\ m = 3$, and (ii) $n = 5,\ m = 2$.

EXERCISE 4.55. Carry out all computations in Lemma 4.26 for $n = 5$.

EXERCISE 4.56. Give and *easier* proof of Lemma 4.24 assuming that L is 30-torsion free.

EXERCISE 4.57. Let L be a Lie algebra over a ring of scalar Φ, $30 \in \Phi^*$, and let $a \in L$ be such that $\mathrm{ad}_a^4 = 0$. Give a direct proof that $\mathrm{ad}_a^3 L$ is an abelian inner ideal.

EXERCISE 4.58. Show that any nonzero finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 contains nonzero Jordan elements.

EXERCISE 4.59. Suppose that R is a 2-torsion free ring with involution $*$, $\lambda \in \mathrm{Skew}(Z(R), *)$, and let a be a Jordan element of K. Show that λa is a Jordan element of the Lie algebra R^-.

CHAPTER 8

From Lie Algebras to Jordan Algebras

Introduced by Meyberg, local algebras of Jordan systems have played an important role in the structure of prime nondegenerate Jordan pairs. A similar construction works for Lie algebras, but unlike the Jordan case, only to Jordan elements of a Lie algebra we can attach a Jordan algebra. This Jordan algebra has a behavior similar to that of the local algebra of a Jordan system at an element. Thus many properties can be transferred from the Lie algebra to its Jordan algebras, and in addition the nature of the Jordan element in question is reflected in the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of Jordan theory to Lie algebras. The material of this chapter is organized as follows. Section 8.1 is a brief survey on linear Jordan algebras, i.e. Jordan algebras over a ring of scalars Φ containing $1/2$. We include definitions and basic results on Jordan theory with the purpose of helping the reader to go through them when required. Section 8.2 is the core of the chapter, and maybe the core of the book, since it is in this section where a Lie–Jordan connection is introduced by associating with a Jordan element a of a Lie algebra L (over a ring of scalars in which 6 is invertible) a Jordan algebra L_a. Most properties of the Lie algebra are inherited by its Jordan algebras (we do not know if the simplicity is inherited), and some of the notable elements studied in the previous chapters, as von Neumann regular or extremal elements, are characterized in terms of their associated Jordan algebra. As applications of this Lie–Jordan connection, we give in Section 8.3 a classification-free proof of the fact that every simple finitary Lie algebra over an algebraically closed field of characteristic 0 is spanned by its extremal elements; in Section 8.4 we compute the Jordan algebra at a Clifford element; in Section 8.5 we give the proof of Zelmanov's solution of the Kurosh–Lie problem for the particular case of a Lie algebra over a field of characteristic 0 (in its general version for Lie algebras over an arbitrary field, this Zelmanov's result has important implications in group theory, significatively extending the positive solution of the Restricted Burnside Problem and his work on compact torsion groups); finally, in Section 8.6, we outline a proof of a theorem due to C. Martínez and E. Zelmanov on nil Lie algebras of finite width.

8.1. Linear Jordan algebras

Throughout this section, Φ is a ring of scalars in which 2 is invertible. For basic results and terminology of linear Jordan algebras the reader is referred to [**Jac68**, **McC04**, **ZSSS82**].

Definition and examples.

Definition 8.1. A (linear) *Jordan algebra* is a Φ-algebra J whose product, denoted by $\bullet$, satisfies the following conditions:

(J1) $x \bullet y = y \bullet x$,

(J2) $x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$,

for all $x, y \in J$, where $x^2 = x \bullet x$.

In operator form, the defining relations (J1) and (J2) give $[l_x, l_{x^2}] = 0$, which is in turn yields the identity (see [**Jac68**, I.7(54)]):

$$[[l_a, l_b], l_c] = l_{[l_a,l_b]c}, \ a, b, c \in J, \tag{8.1}$$

proving that the linear map $[l_a, l_b]$ is a derivation of the Jordan algebra J.

– For each $x \in J$, the linear map $U_x : J \to J$, defined by $U_x = 2l_x^2 - l_{x^2}$, satisfies the *fundamental Jordan identity*:

(J3) $U_{U_x y} = U_x U_y U_x$, for all $x, y \in J$.

– Triple product is defined by

$$V(x, y)z = V_{x,y}z = \{x, y, z\} := U_{x,z}y := U_{x+z}y - U_x y - U_z y.$$

EXAMPLE 8.2. Let R be an associative Φ-algebra. Then R with the new product defined by $x \bullet y = \frac{1}{2}(xy + yx)$ becomes a Jordan algebra R^+, with $U_x z = xzx$, $\{x, y, z\} = xyz + zyx$. If R has an involution $*$, then $\mathrm{Sym}(R, *) = \{x \in R : x = x^*\}$ is a subalgebra of the Jordan algebra R^+.

– By [**Her69**, Theorem 17], if R is simple then the Jordan algebra R^+ is simple, and the same is for the Jordan algebra $\mathrm{Sym}(R, *)$ if R is simple with involution by [**Her69**, Theorem 21].

DEFINITION 8.3. A Jordan algebra is said to be *special* if it is isomorphic to a subalgebra of R^+ for an associative algebra R. Non-special Jordan algebra are called *exceptional*.

Jordan algebras of the form $\mathrm{Sym}(R, *)$ are special. An example of exceptional Jordan algebras is the *split Albert algebra*, introduced in Section 1.6, $\mathrm{Her}_3(\mathcal{C})$ of Hermitian 3×3 matrices on the split Cayley algebra $\mathcal{C}$ (over a field $\mathbb{F}$) with standard involution (see [**Jac68**, I.5.Theorem 4]). E. Zelmanov proved in [**Zel79b**] that under a scalar extension every simple exceptional Jordan algebra is a split Albert algebra.

It could be said that exceptionality is a triune phenomenon in the heavenly kingdom of the nonassociative algebras. Starting with the split Cayley Lie algebra $\mathcal{C}$, [1] we get, as its algebra of derivations, the split simple Lie algebra of type G_2, and by means of Hermitian 3×3-matrices, the split exceptional Jordan algebra $\mathrm{Her}_3(\mathcal{C})$. Then $\mathrm{Der}(\mathrm{Her}_3(\mathcal{C}))$ is the split simple Lie algebra of type F_4, and we can still enlarge $\mathrm{Der}(\mathrm{Her}_3(\mathcal{C}))$ by addition of the multiplication operators l_X, $X \in \mathrm{Her}_3(\mathcal{C})$, to get, as a subalgebra of $\mathfrak{gl}(\mathrm{Her}_3(\mathcal{C}))$, the split simple Lie algebra of type E_6. For details of these miraculous constructions the reader is referred to [**Jac79**].

EXAMPLE 8.4. Let X be a vector space over a field $\mathbb{F}$ of characteristic not 2 which is equipped with a symmetric bilinear form $\langle\, , \rangle$. Then the vector space $\mathbb{F} \oplus X$ becomes a Jordan algebra, denoted by $J(X, \langle\, , \rangle)$, with the product defined by

$$(\alpha, x) \bullet (\beta, y) = (\alpha\beta + \langle x, y\rangle, \alpha y + \beta x)$$

for $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$. This Jordan algebra is special; in fact, it is isomorphic

[1] Under a scalar extension, any simple alternative algebra that is not associative is a split Cayley algebra (Kleinfeld's theorem) [**ZSSS82**, 7.3 Corollary 1]

to a Jordan subalgebra of the Clifford (associative) algebra defined by the bilinear form $\langle\, ,\rangle$ [**Jac68**, II.3.Example 4]. For this reason, $J(X, \langle\, ,\rangle)$ is sometimes called a *Clifford Jordan algebra.*

Note that if R is an associative Φ-algebra, then R is unital if and only if R^+ is unital (with the same unit element). Note also that the Jordan algebra $J(X, \langle\, ,\rangle)$ in Example 8.4 is unital with $(1, 0)$ as unit element, and that it is simple if the symmetric bilinear form $\langle\, ,\rangle$ is nondegenerate and $\dim_{\mathbb{F}} X > 2$.

As a corollary of his astonishing structure theorem for prime nondegenerate Jordan algebras [**Zel83b**] (*The Russian Revolution of Jordan Algebras* in McCrimmon's words), E. Zelmanov proved that the four types of Jordan algebras given in the above examples describe all the simple Jordan algebras.

Homotopes and Local algebras of Jordan algebras. Let J be a Jordan algebra and let $a \in J$. In the Φ-module J a new product $\bullet_a$ is defined by setting $x \bullet_a y = \frac{1}{2}\{x, a, y\}$ for all $x, y \in J$. The resulting algebra is also a Jordan algebra denoted by $J^{(a)}$ and called the *a-homotope* of J at a. Then

$$\mathrm{Ker}_J\{a\} = \{z \in J : U_a z = 0\}$$

is an ideal of $J^{(a)}$, and the algebra $J_a = J^{(a)} / \mathrm{Ker}_J\{a\}$ is called the *local algebra* of J at a. (See [**Mey72**] or [**DM95**]).

As will be seen in the next section, a similar process works for any Lie algebra, but with the restriction that the element is required to be Jordan. In Exercise 8.106, we illustrate this process of localization in the case of an associative algebra.

Macdonald Principle. [**McC04**, II.5.1.2] *Any Jordan polynomial in three variables which has degree* ≤ 1 *in one variable and vanishes in all special Jordan algebras necessarily vanishes in all Jordan algebras, i.e. is an identity for Jordan algebras.*

Shirshov–Cohn Principle. [**McC04**, II.5.1.2] *Let J be a unital Jordan algebra* (*we may always suppose that J is unital by taking its unital hull $\hat{J}$*). *To verify that certain relations between elements x, y in J always imply certain other relations among the elements $f_1(x, y), \ldots, f_n(x, y)$ in the unital subalgebra generated by x, y, it is sufficient to establish the implication in a Jordan algebra* $\mathrm{Sym}(R, *)$, *where R is a unital associative algebra with involution.*

Power associativity, algebraic and nilpotent elements. An algebra A over an arbitrary ring of scalars Φ is called *power-associative* if the subalgebra $\langle x\rangle$ generated by an arbitrary element $x \in A$ is associative. Using Macdonald's Principle it is easily verified that Jordan algebras are power associative. Thus the notion of nilpotency makes sense for elements of a Jordan algebra J: an element $a \in J$ is nilpotent if $a^n = 0$ for some $n \geq 1$. If any $x \in J$ is nilpotent, then J is called a *nil Jordan algebra.* The same is true for the notion of algebraic element: an element a in a Jordan algebra J over a field $\mathbb{F}$ is said to be *algebraic* if it is a root of a nonzero polynomial in $\mathbb{F}[\xi]$, equivalently, the subalgebra of J generated by a is finite-dimensional. In this case, $\deg(a) = \dim_{\mathbb{F}} \mathbb{F}[a]$ is the *degree* of a, where $\mathbb{F}[a]$ denotes the *unital* subalgebra of $\hat{J}$ generated by a. Then J is said to be *algebraic* if every $x \in J$ is algebraic, and *algebraic of bounded degree* if it is algebraic and there exists a positive integer n such that $\deg(x) \leq n$ for all $x \in J$.

Idempotents and Peirce decompositions in Jordan algebras. An element e of a Jordan algebra J is called an *idempotent* if $e^2 = e$.

PROPOSITION 8.5. [**McC04**, II.5.2.4] *If e is an idempotent of J with $U_e = 1_J$, then J is unital with e as unit element.*

PROOF. Taking $x = e$ in Exercise 8.107(i), we obtain

$$l_e = l_e 1_J = l_e U_e = \frac{1}{2} U_{e,e} = U_e = 1_J,$$

which proves that e is the unit element of J. □

PROPOSITION 8.6. [**McC04**, II.8.1.2-4] *Any idempotent e of a Jordan algebra J yields the* Peirce decomposition

$$J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e),$$

where $J_i(e) = \{x \in J : e \bullet x = ix\}$, for $i = 1, \frac{1}{2}, 0$. The supplementary projections relative to this decomposition are given by

$$E_1 = U_e, \quad E_{\frac{1}{2}} = U_{e,1-e} = 2l_e - 2U_e, \quad E_0 = U_{1-e} = 1_J - 2l_e + U_e,$$

with $1 - e$ in the unital hull $\hat{J}$ of J.

PROOF. Use Exercise 8.108 to prove that U_e and U_{1-e} are orthogonal projections on J. Hence $U_{e,1-e} = 1_J - U_e - U_{1-e}$ is also a projection. Then $l_e U_e = U_e l_e = U_e$ (identity obtained taking $x = e$ in Exercise 8.107(i)) proves that $U_e J = J_1(e)$, and similarly, $U_{1-e} J = J_0(e)$. That $U_{e,1-e} J = J_{\frac{1}{2}}(e)$ is now easily verified. □

An alternative proof. Replacing U_e by $2l_e^2 - l_e$ in the identity $l_e U_e = U_e$ we get that l_e vanishes the polynomial $(\xi - 1)(\xi - \frac{1}{2})\xi \in \Phi[\xi]$. Since the ideals $(\xi - 1)$, $(\xi - \frac{1}{2})$, (ξ) of $\Phi[\xi]$ are comaximal (because $\frac{1}{2} \in \Phi$), a standard application of the Chinese remainder theorem yields the decomposition $J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e)$.

EXAMPLE 8.7. Let e be an idempotent of an associative algebra R. Then e is an idempotent of the Jordan algebra $J = R^+$ and

$$J_1(e) = eRe, \; J_{\frac{1}{2}}(e) = eR(1-e) + (1-e)Re, \text{ and } J_0(e) = (1-e)R(1-e).$$

DEFINITION 8.8. Let J be a Jordan algebra. An element $z \in J$ is said to be an *absolute zero divisor* if $U_z J = 0$, and J is said to be *nondegenerate* if it has no nonzero absolute zero divisors. Note that R^+ is nondegenerate if and only if R is semiprime.

LEMMA 8.9. [**McC04**, II.8.10.2]. *Let e be an idempotent of a nondegenerate Jordan algebra J. If $J_0(e) = 0$, then J is unital with e as unit element.*

LEMMA 8.10. [**Jac68**, III.7.Lemma 1]. *Let J be a Jordan algebra and let x be a non-nilpotent algebraic element of J. Then the subalgebra $\langle x \rangle$ of J generated by x contains a nonzero idempotent.*

Recall that an algebra A is said to be nilpotent if there exists a positive integer such that any product (in any association) of n elements in A vanishes.

THEOREM 8.11. (Albert) *Any finite-dimensional Jordan nil algebra is nilpotent.*

PROOF. It follows from [**Jac68**, V.2. Corollary 1 and V.3.Theorem 3]. □

Corollary 8.12. *Any finite-dimensional Jordan algebra J which is not nilpotent contains a nonzero idempotent.*

Proof. By Albert's theorem J contains an element which is algebraic but not nilpotent. So J contains a nonzero idempotent by Lemma 8.10. $\square$

Von Neumann regularity in Jordan algebras. Recall that an element a of an associative algebra R is von Neumann regular if there exists $b \in R$ such that $a = aba$. This associative notion of von Neumann regularity has a natural extension to Jordan algebras.

Definition 8.13. Let J be a Jordan algebra. An element $a \in J$ is said to be *von Neumann regular* if there exists $b \in J$ such that $a = U_a b$.

If $J = R^+$, then both notions of von Neumann regularity, the associative and the Jordan, agree. It is well known that if $a \in R$ is von Neumann regular, a=axa, then by replacing x by $b = xax$, we get $a = aba$ and $b = bab$. The same is true for elements of a Jordan algebra.

Lemma 8.14. *Let $a \in J$ be von Neumann regular. Then there exists $b \in J$ such that $U_a b = a$ and $U_b a = b$.*

Proof. Let $a = U_a x$. Taking $b = U_x a$ we have:

$$U_a b = U_a U_x a = U_a U_x U_a x = U_{U_a x} x = U_a x = a \quad \text{and}$$
$$U_b a = U_{U_x a} a = U_x U_a U_x U_a x = U_x U_{U_a x} x = U_x U_a x = U_x a = b.$$

$\square$

It is also well known that every von Neumann regular element $a = aba$ in an associative algebra R gives rise to two idempotents: $e = ab$ and $f = ba$. A similar fact does not hold in general for von Neumann regular elements of a Jordan algebra, although, as will be seen in Chapter 11, there is a natural extension of this result for Jordan pairs. Yet, in spite of what we have just said, it is still possible in some occasions to produce idempotents in Jordan algebras by means of von Neumann regular elements.

Lemma 8.15. (E. García Rus) *Let $x \in J$. If x^2 is von Neumann regular and nonzero, then there exists an element $y \in J$ such that $U_x y$ is a nonzero idempotent of J.*

Proof. Since x^2 is von Neumann regular, we have by Lemma 8.14 that there exists $y \in J$ such that $x^2 = U_{x^2} y$ and $y = U_y x^2$. Taking $e := U_x y$ and using Macdonald Principle, we get

$$(U_x y)^2 = U_x U_y x^2 = U_x y,$$

with $U_x e = U_x(U_x y) = U_{x^2} y = x^2 \neq 0$, which proves that e is a nonzero idempotent.
. $\square$

Division Jordan algebras and inner ideals. Let J be a unital Jordan algebra with 1 denoting its unit element. An element $x \in J$ is called *invertible* if there exists $y \in J$ satisfying:

(Jinv) $x \bullet y = 1$ and $x^2 \bullet y = x$.

In this case, U_x is invertible and the *inverse* of x, denoted by x^{-1}, is uniquely determined: $x^{-1} = U_x^{-1} x$. (See [**McC04**, II.6.1].)

DEFINITION 8.16. A unital Jordan algebra in which every nonzero element is invertible is called a *division Jordan algebra*.

E. Zelmanov proved in [**Zel79a**] that the division Jordan algebras are precisely those of classical type:

- R^+ where R is a division associative algebra.
- $\mathrm{Sym}(R,*)$, where R is a division algebra with involution.
- $J(X,\langle\,,\rangle)$ where $\langle x,x\rangle$ is not a square in $\mathbb{F}$ for any $0\neq x\in X$.
- A form of a split Albert algebra defined by an anisotropic cubic form (see [**McC04**, I.2.3.8] for definition).

Let J be a unital Jordan algebra and let $x\in J$ be an algebraic element. Then (see [**Jac68**, page 54]), x is invertible if and only if its minimum polynomial is irreducible. Hence we easily obtain.

PROPOSITION 8.17. *Any algebraic division Jordan algebra over an algebraicaly closed field is one-dimensional.*

DEFINITION 8.18. A Φ-submodule B of a Jordan algebra J is called an *inner ideal* of J if $U_xJ\subset B$ for all $x\in B$. If actually $U_x\hat{J}\subset B$, then B is called a *strict inner ideal*.

PROPOSITION 8.19. *For any element a of a Jordan algebra J the Φ-submodule U_aJ a strict inner ideal of J, called the* principal inner ideal generated by *a.*

PROOF. It follows from (J3) and the Macdonald Principle $(U_xy)^2=U_xU_yx^2$. □

By [**McC04**, II.18.14], if a Jordan algebra J contains an element b with surjective U-operator, $U_bJ=J$, then J is unital and b is invertible. Hence it follows that J is a division Jordan algebra if it is nondegenerate and has no proper inner ideals. We give here another proof of this result.

PROPOSITION 8.20. *A Jordan algebra. is a division Jordan algebra if and only if it is nondegenerate and has no proper inner ideals.*

PROOF. Clearly, a division Jordan algebra satisfies the above conditions. Suppose then that J is a nondegenerate algebra and has no proper inner ideals. Then for any $0\neq x\in J$, $U_xJ=J$ and $U_{x^2}J=U_x^2J=J$, so $x^2\neq 0$. Since x^2 is von Neumann regular, we have by Lemma 8.15 that there exists $y\in J$ such that $e:=U_xy$ is a nonzero idempotent in J. Now $U_eJ=J$ implies $U_e=1_J$, so e is the unit element of J by Propostion 8.5. Let us now see that every nonzero element $x\in J$ is invertible. Let $y\in J$ be such that $U_xy=e$. By (J3), $1_J=U_eJ=U_{U_xy}=U_xU_yU_x$ which implies that U_x is invertible in $\mathrm{End}_\Phi(J)$. Now let $b\in J$ be such that $U_xb=x$. Again by (J3), $U_xU_bU_x=U_x$ and hence $U_xU_b=U_bU_x$, which proves that $U_x^{-1}=U_b$. We will show that b is the inverse of x by verifying equations (Jinv). Use Macdonald Principle to check the identity $U_x(x\bullet y)=x\bullet U_xy$ and evaluate it in $y=b$. We get $U_x(x\bullet b)=x\bullet U_xb=x^2=U_xe$, which implies $x\bullet b=e$ since U_x is invertible. Similarly we get $U_b(x^2\bullet b)=b\bullet U_bx^2=b\bullet U_x^{-1}U_xe=b\bullet e=b$, which implies $x^2\bullet b=U_xb=x$, so showing that b is the inverse of x. □

Isotopy. Suppose that L is unital and $u\in J$ invertible. Then the u-homotope $J^{(u)}$ is called the *u-isotope*. Note that $J^{(u)}=J_u$, the local of J at u, and that $J^{(u)}$ is unital with $1_{J^{(u)}}=u^{-1}$. While in the associative case, isotopy yields isomorphism, in the Jordan case this is not true in general. Nevertheless, as will be seen in Chapter 11, isotopy in Jordan algebras is connected with isomorphism in Jordan pairs.

Annihilators. The notion of annihilator $\mathrm{ann}_J(X)$ of a subset X of a Jordan algebra J, introduced by E. Zelmanov in [**Zm78**], has played a fundamental role in the structure theory of Jordan algebras. As will be shown now, $\mathrm{ann}_J(X)$ is a strict inner ideal of J and coincides with $\mathrm{Ann}(I)$ when applied to an ideal I of J.

DEFINITION 8.21. Given $X \subset J$, the annihilator of X in J is defined as the set $\mathrm{ann}_J(X) = \{a \in J : \{a, X, \hat{J}\} = 0\}$.

THEOREM 8.22. (Zelmanov) $\mathrm{ann}_J(X)$ *is a strict inner ideal of J. If X is an ideal I, then* $\mathrm{ann}_J(I) = \mathrm{Ann}_J(I)$ *and therefore an ideal of J.*

PROOF. Since the intersection of strict inner ideals is a strict inner ideal, we may assume that X is a single set $\{x\}$. Given $a \in \mathrm{ann}_J(x)$ and $b \in \hat{J}$, we must prove that $\{U_a b, x, \hat{J}\} = 0$. Using Macdonald Principle we get the identities

$$\{y, x, z\} + \{x, y, b\} = 4(y \bullet x) \bullet z, \tag{8.2}$$

$$V(U_y a, a) = V(a, U_a y). \tag{8.3}$$

Replacing y by $x + \lambda b$ in (8.3), we get by linearization

$$V(\{x, a, b\}, x) = V(x, U_a b) + V(b, U_a x). \tag{8.4}$$

Since $a \in \mathrm{ann}_J(x)$, $U_a x = 0$ and also $\{x, a, b\} = 0$ (applying 8.2), so $\{x, U_a b, \hat{J}\} = 0$, which again by (8.2) implies $\{U_a b, x, \hat{J}\} = 0$, as required.

Let now I be an ideal of J. Interchanging the rolles of a, b in (8.4) and using (8.29), we get $V(U_b a, x_1)\hat{J} = 0$ for any $b \in \hat{J}$, which proves that $\mathrm{ann}_J(I)$ is an ideal annihilating I, so equal to $\mathrm{Ann}_J(I)$. □

s-identities and Jordan PI-algebras. A *Jordan polynomial* $p(x_1, \ldots, x_n)$ of the free Jordan Φ-algebra $FJ(X)$ [**Jac68**, I.9] is said to be an *s-identity* if (i) it is *admissible*, i.e. the coefficient of one of the terms of higher degree is 1, and (ii) it is satisfied by all special Jordan algebras but not by all Jordan algebras. A Jordan algebra J satisfying a polynomial identity which is not an s-identity is called a *Jordan PI-algebra.* Given $x_1, x_2, \ldots, x_n \in J$, we put

$$x_1 \bullet x_2 \cdots \bullet x_n := x_1 \bullet (x_2 \bullet \cdots \bullet x_n).$$

For $n > 1$, let $\mathfrak{S}_n$ denote the group of permutations of $1, \ldots, n$.

PROPOSITION 8.23. *A nonzero Jordan polynomial of the form*

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma \in \mathfrak{S}_n} \alpha_\sigma x_{\sigma(1)} \bullet \cdots \bullet x_{\sigma(n)} \bullet x_{n+1} \quad (\alpha_\sigma \in \Phi)$$

is never an s-identity.

PROOF. By relabeling the variables we may assume that $\alpha_\sigma = 1$ for $\sigma = 1$. Let $Y = \{y_1, y_2, \ldots\}$ be a countable set. Denote by S the free semigroup generated by $Y \cup \{0\}$ satisfying the relations $y_i y_j = 0\,(j \neq i+1)$ and $y_i 0 = 0 y_i = 0\,(i \geq 1)$. Let R be the associative algebra defined by taking $S - \{0\}$ as a basis. It is easy to verify that $2^n p(y_1, \ldots, y_n, y_{n+1}) = y_1 \bullet \cdots \bullet y_n \bullet y_{n+1} \neq 0$. Thus the special Jordan algebra R^+ does not satisfies the identity $p(x_1, \ldots, x_n, x_{n+1}) = 0$, and therefore $p(x_1, \ldots, x_n, x_{n+1})$ is not an s-identity. □

LEMMA 8.24. [**ACGGL05**, 1.9] *Let J be algebraic of bounded degree. Then J is a Jordan PI-algebra.*

PROOF. Suppose that every element of J is algebraic of degree less than or equal to a fixed number n. Then J satisfies the admissible Jordan polynomial

$$p(x, y, z) := \mathcal{A}_{n+1}(V_{x^n,y}, \cdots, V_{x,y}, V_{1,y})z$$

for the alternating standard identity

$$\mathcal{A}_{n+1}(x_1, \ldots, x_n, x_{n+1}) := \sum_{\pi \in \mathfrak{S}_{n+1}} sg(\pi) x_{\pi(1)} \cdots x_{\pi(n)} x_{\pi(n+1)},$$

which proves that J is PI. □

The following result, due to E. Zelmanov, is the Jordan analog of Posner's theorem for prime associative PI-algebras [**Coh03**, Theorem 8.6.6].

THEOREM 8.25. [**Zel83b**, Theorem 1] *Any strongly prime Jordan PI-algebra has nonzero (associative) center $Z = Z(J)$ and the Jordan algebra of fractions $Z^{-1}J$ is either a simple finite-dimensional Jordan algebra over the field $Z^{-1}Z$, or the Jordan algebra of a nondegenerate symmetric bilinear form over $Z^{-1}Z$.*

REMARKS 8.26. (1) Using Zelmanov's classification of division Jordan algebras and computing their isotopes in each one of the four types, one checks that the isotope of division Jordan PI-algebra is again a division Jordan PI-algebra.

(2) In their recent paper [**SZ19**, Lemma 4.1], I. Shestakov and E. Zelmanov have proved that finite presentation is also an isotopy invariant.

The McCrimmon radical of a Jordan algebra. A Jordan algebra J is *strongly prime* if it is prime and nondegenerate. By a *nondegenerate* (resp. *strongly prime*) *ideal of* J, we mean an ideal I of J such that J/I is a nondegenerate (resp. strongly prime) Jordan algebra. The intersection of nondegenerate ideals of J is a nondegenerate ideal of J, so there exists a smallest nondegenerate ideal $\mathrm{Mc}(J)$ called the *McCrimmon radical* of J. The McCrimmon radical is a radical in the sense of Amitsur-Kurosh (see [**The85**, Theorem 4]).

Set $\mathrm{Mc}_0(J) = 0$ and let $\mathrm{Mc}_1(L)$ be the ideal of J generated by its absolute zero divisors ($\mathrm{Mc}_1(L)$ is actually spanned by the absolute zero divisors of J [**McC69**, Theorem 9]). Using transfinite induction we define a chain of ideals of J by

$$\mathrm{Mc}_\alpha(J) = \bigcup_{\beta<\alpha} \mathrm{Mc}_\beta(J) \text{ for a limit ordinal } \alpha, \text{ and}$$

$$\mathrm{Mc}_\alpha(J)/\mathrm{Mc}_{\alpha-1}(J) = \mathrm{Mc}_1(J/\mathrm{Mc}_{\alpha-1}(J)) \text{ otherwise,}$$

and put $\mathrm{Mc}(J) = \bigcup_\alpha \mathrm{Mc}_\alpha(J)$. It is clear from the construction that $\mathrm{Mc}(J)$ is the McCrimmon radical of J. Moreover, it is proved in [**Zel84a**] that $\mathrm{Mc}(J)$ is locally nilpotent.

DEFINITION 8.27. An *m-sequence* of a Jordan algebra J is a sequence $\{a_n\}$ of elements of J such that for $n \geq 1$, $a_{n+1} = U_{a_n} b_n$ for some $b_n \in J$. We will say that $\{a_n\}$ has (finite) length k if $a_k \neq 0$ and $a_{k+1} = 0$.

There is a beautiful characterization of the McCrimmon radical in terms of m-sequences which is similar to that of the Baer radical of a ring [**Coh03**, Proposition 8.5.6].

PROPOSITION 8.28. [**The85**, Theorem 2] *An element $x \in J$ belongs to* $\mathrm{Mc}(J)$ *if and only if every m-sequence of J beginning with x has finite length.*

COROLLARY 8.29. [**The85**, Theorem 3] $\mathrm{Mc}(J)$ *is equal to the intersection of the strongly prime ideals of J.*

Capacity. By a *division idempotent* of a Jordan algebra J we mean a nonzero idempotent e of J such that U_eJ is a division Jordan algebra. It follows from Proposition 8.17 that every division idempotent e of an algebraic Jordan algebra J over an algebraically closed field $\mathbb{F}$ generates an one-dimensional inner ideal, i.e. $U_eJ = \mathbb{F}e$, such an idempotent e is called *reduced.*

• A Jordan algebra J has *capacity* n if J is unital and 1 can be written as a sum $1 = e_1 + \cdots + e_n$ of n orthogonal division idempotents. It has *finite capacity* if it has capacity n for some n. By Jacobson's capacity theorem [**McC04**, I.5.2], any nondegenerate Jordan algebra having finite capacity is a direct sum of ideals each of which is a simple Jordan algebra of finite capacity.

• If $J = R^+$, where R is a semiprime associative Φ-algebra, then J has finite capacity if and only if R is Artinian.

• A Jordan algebra over a field $\mathbb{F}$ having finite capacity and such that every division idempotent is reduced is said to be *reduced.* Any finite-dimensional nondegenerate Jordan algebra over an algebraically closed field is reduced.

• A Jordan algebra is said to be *semiprimitive* if it has no quasi-invertible ideals (see [**McC04**, III.1.3.1] for definition), i.e. its Jacobson radical is zero. Every semiprimitive Jordan algebra is nondegenerate [**McC04**, III.1.6.1]. Under finiteness conditions, the converse is also true.

• A Jordan algebra is said to be an *I-algebra* if every non-nil inner ideal of J contains a nonzero idempotent. Algebraic Jordan algebras (Lemma 8.10) and von Neumann regular Jordan algebras (Lemma 8.15) are I-algebras.

THEOREM 8.30. [**McC04**, I.8.1 (I-Finite Capacity Theorem)] *Any semiprimitive I-algebra having no infinite family of nonzero orthogonal idempotents is unital and has finite capacity.*

LEMMA 8.31. (E. Zelmanov) *Let J be a Jordan PI-algebra over a field $\mathbb{F}$ of characteristic* 0 *such that every element of J is a sum of nilpotent elements. Then* $J = \mathrm{Mc}(J)$.

PROOF. By replacing J by $J/\mathrm{Mc}(J)$, we may suppose that J is nondegenerate. Furthermore, since a nondegenerate Jordan algebra is a subdirect product of strongly prime Jordan algebras (Corollary 8.29), we may assume that J is strongly prime. Then, by Theorem 8.25, J has nonzero (associative) center $Z = Z(J)$ and the Jordan algebra of fractions $Z^{-1}J$ is either a simple finite-dimensional Jordan algebra over the field $Z^{-1}Z$, or the Jordan algebra of a nondegenerate symmetric bilinear form over $Z^{-1}Z$. Let $\overline{Z}$ be the algebraic closure of the field $Z^{-1}Z$. Then $\tilde{J} := \overline{Z} \otimes_{Z^{-1}Z} Z^{-1}J$ is a reduced simple Jordan algebra over $\overline{Z}$. Hence, by [**Jac68**, V.4.Theorem 6], $\tilde{J}$ has a nonzero linear trace $\mathrm{tr} : \tilde{J} \to \tilde{J}$. Since every element of $\tilde{J}$ is a sum of nilpotent elements, it follows that $\mathrm{tr}(\tilde{J}) = 0$, a contradiction. This proves the lemma. □

COROLLARY 8.32. *Let J be a Jordan algebra over a field $\mathbb{F}$ of characteristic* 0 *such that every element of J is a sum of nilpotent elements. If L is algebraic of bounded degree, then $J = \mathrm{Mc}(J)$.*

PROOF. By Lemma 8.24, J is PI, so Lemma 8.31 applies. □

The socle of a Jordan algebra. The *socle* $\mathrm{Soc}(J)$ of a Jordan algebra J is defined as the sum of its minimal inner ideals. If J is nondegenerate, then $\mathrm{Soc}(J)$ is a direct sum of ideals each of which is a simple algebra containing a minimal inner ideal [**OR79**, Theorem 17]. As mentioned in the introduction of the chapter, the nature of an element x of a Jordan algebra J is reflected in the structure of the local algebra J_x. An example of this fact is the local characterization of the socle of a nondegenerate Jordan algebra: $x \in \mathrm{Soc}(J)$ if and only if the (nondegenerate) Jordan algebra J_x has finite capacity (see [**Loo89**, Theorem 1] or [**Mon99**, Lemma 0.7]).

As most of the symmetric notions in associative algebras, the socle can be characterized in Jordan terms: Let R be a semiprime associative algebra. Then $\mathrm{Soc}(R) = \mathrm{Soc}(R^+)$ [**FL85**, Proposition 2.6]. Hence it follows easily the local characterization of the socle of a semiprime associative algebra that will be used in Proposition 13.85.

PROPOSITION 8.33. *Let R be a semiprime associative algebra. Then x is in the socle if and only if R_x is Artinian.*

PROOF. Following Exercise 8.106, given $x \in R$, the associative algebra R_x inherits semiprimeness from R and keeps up the Jordan structure: $(R_x)^+ = (R^+)_x$. By the local characterization of the socle of a nondegenerate Jordan algebra quoted above, $x \in \mathrm{Soc}(R) = \mathrm{Soc}(R^+)$ if and only if the Jordan algebra $(R^+)_x \cong R_x^+$ has finite capacity, equivalently, the associative algebra R_x is Artinian. □

Primitive Jordan algebras. Let J be a Jordan Φ-algebra. Following [**HM81**], an inner ideal K of J is called *e-modular* for some $e \in J$ if $\hat{K} = K + \Phi(1-e)$ is an inner ideal of the unital hull of J. It is clear that K is proper if and only if $e \notin K$, and that if J is unital, then any inner ideal of J is 1-modular. Moreover, using (iii) of Exercise 8.107, it is easy to see that given $e \in J$, the inner ideal $U_{1-e}J$ of J is e-modular if and only if $e^2 - e \in U_{1-e}J$. In particular, if e is an idempotent, then $U_{1-e}J$ is an e-modular inner ideal of J.

DEFINITION 8.34. A Jordan algebra J is called *primitive* if it contains a proper e-modular inner ideal K such that $K + I = J$ for any nonzero ideal I of J.

EXAMPLES 8.35. (1) Let R be an associative algebra. Then R is (left or right) primitive if and only if the Jordan algebra R^+ is primitive [**AMC94**, Theorem 2.2].

(2) Let R be a prime associative algebra with involution $*$. Then R is primitive if and only if the Jordan algebra $\mathrm{Sym}(R, *)$ is primitive [**AMC94**, Theorem 2.3].

(3) Let J be a strongly prime Jordan algebra with nonzero socle. Then J is primitive [**FLRP86b**, Theorem 12].

PROPOSITION 8.36. *Let J be a Jordan algebra and let I be a nonzero ideal of J. If J is primitive, then J is strongly prime and I is primitive. Conversely, if J is strongly prime and I is primitive, then J is primitive.*

PROOF. If J is primitive, then so is I by [**AMC94**, Theorem 1.1], and J is prime by [**HM81**, Proposition 5.5]. Moreover, J is nondegenerate. Otherwise $\mathrm{Mc}(J)$ would be primitive and locally nilpotent (see Remark 3.19(2)), which leads to a contradiction, since if e is a modulus for an inner ideal, then so is any power e^n [**HM81**, Proposition 2,10]. Conversely, if J is strongly prime and I is primitive, then J is primitive by [**AMC94**, Theorem 1.5]. □

The notion of primitivity in Jordan algebras can be characterized locally.

PROPOSITION 8.37. *Let J be a Jordan algebra.*

(1) *If J is primitive, then so is J_a for any $0 \neq a \in J$.*
(2) *If J is strongly prime and J_a is primitive for some $0 \neq a \in J$, then J is primitive.*

PROOF. (1) is proved [**ACM95**, Theorem 4.1], and (2) in [**AC98**, Theorem 4.1]. □

PROPOSITION 8.38. *Any primitive Jordan PI-algebra J is simple and has finite capacity.*

PROOF. By [**Zel83b**, Theorems 7 and 8], J is either finite-dimensional over its center, or isomorphic to the Jordan algebra of a nondegenerate symmetric bilinear form. In both cases, J is simple and has finite capacity. □

Algebraic Jordan algebras of bounded degree.

LEMMA 8.39. *Let J be algebraic of bounded degree. If J is nondegenerate, then it is semiprimitive.*

PROOF. By Lemma 8.24, J is PI and hence, by Zelmanov's PI-Radical Theorem [**Zel82**, Theorem 4], J does not contain any nonzero nil ideal. This, together with the fact that J is an I-algebra, implies that J is semiprimitive, since otherwise the Jacobson radical of J would contain a nonzero idempotent, a contradiction. □

LEMMA 8.40. *Let J be a Jordan algebra over a field $\mathbb{F}$ of characteristic 0. If J is algebraic of bounded degree n, then any family of nonzero orthogonal idempotents of J has a cardinal less than or equal to n.*

PROOF. Given $m \geq 1$, let $(e_1, \ldots, e_m)$ be a sequence of nonzero orthogonal idempotents of J, let $\lambda_1, \ldots, \lambda_m$ be nonzero elements of $\mathbb{F}$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$, and set $a := \lambda_1 e_1 + \cdots + \lambda_m e_m$. Vandermonde determinant says to us that the vectors $a, a^2, \ldots, a^m$ are linearly independent. This proves that any sequence of nonzero orthogonal idempotents of J has a cardinal less than or equal to n. □

THEOREM 8.41. *Let J be a nondegenerate algebraic Jordan algebra of bounded degree over a field $\mathbb{F}$ of characteristic 0. Then J has finite capacity.*

PROOF. L is an I-algebra (Lemma 8.10), semiprimitive (Lemma 8.39), with no infinite family of nonzero orthogonal idempotents (Lemma 8.40). This proves (Theorem 8.30) that J is unital and has finite capacity. □

REMARK 8.42. E. Zelmanov shows in [**Zel82**] that any strongly prime algebraic Jordan PI-algebra over a field of characteristic different from 2 is simple and has finite capacity. We encourage the reader to glance through the proof and observe the amazing reduction to the case of a special Jordan algebra [**Zel82**, Lemma 20] to prove the idempotent-finite condition [**Zel82**, Lemma 24].

8.2. The Jordan algebra attached to a Jordan element

Throughout this section we will deal with Lie algebras over a ring of scalars Φ in which 6 is invertible. As in preceding chapters, and in order to simplify the notation, we will often make use of capital letters to denote the adjoint operators.

Definition, examples, and basic results. Let a be a Jordan element of L. In the Φ-module L a new product, denoted by $\bullet$, is defined by setting $x \bullet y := [[x, a], y]$ for all $x, y \in L$. The resulting (nonassociative) algebra is denoted by $L^{(a)}$.

THEOREM 8.43. *Let L be a Lie algebra and let $a \in L$ be a Jordan element. We have:*

(i) $\mathrm{Ker}_L\{a\} = \{z \in L : [a, [a, z]] = 0\}$ *is an ideal of* $L^{(a)}$.
(ii) $L_a := L^{(a)}/\,\mathrm{Ker}_L\{a\}$ *is a Jordan algebra, with*
(iii) $l_{\overline{x}}\overline{y} = \overline{XAy}, \quad U_{\overline{x}}\overline{y} = \overline{X^2A^2y}, \quad \{\overline{x}, \overline{y}, \overline{z}\} = 2\overline{XZA^2y}$

for all $x, y \in L$, where $\overline{x}$ denotes the coset $x + \mathrm{Ker}_L\{a\}$.

PROOF. (i) It suffices to show that $[[z, a], y] \in \mathrm{Ker}_L\{a\}$ for all $z \in \mathrm{Ker}_L\{a\}$ and $y \in L$. But this follows from the Leibniz rule and 4.4(6):

$$A^2[[z, a], y] = -[Az, A^2y] = [A^2z, Ay] = 0.$$

(ii) Let $x, y \in L$. Using (L1) and (L2) we get

$$x \bullet y - y \bullet x = [[x, a], y] - [[y, a], x] = [[x, y], a] \in \mathrm{Ker}_L\{a\}, \tag{8.5}$$

which proves that L_a is commutative. Thus we only need to verify that for all $x \in L$, the operators $l_{\overline{x}}$ and $l_{\overline{x}^2}$ commute.

In the multiplication ideal $\mathrm{Ad}(L)$, define an equivalence relation by $T_1 \equiv T_2$ if $T_1 - T_2$ vanishes on $\mathrm{Ker}_L\{a\}$, i.e. if $A^2T_1 = A^2T_2$, $T_1, T_2 \in \mathrm{Ad}(L)$, and denote by $\overline{T}$ the equivalence class of T. Clearly, $T_1 \equiv T_2$ implies $T_1S \equiv T_2S$ for all $S \in \mathrm{Ad}(L)$, and since $A^3 = 0$, we have

$$\overline{AT} = \overline{0}. \tag{8.6}$$

for all $T \in \mathrm{Ad}(L)$. Moreover, by 4.4(2),

$$\overline{XA^2T} = \overline{0}. \tag{8.7}$$

Let $x \in L$. Then $l_x = [X, A]$, $\;x^2 = [[x, a], x]$ and $l_{x^2} = [[[X, A], X], A]$. Using (8.6) and (8.7), we get:

$$l_x l_{x^2} \equiv XA[[X, A], X]A \equiv 2(XA)^3 - XAX^2A^2$$

and

$$l_{x^2} l_x \equiv [[X, A], X]A(XA - AX) \equiv 2(XA)^3 - X^2A^2XA,$$

since by 4.4(1) and (8.7), $X(AXA^2)X = (XA^2)XAX \equiv 0$. Now it follows from (1) and (5) of Lemma 4.4 that

$$A^2XAX^2A^2 = A^2X^2AXA^2 = A^2X^2A^2XA,$$

so completing the proof that L_a is a Jordan algebra.

Similarly we get $l_x^2 \equiv (XA)^2$ and $l_{x^2} \equiv 2(XA)^2 - X^2A^2$, so $2l_x^2 - l_{x^2} \equiv X^2A^2$, which implies

$$U_{\overline{x}}\overline{y} = \overline{X^2A^2y}, \quad \text{and}$$

$$\{\overline{x}, \overline{y}, \overline{z}\} = U_{\overline{x}+\overline{z}}\overline{y} - U_{\overline{x}}\overline{y} - U_{\overline{z}}\overline{y} = \overline{(X+Z)^2A^2y} - \overline{X^2A^2y} - \overline{Z^2A^2y} = 2\overline{XZA^2y}$$

since $A^2[X, Z]A^2 = 0$ by 4.4(1), which completes the proof. □

Definition 8.44. For any Jordan element a of a Lie algebra L, the Jordan algebra L_a defined above is called the *Jordan algebra of L at a.*

Lemma 8.45. *Let R be a semiprime associative algebra over a ring of scalars Φ in which 6 is invertible and let $b \in R$ be such that $b^2 = 0$. Denote by π the canonical homomorphism of R^- onto $\overline{R} = R^-/Z(R)$. Then we have the following chain of isomorphisms of Jordan algebras: $R_b^+ \cong R_b^- \cong \overline{R}_{\pi(b)}$.*

Proof. From $b^2 = 0$ it easily follows that the linear isomorphism $x \mapsto \frac{1}{2}x$ of R induces a Jordan isomorphism of R_b^+ onto R_b^-. And we also have that the Lie epimorphism $\pi : R^- \to \overline{R}$ induces the Jordan epimorphism $\overline{\pi} : R_b^- \to \overline{R}_{\pi(b)}$. We claim that $\overline{\pi}$ is actually an isomorphism. Let $x \in R$ be such that $[\pi(b), [\pi(b), \pi(x)]] = \pi(0)$, Then $[b, [b, x]] \in Z(R)$ and hence $[b, [b, x]] = -bxb = 0$ since $Z(R)$ has no nonzero nilpotente elements. Now we have by above that $R_b^+ \cong R_b^- \cong \overline{R}_{\pi(b)}$. □

Corollary 8.46. *Let R be a centrally closed prime ring of characteristic different from 2 and 3, and let a be a Jordan element of the Lie algebra R^-. Then $R_a^- \cong R_b^+$ for some $b \in R$, $b^2 = 0$.*

Proof. By Theorem 4.35, there exists $z \in Z(R)$ such that $(a - z)^2 = 0$. Set $b = a - z$. It is clear that the identity on R induces an isomorphism of R_a^- onto R_b^-. Now Lemma 8.45 applies. □

Remark 8.47. There is a natural functor from the *pointed Lie algebra category* $(\mathcal{L}, \bullet)$ (whose objects are the pairs (L, a), where L is a Lie algebra and a is a Jordan element, and whose morphisms $f : (L, a) \to (M, b)$ are the homomorphism of Lie algebras such that $f(a) = b$) to the category of the Jordan Φ-algebras, assigning to each pair (L, a) the Jordan algebra L_a, and to each morphism $f : (L, a) \to (M, b)$ the homomorphism $\overline{f} : L_a \to M_b$ given by $\overline{f}(\overline{x}) = \overline{f(x)}$, $x \in L$, which is well defined since $f(\mathrm{Ker}_L\{a\}) \subset \mathrm{Ker}_L\{b\}$.

Proposition 8.48. *Let $a \in L$ be a Jordan element. We have:*

(1) *For any inner ideal M of L, $M_a = M/(\mathrm{Ker}_L\{a\} \cap M)$ is a subalgebra of L_a.*

(2) *Let $L = B \oplus C$ be a direct sum of ideals and let $a = b + c$ with respect to this decomposition, then b and c are Jordan elements of the Lie algebras B and C respectively, and $L_a \cong B_b \times C_c$.*

Proof. (1) Regarded M as a subalgebra $M^{(a)}$ of $L^{(a)}$, denote by $x \mapsto \overline{x}$ the injection of $M^{(a)}$ into $L^{(a)}$. It is clear that $M_a = M^{(a)}/\mathrm{Ker}_M\{a\}$ is a subalgebra of L_a.

(2) $0 = \mathrm{ad}_a^3 L = \mathrm{ad}_b^3 B \oplus \mathrm{ad}_c^3 C$ implies that b is a Jordan element in B and c is a Jordan element in C. Let $x = y + z$, where $y \in B$ and $z \in C$, and denote by $\overline{y}_B$ (resp. $\overline{z}_C$) the coset of y in B_b (resp. the coset of z in C_c). Then the map $\overline{x} \mapsto (\overline{y}_B, \overline{z}_C)$, is an isomorphism of L_a onto $B_b \times C_c$. □

The process of forming the Jordan algebra of a Lie algebra at a Jordan element and then a local algebra of the resulting Jordan algebra (Section 8.1) yields a Jordan algebra of the Lie algebra in a new Jordan element.

PROPOSITION 8.49. *Let a be a Jordan element of a Lie algebra L. For any $y \in L$, $b = [[a, y], a]$ is a Jordan element and $(L_a)_{\overline{y}} \cong L_b$.*

PROOF. We already know by 4.4(4) that $b = [[a, y], a]$ is a Jordan element. For any $x \in L$, denote by $\overline{x}$ and $\widetilde{x}$ the cosets of x in L_a and L_b respectively. We have

$$\overline{x} = 0 \Leftrightarrow \operatorname{ad}_a^2 x = 0 \Rightarrow \operatorname{ad}_b^2 x = \operatorname{ad}_a^2 \operatorname{ad}_y^2 \operatorname{ad}_a^2 x = 0,$$

which proves that $\overline{x} \mapsto \widetilde{x}$ defines a linear map φ of L_a onto L_b. We claim that φ is actually a homomorphism of the $\overline{y}$-homotope $L_a^{(\overline{y})}$ onto L_b, with $\ker(\varphi) = \ker(U_{\overline{y}})$. By 4.4(2), $\varphi(\overline{x}^2) = \varphi(U_{\overline{x}}\overline{y}) = \widetilde{\operatorname{ad}_x^2 \operatorname{ad}_a^2 y} = -\widetilde{\operatorname{ad}_x^2 b} = \widetilde{[[x, b], x]} = \widetilde{x} \bullet \widetilde{x}$, with $\widetilde{x} = 0$ if and only if $\operatorname{ad}_b^2 x = \operatorname{ad}_a^2 \operatorname{ad}_y^2 \operatorname{ad}_a^2 x = 0$, equivalently, $U_{\overline{y}}\overline{x} = 0$, i.e. $\overline{x} \in \ker(U_{\overline{y}})$, which proves the claim. Hence φ induces an isomorphism of $(L_a)_{\overline{y}}$ onto L_b. □

Inner inheritance. Recall (Proposition 4.6) that any Jordan element a of a Lie algebra L yields two abelian inner ideals: the principal inner ideal $[a]_L = \operatorname{ad}_a^2 L$, and the inner ideal $(a)_L = \Phi a + [a]_L$ generated by a. Similar notations are used for the principal inner ideal $[a]_J = U_a J$, and the inner ideal $(a)_J = \Phi a + U_a J$ generated by an element of a Jordan algebra J.

LEMMA 8.50. *Let a be a Jordan element of a Lie algebra L. We have:*

(i) *The map $\overline{x} \mapsto \operatorname{ad}_a^2 x$ is a Φ-module isomorphism of L_a onto the inner ideal $[a]_L$ of L.*
(ii) *This isomorphism induces an isomorphism from the lattice of the inner ideals of the Jordan algebra L_a onto the lattice of the inner ideals of the Lie algebra L contained in $[a]_L$.*

PROOF. Only statement (ii) requires some attention. Let $\overline{M}$ be an inner ideal of L_a. For $\overline{x} \in \overline{M}$ and $b \in L$, $U_{\overline{x}}\overline{b} = \overline{\operatorname{ad}_x^2 \operatorname{ad}_a^2 b} \subset \overline{M}$, equivalently,

$$\operatorname{ad}_{\operatorname{ad}_a^2 x}^2 b = \operatorname{ad}_a^2 \operatorname{ad}_x^2 \operatorname{ad}_a^2 b \subset \operatorname{ad}_a^2 M.$$

□

Global to Local Inheritance. We have seen (Theorem 7.10) that a Lie algebra L is strongly prime if and only if $[[x, L], y] = 0$ implies $x = 0$ or $y = 0$ for any $x, y \in L$. Strongly prime Jordan algebras are characterized in a similar way [**BMS87**]: A (linear) Jordan algebra J is strongly prime if and only if $\{x, J, y\} = 0$ implies $x = 0$ or $y = 0$, $x, y \in J$.

PROPOSITION 8.51. *Let a be a Jordan element of a Lie algebra L. We have:*

(1) *If $\overline{x}$ is a nonzero absolute zero divisor of the Jordan algebra L_a, then $\operatorname{ad}_a^2 x$ is a nonzero absolute zero divisor of L. Thus, if L is nondegenerate, then so is L_a.*
(2) *If L is strongly prime, then so is L_a.*
(3) *If L is strongly prime with nonzero socle, then so is L_a.*
(4) *Let $x \in L$. Then $\overline{x}^n = \overline{\operatorname{ad}_{[x,a]}^{n-1} x}$. Hence if $[x, a]$ is ad-nilpotent, then $\overline{x} \in L_a$ is nilpotent. In particular, if L is nil, then the Jordan algebra L_a is nil.*

Suppose now that the ring of scalars of L is a field of characteristic not 2 *or* 3. *We have:*

(5) *If L is algebraic, then L_a is an algebraic Jordan algebra.*
(6) *If L is locally finite, then L_a is a locally finite Jordan algebra.*

PROOF. (1) Let $\overline{x}$ be an absolute zero divisor of L_a. By 4.4(3),

$$\operatorname{ad}^2_{\operatorname{ad}^2_a x} L = \operatorname{ad}^2_a \operatorname{ad}^2_x \operatorname{ad}^2_a L = 0,$$

which implies $\operatorname{ad}^2_a x = 0$ by nondegeneracy of L, i.e. $\overline{x} = \overline{0}$.

(2) It suffices to show that $\{\overline{x}, L_a, \overline{z}\} = 0$ implies $\overline{x} = 0$ or $\overline{z} = 0$. Suppose then that $\{\overline{x}, \overline{y}, \overline{z}\} = 0$ for all $y \in L$, i.e. $A^2XZA^2 = 0$. By 4.4(8),

$$\operatorname{ad}_{A^2x} \operatorname{ad}_{A^2z} = A^2XZA^2 = 0,$$

and hence, by Theorem 7.10, $\operatorname{ad}^2_a x = 0$ or $\operatorname{ad}^2_a z = 0$, i.e. $\overline{x} = 0$ or $\overline{z} = 0$.

(3) We only need to prove that L_a contains a minimal inner ideal. Since L is strongly prime with nonzero socle, there exists a minimal abelian inner ideal B of L such that $\operatorname{ad}^2_a B \neq 0$ and therefore a minimal abelian inner ideal (Corollary 4.20). Then, by Lemma 8.50(ii), $\overline{B}$ is a minimal inner ideal of L_a.

It follows by induction that for each $x \in L$ and any integer $n > 1$,

$$\overline{x}^n = \overline{\operatorname{ad}^{n-1}_{[x,a]} x}.$$

Items (4) and (5) now follow easily.

(6) It suffices to prove that $L^{(a)}$ is locally finite. Suppose that $\{x_1, \dots, x_n\}$ is a finite subset of L. Clearly, the subalgebra of $L^{(a)}$ generated by $\{x_1, \dots, x_n\}$ is contained in the subalgebra of L generated by $\{x_1, \dots, x_n, a\}$, which is finite-dimensional. □

OPEN QUESTION 8.52. Let L be a simple nondegenerate Lie algebra and let $a \in L$ be a nonzero Jordan element. Is the Jordan algebra L_a simple?

The answer is affirmative for the following types of Lie algebras: $L = \overline{R'}$ where R is a simple ring [**FLGGL07**, Theorem 3.3], $L = \overline{K'}$ where $K = \operatorname{Skew}(R, *)$ and R is a simple ring with involution [**FLGGL07**, Theorems 3.6 and 3.10], and L is a simple nondegenerate Lie algebra of characteristic not 2 or 3 which is finite-dimensional over its centroid (L is strongly prime and hence so is L_a by Proposition 8.51(2), since L_a is also finite-dimensional over the centroid of L, L_a coincides with its socle and hence it is simple). Using Zelmanov's theorem for simple Lie algebras with a finite $\mathbb{Z}$-grading (Section 14.1) and the previous three cases, it can be proved that the answer is also affirmative for simple Lie algebras of characteristic 0 or greater than 7 containing a nonzero von Neumann regular element.

REMARK 8.53. Inheritance of nondegeneracy was used in [**GGLN11**] to show that a Lie triple system is nondegenerate if and only if its standard envelope is nondegenerate and hence that a Kantor pair is nondegenerate if and only if its standard envelope is nondegenerate.

PI-inheritance. We show in this subsection that if L is a Lie PI-algebra, then L_a is a Jordan PI-algebra for any Jordan element a of L.

DEFINITION 8.54. Let $p = p(x_1, \ldots, x_n)$ be an element of a free Lie Φ-algebra $\mathfrak{L}\langle X\rangle$. We say that a Lie algebra L satisfies the identity p if $p(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n$ in L. A Lie algebra satisfying a nontrivial polynomial identity is called a *Lie PI-algebra.*

Recall that left commutators $[x_1, x_2, \ldots, x_n]$ are defined recursively as follows: $[x_1] = x_1$ and $[x_1, x_2, \ldots, x_n] = [x_1, [x_2, \ldots, x_n]] = \mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_{n-1}} x_n$ for $n > 1$, $x_1, x_2, \ldots, x_n \in L$. We also recall that by the Jacobi identity any monomial of the Φ-algebra $\mathfrak{L}\langle X\rangle$ over a countable set of indeterminates X can be written as a linear combination of left commutators.

LEMMA 8.55. *For $n \geq 1$ there exists a function $\epsilon_n : \mathfrak{S}_n \to \{-1, 0, 1\}$ such that, for any $x_1, \ldots, x_n, x_{n+1}$ in X,*

$$\mathrm{ad}_{[x_1,\ldots,x_n]}\, x_{n+1} = \sum_{\sigma\in\mathfrak{S}_n} \epsilon_n(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}].$$

PROOF. By induction on n. The case $n = 1$ is trivial. Now

$$\begin{aligned}\mathrm{ad}_{[x_1,x_2,\ldots,x_{n+1}]}\, x_{n+2} &= \mathrm{ad}_{[x_1,[x_2,\ldots,x_{n+1}]]}\, x_{n+2} \\ &= \mathrm{ad}_{x_1} \mathrm{ad}_{[x_2,\ldots,x_{n+1}]}\, x_{n+2} - \mathrm{ad}_{[x_2,\ldots,x_{n+1}]} \mathrm{ad}_{x_1} x_{n+2}.\end{aligned}$$

Hence, by the induction hypothesis,

$$\begin{aligned}\mathrm{ad}_{[x_1,\ldots,x_{n+1}]}\, x_{n+2} &= \sum_{\sigma\in\mathfrak{S}_n} \epsilon_n(\sigma)[x_1, x_{\sigma(1)+1}, \ldots, x_{\sigma(n)+1}, x_{n+2}] \\ &- \sum_{\sigma\in\mathfrak{S}_n} \epsilon_n(\sigma)[x_{\sigma(1)+1}, \ldots, x_{\sigma(n)+1}, x_1, x_{n+2}] \\ &= \sum_{\tau\in\mathfrak{S}_{n+1}} \epsilon_{n+1}(\tau)[x_{\tau(1)}, \ldots, x_{\tau(n+1)}, x_{n+2}].\end{aligned}$$

where ϵ_n is defined inductively by

$$\epsilon_{n+1}(\tau) = \begin{cases} 0 & \text{if } \tau(1) \neq 1 \text{ and } \tau(n+1) \neq 1 \\ \epsilon_n(\sigma) & \text{if } \tau(1) = 1 \text{ and } \sigma \in \mathfrak{S}_n \text{ is defined by } \sigma(i) = \tau(i+1) - 1 \\ -\epsilon_n(\sigma) & \text{if } \tau(n+1) = 1 \text{ and } \sigma \in \mathfrak{S}_n \text{ is defined by } \sigma(i) = \tau(i) - 1. \end{cases}$$

□

PROPOSITION 8.56. *Any Lie PI-algebra L satisfies the multilinear identity of the form*

$$p(x_1, \ldots, x_n, x_{n+1}) = \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma [x_{\sigma(1)}, \ldots, x_{\sigma(n)}, x_{n+1}] \quad (\alpha_\sigma \in \Phi).$$

PROOF. As pointed out above, we may assume that L satisfies a polynomial identity $p = 0$, where p is a linear combination of left commutators $[x_{i_1}, \ldots, x_{i_m}]$. Moreover, by [**ZSSS82**, Corollary of Theorem 1.5.7], we can assume that p is multilinear. Finally, by Lemma 8.55, we can replace p by a polynomial having the required form. □

PROPOSITION 8.57. *Let a be a Jordan element of a Lie algebra L. If L is PI, then L_a is a Jordan PI-algebra.*

PROOF. By Proposition 8.56, L satisfies a multilinear identity of the form

$$p(x_1,\dots,x_n,x_{n+1}) = \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma[x_{\sigma(1)},\dots,x_{\sigma(n)},x_{n+1}].$$

For each integer $1 \le i \le n$, replace x_i by $[x_i, a]$. We obtain the Jordan polynomial

$$\begin{aligned} q(x_1,\dots,x_n,x_{n+1}) &= \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma x_{\sigma(1)} \bullet \cdots \bullet x_{\sigma(n)} \bullet x_{n+1} \\ &= \sum_{\sigma\in\mathfrak{S}_n} \alpha_\sigma[[x_{\sigma(1)},a],\dots,[x_{\sigma(n)},a],x_{n+1}], \end{aligned}$$

which vanishes on the Jordan algebra L_a. Since by Proposition 8.23, this Jordan polynomial is never an s-identity, L_a is PI. □

REMARK 8.58. Let W_m denote the Lie algebra of derivations of $\mathbb{F}[x_1,\dots,x_m]$, where $\mathbb{F}$ is a field of characteristic 0 (see Exercise 2.88). It is proved in [**Bah76**, 2.1.3(iv and v)]:

(1) $s_n(x_1,\dots,x_n,x_{n+1}) = \sum_{\sigma\in\mathfrak{S}_n} \epsilon_\sigma[x_{\sigma(1)},\dots,x_{\sigma(n)},x_{n+1}]$, where $\epsilon_\sigma = 1$ if σ is an even permutation and $\epsilon_\sigma = -1$ otherwise, is a nontrivial polynomial identity,
(2) W_m satisfies the identity $s_n(x_1,\dots,x_n,x_{n+1})$ for $n = m^2 + 2m + 2$.

Von Neumann regular elements in Lie algebras revisited. Recall that a Jordan element a in a Lie algebra L is von Neumann regular if $a \in \operatorname{ad}_a^2 L$. And that an element x of a Jordan algebra J is von Neumann regular if $x \in U_x J$.

PROPOSITION 8.59. *Let a be a Jordan element of a Lie algebra L. For any $x \in L$ the following conditions are equivalent:*

(i) $\operatorname{ad}_a^2 x$ *is a von Neumann regular element of* L.
(ii) $\overline{x}$ *is a von Neumann regular element of the Jordan algebra* L_a.

PROOF. By Theorem 8.43, for any $x, y \in L$,

$$U_{\overline{x}}\overline{y} = \overline{x} \quad\text{if and only if}\quad A^2X^2A^2y = A^2x,$$

so $\overline{x}$ is von Neumann regular in the Jordan algebra L_a if and only if $\operatorname{ad}_a^2 x = A^2x$ is von Neumann regular in L. □

PROPOSITION 8.60. *Let L be a nondegenerate Lie algebra, let I be an ideal of L, and let $a \in I$ be von Neumann regular. Then the Jordan algebras I_a and L_a agree.*

PROOF. By Proposition 8.48, I_a is contained in L_a, so it suffices to show that any coset $\overline{x}$ in L_a is equal to a coset $\overline{y}$ in I_a. Since a is von Neumann regular, $a = \operatorname{ad}_a^2 b$ for some $b \in L$. Then we have by 4.4(3) $\operatorname{ad}_a^2 = \operatorname{ad}^2_{\operatorname{ad}_a^2 b} = \operatorname{ad}_a^2 \operatorname{ad}_b^2 \operatorname{ad}_a^2$. Hence, for any $x \in L$, $\operatorname{ad}_a^2 x = \operatorname{ad}_a^2 y$, where $y = \operatorname{ad}_b^2 \operatorname{ad}_a^2 x \in I$. □

We have already seen the important role played by von Neumann regular elements in the structure of Lie algebras, so it is very useful to have criterions guaranteeing the existence of nontrivial von Neumann regular elements. The next proposition provides us with such a criterion in terms of Jordan algebras. As an example of the usefulness of this Jordan criterion, we will prove later that every locally finite simple Lie algebra over an algebraically closed field of characteristic 0 containing a nonzero Jordan element is linearly spanned by von Neumann regular elements.

PROPOSITION 8.61. *Let $a \in L$ be a Jordan element. We have:*

(i) *If $a = [[a, e], a]$ for some $e \in L$, i.e. a is von Neumann regular, then L_a is unital with $\overline{e}$ as unit element.*

(ii) *If L is nondegenerate and L_a is unital, then a is von Neumann regular.*

PROOF. (i) Let $a = [[a, e], a]$. Then $A = [[A, E], A] = 2AEA - EA^2 - A^2E$ and hence $A^2 = 2A^2EA - AEA^2 = A^2EA$ since $A^3 = 0$ and $AEA^2 = A^2EA$ by 4.4(1), which proves $l_{\overline{e}} = 1_{L_a}$.

(ii) Suppose now that L is nondegenerate and that L_a is unital, with $\overline{e}$ as unit element, i.e. $l_{\overline{e}} = U_{\overline{e}} = 1_{L_a}$. We claim that $a = [[a, e], a]$. Set $z = [[a, e], a] - a$. Since L is nondegenerate, it suffices to show that $Z^2 = 0$. But $Z = 2AEA - EA^2 - A^2E - A$ implies $Z^2 = 0$, since $A^3 = 0$, $A^2 = A^2EA$ and $A^2 = A^2E^2A^2$ by hypothesis, and $AEA^2 = A^2EA$, $A^2EA^2 = 0$ and $A^2E^2A^2 = \mathrm{ad}^2_{A^2(e)} = \mathrm{ad}^2_{-a} = A^2$ by 4.4(1-3). $\square$

Next we compute the Jordan algebra L_a of a Lie algebra L when a is actually a von Neumann regular element. This particular case was already known in [**Ben77**, Lemma 2.2].

PROPOSITION 8.62. *Let $a \in L$ be von Neumann regular, $a = [[a, b], a]$ for some $b \in L$. Then L_a is isomorphic to the Jordan algebra, $J(a, b)$, defined in the Φ-module $\mathrm{ad}^2_a L$ by the product $x \bullet y = [[x, b], y]]$ for all $x, y \in \mathrm{ad}^2_a L$.*

PROOF. The map $\varphi : L_a \to J(a, b)$, defined by $\varphi(\overline{x}) := -A^2x$, where $\overline{x}$ denotes the coset of x in L_a, is an algebra-isomorphism. Clearly, φ is a linear isomorphism, and since both algebras are commutative and $\frac{1}{2} \in \Phi$, it suffices to check that $\varphi(\overline{x})^2 = \varphi(\overline{x}^2)$ for every $x \in L$, which follows from the definitions of the involved products:

$$\varphi(\overline{x})^2 = [[A^2x, b], A^2x] = -\mathrm{ad}^2_{A^2x} b = -A^2X^2A^2b = A^2X^2a = -A^2XAx = \varphi(\overline{x}^2),$$

where we have used the identity $XAx = [x, [a, x]] = -X^2a$. $\square$

Division elements of Lie algebras. By a *division element* of a Lie algebra L we mean a nonzero Jordan element $a \in L$ such that L_a is a division Jordan algebra.

PROPOSITION 8.63. *Let L be a nondegenerate Lie algebra and let $a \in L$ be a Jordan element. Then a is a division element of L if and only if $\mathrm{ad}^2_a L$ is a minimal inner ideal.*

PROOF. Since L_a is nondegenerate (Proposition 8.51(i)), it suffices to prove, by the inner characterization of division Jordan algebras (Proposition 8.20), that L_a contains no nontrivial inner ideal if and only if the principal inner ideal $\mathrm{ad}^2_a L$ is minimal, but this follows from Lemma 8.50(ii). $\square$

Extremal elements revisited. In this subsection, L denotes a Lie algebra over a field $\mathbb{F}$ of characteristic not 2 or 3. *By an extremal element we will mean here one which is not an absolute zero divisor, i.e. a nonzero element $e \in L$ such that $\mathrm{ad}^2_e L = \mathbb{F}e$.*

PROPOSITION 8.64. *A nonzero Jordan element $e \in L$ is extremal if and only if the Jordan algebra L_e is one-dimensional.*

PROOF. It follows from the vector space isomorphism $\overline{x} \mapsto \mathrm{ad}^2_e x$ from L_e onto $\mathrm{ad}^2_e L$ (Lemma 8.50(i)). $\square$

COROLLARY 8.65. *Let L be algebraic over an algebraically closed field. Then every division element of L is extremal.*

PROOF. Let $b \in L$ be a division element. By Proposition 8.51(5), L_b is algebraic and hence, by Proposition 8.17, one-dimensional, so b is an extremal element by Proposition 8.64. □

Primitive Lie algebras. As we have seen in Proposition 8.37, a Jordan algebra J is primitive if and only if J is strongly prime and J_a is primitive for some $0 \neq a \in J$. In this subsection we take this local approach to define a notion of primitivity for Lie algebras. All the algebras considered here are over a ring of scalars Φ in which 6 is invertible.

DEFINITION 8.66. Let L be a Lie algebra and let $0 \neq a \in L$ be a Jordan element. Then L is called *primitive at a* if L is strongly prime and the Jordan algebra L_a is primitive. By a *primitive Lie algebra* we will mean a primitive Lie algebra L at some nonzero Jordan element.

PROPOSITION 8.67. *Let I be a nonzero ideal of a Lie algebra L. If L is primitive, then so is I. Conversely, if L is strongly prime and I is primitive, then L is primitive.*

PROOF. Let L be primitive. By Corollary 4.9, I is strongly prime. Let a be a nonzero Jordan element of L such that L_a is primitive. By Proposition 4.10, there exists $y \in I$ such that $b = [[a, y], a]$ is a nonzero *Jordan* (4.4(4)) element of I. Since I_b can be regarded as an ideal of the Jordan algebra $L_b \cong (L_a)_{\overline{y}}$ (Proposition 8.49), it follows from Proposition 8.36 that I_b is a primitive Jordan algebra, so I is primitive.

Conversely, suppose that L is strongly prime and I primitive at some nonzero Jordan element $b \in I$. Then b is as a Jordan element of L (Proposition 4.13), I_b is a nonzero ideal of L_b, and L_b is strongly prime (Proposition 8.51(2)). Hence it follows from Proposition 8.36 that L_b is primitive, so L is primitive. □

As mentioned in Proposition 8.37, in any strongly prime Jordan algebra, primitivity of one its local algebras implies that of the global algebra, and therefore of all its local algebras. Since at present we don't know whether a similar result holds for Lie algebras, we will call those Lie algebras enjoying this property *strongly primitive.*

PROPOSITION 8.68. *Let L be a strongly prime Lie algebra with nonzero socle. Then L is strongly primitive.*

PROOF. Let a be a nonzero Jordan element of L (whose existence is guaranteed by Lemma 4.5). By Proposition 8.51(3), L_a is a strongly prime Jordan algebra with nonzero socle. Then L_a is primitive by Example 8.35(3), which proves that L is strongly primitive. □

A partial converse of this result holds for primitive Lie PI-algebras.

PROPOSITION 8.69. *Let L be a primitive Lie PI-algebra. Then it has nonzero socle.*

PROOF. Let a be a nonzero element of L such that the Jordan algebra L_a is primitive. By Proposition 8.57, L_a is PI. So, by Proposition 8.38, L_a is simple and has finite capacity. Hence, by Lemma 8.50, L has nonzero socle. □

In the remainder of this section, R will be an associative algebra over ring of scalars Φ in which 6 is invertible, and $\overline{R}$ will stand for the Lie algebra $R^-/Z(R)$. Primitive associative algebra are understood to be left or right primitive.

PROPOSITION 8.70. *If R is primitive, then $\overline{R}$ is strongly prime and all its Jordan algebras are primitive. Conversely, if R is prime and $\overline{R}$ is primitive, then R is primitive.*

PROOF. Let $\pi : R^- \to \overline{R}$ be the canonical Lie epimorphism and suppose that R is primitive. By [**Coh03**, Proposition 8.5.1], R is prime, and hence $\overline{R}$ is strongly prime (Proposition 3.35(3)). Let $\pi(a)$ be a nonzero Jordan element of $\overline{R}$, i.e. $a \in R \setminus Z(R)$ and $\mathrm{ad}_a^3 R \subset Z(R)$. It follows from Proposition 3.35(1) that there exists $x \in R$ such that $b = [[a, x], a]$ is nilpotent of index 2. Since R_b is primitive [**ACM95**, Proposition 0.1(iii)], and therefore so is the Jordan algebra R_b^+ (Example 8.35(1)), we have by Lemma 8.45 that $\overline{R}_{\pi(b)} \cong R_b^+$ is primitive.

Conversely, suppose that R is prime and $\overline{R}$ is primitive. Let $\pi(a)$ be a nonzero Jordan element of $\overline{R}$ such that the Jordan algebra $\overline{R}_{\pi(a)}$ is primitive. As before, there exists $x \in R$ such that $b = [[a, x], a]$ is nilpotent of index 2. Now it follows from Lemma 8.45, Proposition 8.49, and Proposition 8.37(1) that the Jordan algebra $R_b^+ \cong \overline{R}_{\pi(b)} \cong (\overline{R}_{\pi(a)})_{\overline{\pi(x)}}$ is also primitive, which implies by Example 8.35(1) that R_b is primitive. Then R is primitive by [**AC98**, Theorem 1.1]. □

COROLLARY 8.71. *If R is primitive and contains nonzero nilpotent elements, then $\overline{R}$ is strongly primitive.*

PROOF. It follows from Proposition 8.70 since R contains nilpotent elements of index 2, and therefore $\overline{R}$ has nonzero Jordan elements (Example 4.2). □

Suppose in addition that R has an involution $*$ and contains nonzero nilpotent skew-symmetric elements. Set $K = \mathrm{Skew}(R, *)$ and $\overline{K} = K/K \cap Z(R)$.

PROPOSITION 8.72. *If R is primitive, then $\overline{K}$ is primitive whenever one of the following two conditions holds:*

(i) *The involution is of the second kind.*
(ii) *R is not an order in a simple ring of dimension less than* 16 *over its center.*

PROOF. $\overline{K}$ is strongly prime by Propositions: 3.36 if (i), and 3.39 if (ii). Choose a nonzero element $a \in K$ such that $a^2 = 0$. Denote by π the canonical epimorphism of R^- onto $\overline{R}$ and its corresponding restriction of K onto $\overline{K}$. By [**ACM95**, Proposition 0.1] and Example 8.35(2), the Jordan algebra $\mathrm{Sym}(R_a, *)$ is primitive. Since $\overline{R}_{\pi(a)} \cong R_a^+$ by Lemma 8.45, and hence $\overline{K}_{\pi(a)} \cong \mathrm{Sym}(R_a, *)$, we have that $\overline{K}_{\pi(a)}$ is primitive, which proves that $\overline{K}$ is primitive. □

8.3. Extremal elements and finitary Lie algebras

The results of this section are taken from [**FL16**]. Unless specified otherwise, $\mathbb{F}$ will stand for an arbitrary field. Given a vector space X over $\mathbb{F}$, we denote by $\mathcal{L}(X)$ the associative $\mathbb{F}$-algebra of all linear maps of X, and by $\mathcal{F}(X)$ the ideal of $\mathcal{L}(X)$ consisting of all finite rank linear maps. As in the above subsection, by an extremal element we will mean one which is not an absolute zero divisor.

DEFINITION 8.73. A Lie $\mathbb{F}$-algebra L is called *finitary* if it is isomorphic to a subalgebra of the Lie algebra $\mathfrak{f}gl(X) = \mathcal{F}(X)^-$ for some vector space X over $\mathbb{F}$.

PROPOSITION 8.74. *Every finitary Lie algebra $L \leq \mathcal{F}(X)^-$ is locally finite.*

PROOF. It is enough to show that the associative algebra $\mathcal{F}(X)$ is locally finite. But this is a direct consequence of Lifoff' theorem [**BMM96**, Theorem 4.3.11]. □

Let L be a simple infinite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. According to Baranov's structure theorem [**Bar99**, Corollary 1.2], L is finitary if and only if it is isomorphic to one of the following (see Section 2.2 for notation): $\mathfrak{f}sl_Y(X)$, $\mathfrak{f}o(X, \langle\,,\rangle)$, or $\mathfrak{f}sp(X, \langle\,,\rangle)$. As mentioned in Example 6.2 (or in [**FLGGL04**, Proposition 6.4]), each one of these Lie algebras contains an extremal element. In fact, they are spanned by extremal elements. In this section we provide a classification-free proof of this result by using Jordan theory instead of representation theory.

• For any $a \in \mathcal{F}(X)$, $\mathrm{rank}(a) := \dim(Xa)$ denotes the rank of a. The following properties of the rank are immediate. Let $a, b \in \mathcal{F}(X)$ and $c \in \mathcal{L}(X)$. Then:

(i) $\mathrm{rank}(a+b) \leq \mathrm{rank}(a) + \mathrm{rank}(b)$,
(ii) $\max\{\mathrm{rank}(ac), \mathrm{rank}(ca)\} \leq \mathrm{rank}(a)$,
(iii) $\mathrm{rank}([a,c]) \leq 2\,\mathrm{rank}(a)$.

LEMMA 8.75. *Let R be an associative $\mathbb{F}$-algebra and let a, b be algebraic elements of R such that $ab = ba$. Then for any $c \in \langle a, b\rangle$, we have that c is algebraic with $\deg(c) \leq \deg(a)\deg(b)$.*

PROOF. Let $r = \deg(a)$ and $s = \deg(b)$. Then $\dim_{\mathbb{F}}\langle a, b\rangle \leq rs$. □

LEMMA 8.76. *Let $a \in \mathcal{F}(X)$ with $\mathrm{rank}(a) = r > 0$. We have:*

(i) *a is algebraic with $\deg(a) \leq r^2 + 1$.*
(ii) *l_a and r_a are algebraic with $\max\{\deg(l_a), \deg(r_a)\} \leq r^2 + 1$.*
(iii) *ad_a is algebraic with $\deg(\mathrm{ad}_a) \leq (r^2+1)^2$.*

PROOF. (i) The restriction $\hat{a}$ of a to the r-dimensional subspace Xa is algebraic with $\deg(\hat{a}) \leq r^2$, so a is algebraic with $\deg(a) \leq r^2 + 1$.

(ii) For any associative $\mathbb{F}$-algebra R, the map $x \mapsto l_x$ (resp. $x \mapsto r_x$) is a homomorphism (resp. anti-homomorphism) of R into $\mathcal{L}(A)$. Hence, by (i), both l_a and r_a are algebraic of degree less than or equal to $r^2 + 1$.

(iii) Since $[l_a, r_a] = 0$, $\deg(\mathrm{ad}_a) = \deg(l_a - r_a) \leq (r^2+1)^2$ by Lemma 8.75. □

PROPOSITION 8.77. *Let $L \leq \mathfrak{f}gl(X)$ be a finitary Lie algebra over a field $\mathbb{F}$ of characteristic different from 2 and 3, and let $a \in L$ be a Jordan element. Then L_a is algebraic of bounded degree.*

PROOF. Let $\mathrm{rank}(a) = r$. By Proposition 8.51(4), for each $c \in L$ and any integer $n \geq 1$, $\overline{c}^n = \overline{\mathrm{ad}^{n-1}_{[c,a]}\, c}$. Then, by Lemma 8.76(iii), we have that $\overline{c}$ is algebraic with $\deg(\overline{c}) \leq (4r^2+1)^2 + 1$. This proves that the Jordan algebra L_a is algebraic of bounded degree. □

THEOREM 8.78. *Let L be a nondegenerate finitary Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then L contains extremal elements.*

PROOF. By Lemma 8.76(iii), the adjoint of every $a \in L$ is algebraic, so L contains a nonzero Jordan element by Corollary 4.32. Let $a \in L$ be a nonzero Jordan element. By Propositions 8.51(1) and 8.77, L_a is nondegenerate and algebraic of bounded degree. Thus L_a has finite capacity (Theorem 8.41). Let $\overline{e}$ be a division idempotent of L_a. Since $\mathbb{F}$ is algebraically closed, L_e is one-dimensional, so e is an extremal element of L by Proposition 8.64. □

COROLLARY 8.79. *Let L be a simple finitary Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then L is spanned by its extremal elements.*

PROOF. Any simple Lie algebra over a field of characteristic 0 is nondegenerate (Corollary 3.24), so L contains extremal elements by Theorem 8.78. Let $e \in L$ be an extremal element. By Proposition 6.17, L has a nontrivial finite grading and therefore it is generated by ad-nilpotent elements. Thus any subspace of L which is invariant under automorphisms is an ideal. In particular, the linear span of the extremal elements of L is a nonzero ideal and therefore equal to L. □

8.4. Clifford elements

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic not 2 or 3. A Jordan element $a \in L$ is called a *Clifford element of L* if L_a is a Clifford Jordan algebra (see Example 8.4 for definition).

DEFINITION 8.80. Let R be a centrally closed prime ring of characteristic different from $2, 3$ with involution $*$, and let K be the Lie algebra of its skew-symmetric elements. By a *Clifford element of R* we mean an element $c \in K$ such that

$$c^3 = 0,\ c^2 \neq 0 \ \text{and}\ c^2kc = ckc^2 \ \text{ for all }\ k \in K.$$

Any Clifford element of R is a Jordan element of K: for any $k \in K$ we have $\mathrm{ad}_c^3 k = c^3k - 3c^2kc - 3ckc^2 - kc^3 = 0$. Conversely, if c is a Jordan element of K such that $c^3 = 0$, then $c^2kc = ckc^2$ since $\mathrm{char}(R) \neq 3$. Note also that by Lemma 4.42, if K is not abelian and $*$ is of the first kind, then any Jordan element of K is of cube 0.

Following [**BFLL17**], we prove in this section that every Clifford element c of the ring R is a Clifford element of the Lie algebra K.

EXAMPLE 8.81. [**FLGGL06**, Lemma 3.7] Following the notation of Exercise 2.99, let X be a vector space of dimension greater than 2 with a nondegenerate symmetric bilinear form $\langle\, ,\rangle$ over a field $\mathbb{F}$ of characteristic not 2, let $H = \mathbb{F}x \oplus \mathbb{F}y$ be a hyperbolic plane of X: $\langle x, x\rangle = \langle y, y\rangle = 0$, $\langle x, y\rangle = 1$, let $z \in H^\perp$, and set $c := [x, z] = x^*z - z^*x$. We have:

(i) $c^3 = 0$, with $c^2 = 0$ if and only if the vector z is isotropic.
(ii) If z is anisotropic, then c is a Clifford element of the ring $\mathcal{L}_X(X)$ with the adjoint involution, and $\mathrm{ad}_c^2\, \mathfrak{fo}(X, \langle\, ,\rangle) = [x, H^\perp]$.

PROOF. (i) $c^2 = (x^*z - z^*x)^2 = -x^*\langle z, z\rangle x = 0$ if and only if z is isotropic, and $c^3 = -\langle z, z\rangle x^*x(x^*z - z^*x) = 0$, since x, y are isotropic and $z \in H^\perp$.

(ii) Suppose that $\alpha := -\langle z, z\rangle \neq 0$ (the existence of such a vector z is guaranteed since $\dim_{\mathbb{F}} X > 2$ and $\langle\, ,\rangle$ is nondegenerate), and let $a \in \mathfrak{o}(X, \langle\, ,\rangle)$. We have:

$$c^2ac - cac^2 = \alpha((x^*x)ac - ca(x^*x)) = 0, \tag{8.8}$$

since

$$\begin{aligned}(x^*x)ac &= (x^*x)a(x^*z - z^*x) = -x^*\langle xa, z\rangle x = x^*\langle za, x\rangle x \\ &= (x^*z - z^*x)a(x^*x) = ca(x^*x),\end{aligned}$$

because $\langle xa, x\rangle = \langle x, xa^*\rangle = -\langle x, xa\rangle = -\langle xa, x\rangle$ implies $\langle xa, x\rangle = 0$. This, together with (i), proves that c is a Clifford element of $\mathcal{L}_X(X)$.

Now we show that for every $v \in H^\perp$ there exists $a \in \mathfrak{f}o(X, \langle\, ,\rangle)$ such that $\mathrm{ad}_c^2\, a = [x, v]$. For any $a \in \mathfrak{o}(X, \langle\, ,\rangle)$, we have

$$\begin{aligned}\mathrm{ad}_c^2\, a &= c^2a - 2cac + ac^2 = \alpha x^*xa + 2\langle xa, z\rangle[x, z] - \alpha(xa)^*x \\ &= \alpha[x, xa] + 2\langle xa, z\rangle[x, z].\end{aligned} \tag{8.9}$$

Taking $a = [y, \lambda z + \mu v] \in \mathfrak{f}o(X, \langle\, ,\rangle)$, $\lambda, \mu \in \mathbb{F}$, in (8.9), we get

$$\mathrm{ad}_c^2\, a = (2\mu\langle v, z\rangle - \alpha\lambda)[x, z] + \alpha\mu[x, v] = [x, v]$$

for $\mu = \alpha^{-1}$ and $\lambda = 2\alpha^{-2}\langle v, z\rangle$, which completes the proof. □

REMARK 8.82. As will be shown in Proposition 13.53, every Clifford element of the ring $\mathcal{L}_X(X)$ is actually one of those described in the example above. Note also that in this example it is proved that $[x, H^\perp]$ is an abelian inner ideal of $\mathfrak{o}(X, \langle\, ,\rangle)$ contained in $\mathfrak{f}o(X, \langle\, ,\rangle)$ (question proposed in Exercise 2.99(iii)). For another proof of this fact, using (8.9) one can prove that $[x, xa] \in [x, H^\perp]$ as follows: Write $xa = \beta x + \gamma y + w$ according to the decomposition $X = H \oplus H^\perp$. From $\langle xa, x\rangle = 0$, we get $\beta = 0$ and hence $[x, xa] = [x, \beta x + w] = [x, w] \in H^\perp$.

The square of a Clifford element.

PROPOSITION 8.83. *Let R be a centrally closed prime ring of characteristic different from $2, 3$ with involution $*$, denote by $\mathcal{C}$ the extended centroid of R (equals its centroid), and let $c \in K$ be a Clifford element of R. We have:*

(1) $c^2Kc^2 = 0$.
(2) $(c^2xc^2)^* = c^2x^*c^2 = c^2xc^2$ *for all* $x \in R$.
(3) $c^2Rc^2 = \mathcal{C}c^2$, *so* Rc^2 *is a minimal left ideal of* R.
(4) *The involution $*$ is of the first kind.*
(5) *R is a subring of $\mathcal{L}_(X)$ containing $\mathcal{F}_X(X)$ where X is a vector space with a nondegenerate symmetric bilinear form over the field $\mathcal{C}$.*
(6) *Regarded c^2 as an element of $\mathcal{L}_X(X)$,* $\mathrm{rank}(c^2) = 1$

PROOF. (1) $c^2kc^2 = c(ckc^2) = c(c^2kc) = c^3kc = 0$.

(2) By (1), $c^2(x - x^*)c^2 = 0$ and hence $c^2xc^2 = c^2x^*c^2 = (c^2xc^2)^*$.

(3) Let $x, y \in R$. Since c^2 is symmetric it follows from (2) that

$$(c^2xc^2)yc^2 = c^2x^*c^2y^*c^2 = c^2(yc^2x)^*c^2 = c^2y(c^2xc^2),$$

with $c^2 \neq 0$. Then, by Martindale's Lemma [**BMM96**, Theorem 2.3.4], for each $x \in R$ there is a $\lambda_x \in \mathcal{C}$ such that $c^2xc^2 = \lambda_x c^2$. Since $c^2 \neq 0$ and R is prime, $c^2Rc^2 \neq 0$ and hence $c^2Rc^2 = \mathcal{C}c^2$, since $\mathcal{C}$ is a field.

(4) By (3), given $\alpha \in \mathcal{C}$ there exists $a \in R$ such that $\alpha c^2 = c^2ac^2$. Then, by (2), $\alpha^*c^2 = c^2a^*c^2 = c^2ac^2 = \alpha c^2$, so $\alpha^* = \alpha$, proving that $*$ is of the first kind.

(5) By (3), $c^2 = c^2ac^2$ for some $a \in R$ and hence $c^2R = eR$, where $e = c^2a$ is an idempotent of R. Then $eRe = c^2Rc^2a = \mathcal{C}c^2a = \mathcal{C}e$, which proves by Exercise 2.96 that eR is a minimal right ideal of R, so R has nonzero socle with associated

division ring isomorphic to the field $\mathcal{C}$ ([**BMM96**, Theorem 4.3.7]). Now it follows from Kaplansky's Theorem (2.31) that the involution $*$ of R is either of transpose type or of symplectic type; but the latter cannot occur because c^2 is a symmetric rank-one linear map, so $*$ is of transpose type.

(6) By Lemma 13.40, c^2R is a minimal right ideal of R, so $\mathrm{rank}(c^2) = 1$. □

Let c be a Clifford element of R. Since c^2 is symmetric, of square zero and von Neumann regular (Proposition 8.83(3)), we have by Lemma 5.17 that there exists $d \in R$ such that $d^* = d$, $d^2 = 0$, $c^2dc^2 = c^2$ and $d = dc^2d$. Such an element d will be called a *regular partner* of c^2. Note that then $e := dc^2$ is a $*$*-orthogonal idempotent*, i.e. $e^2 = e$ and $ee^* = e^*e = 0$.

Proposition 8.84. *Let c be a Clifford element of R, let d be a regular partner of c^2, and put $e = dc^2$. We have:*

(1) $dKd = 0$.
(2) $dRd = \mathcal{C}d$.
(3) $eRe = \mathcal{C}e$, $e^*Re = \mathcal{C}c^2$, $eRe^* = \mathcal{C}d$ *and* $eKe^* = e^*Ke = 0$.
(4) $ec = ce^* = 0$, $e^*c^2 = c^2e = c^2$ *and* $de^* = ed = d$.
(5) $[K, K] \neq 0$.
(6) $e + e^* \neq 1$ *in the unital hull* $\hat{R} = \mathcal{C}1 + R$ *of* R.

Proof. Note that by Proposition 8.83(3), $c^2Mc^2 = \mathcal{C}c^2$ for any subspace M of R such that $c^2Mc^2 \neq 0$. A fact which will be used in what follows without further mention.

(1) $dKd = dc^2(dKd)c^2d = 0$, where we have used Proposition 8.83(1) and the fact that dkd is skew-symmetric for every $k \in K$.

(2) Similarly, we have

$$dRd = (dc^2d)R(dc^2d) = dc^2(dRd)c^2d = d\mathcal{C}c^2d = \mathcal{C}dc^2d = \mathcal{C}d,$$

since $c^2dc^2 = c^2$ and $dc^2d = d$ imply $c^2(dRd)c^2 \neq 0$.

(3) $eRe = dc^2(Rd)c^2 = d\mathcal{C}c^2 = \mathcal{C}e$, since $c^2 = c^2(dc^2d)c^2 \in c^2(Rd)c^2$ and therefore the latter is not zero. In a similar way it is proved that $e^*Re = \mathcal{C}c^2$ and $eRe^* = \mathcal{C}d$. Now $eKe^* = dc^2Kc^2d = 0$ by 8.83(1), and $e^*Ke = 0$ is obtained in a similar way.

(4) The identities of this item follow straightforwardly from the very definition of e.

(5) By (4), $[c, e - e^*] = ce + e^*c = cdc^2 + c^2dc \neq 0$. Otherwise $cdc^2 = -c^2dc$ would lead to the contradiction $c^2 = c^2dc^2 = -c^3dc = 0$. Since $[c, e - e^*] \in [K, K]$, $[K, K] \neq 0$.

(6) It follows from (3) and (4) that $(e + e^*)c(e + e^*) = 0$, so $e + e^* \neq 1$. □

As we have seen in the above proposition, any Clifford element c of the ring R gives rise to two nonzero orthogonal elements e and e^* associated to any regular partner d of c^2. Furthermore, the idempotent $e + e^*$ is no complete (8.84(6)), i.e. the symmetric idempotent $g := 1 - e - e^*$ in the unital hull $\hat{R} = \mathcal{C}1 + R$ of R is not zero. We next prove that the complete system $\{e, e^*, g\}$ induces a 3-grading in the Lie algebra K.

- For any nonempty subset S of R, set $\kappa(S) = \{x - x^* : x \in S\}$.

PROPOSITION 8.85. *Let c be a Clifford element of R, $e = dc^2$ and $g = 1-e-e^*$, where d is a regular partner of c^2. Then $K = K_{-1} \oplus K_0 \oplus K_1$ is a 3-grading of K, with $K_{-1} = \kappa((1-e)Ke) = \kappa((1-e)Re) = \kappa(gRe)$, $\mathrm{K}_0 = \kappa(eRe) \oplus gKg$ and $K_1 = \kappa(eK(1-e)) = \kappa(eR(1-e)) = \kappa(eRg)$.*

PROOF. Consider the complete system $\{e_0 = e^*,\ e_1 = g,\ e_2 = e\}$ of orthogonal idempotents of $\hat{R}$ and set $R_i = \oplus_{m-n=i} e_m R e_n$, $-2 \leq i \leq 2$. Then (see [**Smi97**, p.174] or Exercise 1.38), $R = \oplus_{-2\leq i \leq 2} R_i$ is an (associative) 5-grading of R. Explicitly,

$$R = e^*Re \oplus (e^*Rg \oplus gRe) \oplus (e^*Re^* \oplus gRg \oplus eRe) \oplus (gRe^* \oplus eRg) \oplus eRe^*.$$

Since all the components R_i are $*$-invariant subspaces, $K = \oplus_{-2\leq i\leq 2} K_i$, where $K_i = R_i \cap K = \mathrm{Skew}(R_i, *)$ for each index i and $[K_i, K_j] \subset \mathrm{Skew}[R_i, R_j] \cap [K, K] \subset R_{i+j} \cap K = K_{i+j}$. Thus $K = \oplus_{-2\leq i\leq 2} K_i$ is (a priori) a 5-grading of the Lie algebra K. But $K_{-2} = \kappa(e^*Re) = e^*\kappa(R)e = e^*Ke = 0$ and similarly $K_2 = e^*Ke = 0$. Moreover, the i-th homogenous component k_i of any $k \in K$ coincides with $\oplus_{m-n=i} \kappa(e_m k e_n)$, so $k \in K_{-1}$ if and only if

$$\begin{aligned} gke + e^*kg &= (1-e-e^*)ke + e^*k(1-e-e^*) = (1-e)ke + e^*k(1-e^*) \\ &= (1-e)ke - ((1-e)ke)^* = \kappa((1-e)ke), \end{aligned}$$

since $e^*Ke = 0$ by Proposition 8.84(3), which proves $K_{-1} = \kappa(gRe) = \kappa((1-e)Ke)$. Similarly, $K_1 = \kappa(eRg) = \kappa(eK(1-e))$. Therefore

$$K = \kappa((1-e)Ke) \oplus (\kappa(eRe) \oplus gKg) \oplus \kappa(eK(1-e))$$

is a 3-grading of K. Now, for any $x \in R$,

$$\kappa(gxe) = \kappa((1-e)xe) - \kappa(e^*xe) = \kappa((1-e)xe) - e^*\kappa(x)e = \kappa((1-e)xe)$$

since $e^*\kappa(x)e \in e^*Ke = 0$, which proves that $K_{-1} = \kappa((1-e)Re)$. Similarly we obtain that $K_1 = \kappa(eR(1-e))$. □

Although the 3-grading of K has been defined by choosing a regular partner d of c^2, it will be seen now that the component K_{-1} only depends on the Clifford element c.

- Set $x \circ y = xy + yx$ for $x, y \in R$.

PROPOSITION 8.86. *Let c be a Clifford element of R, let $e = dc^2$, where d is a regular partner of c^2, and set $B = \kappa((1-e)Ke)$. We have:*

(1) *If $b \in B$ then $eb = 0$ and $b = e^*b + be = \kappa((1-e)be)$.*
(2) *$B = c^2 \circ K$.*
(3) *$c = e^*c + ce = c^2dc + cdc^2$.*
(4) *$c \in B$.*
(5) *$cKc = \mathcal{C}c$.*

PROOF. (1) Let $b = (1-e)ke + e^*k(1-e^*) \in B$. Then

$$eb = e((1-e)ke + e^*k(1-e^*)) = 0 \quad \text{and} \quad e^*b = e^*k(1-e^*),$$

since $e^*e = 0$ and $e^*Ke = 0$. By symmetry, we also have that $be^* = 0$ and $be = (1-e)ke$. Hence $b = e^*b + be = e^*b(1-e^*) + (1-e)be = \kappa((1-e)be)$.

(2) It follows directly from the definitions:

$$c^2 \circ k = \kappa(kc^2) = \kappa(ke^*c^2 - (eke^*)c^2) = \kappa((1-e)k(e^*c^2))$$
$$= \kappa((1-e)k(c^2e)) \in \kappa((1-e)Ke) = B.$$

Conversely, let $b \in B$. Then

$$b = e^*b + be = (c^2d)b + b(dc^2) = c^2(d \circ b) + (d \circ b)c^2 = c^2 \circ (d \circ b) \in c^2 \circ K,$$

since $e^* = c^2d$, $c^2 = c^2e$ and $c^2b = (c^2e)b = c^2(eb) = 0$.

(3) Set $z := c - c^2dc - cdc^2$. We must prove that $z = 0$. For any $k \in K$ we have

$$c^2kz = c^2kc - (c^2kc^2)dc - (c^2kc)dc^2 = ckc^2 - ck(c^2dc^2) = ckc^2 - ckc^2 = 0$$

since $c^2kc = ckc^2$ and $c^2kc^2 = 0$ by Proposition 8.83(1), and d is a regular partner of c^2. We also have $zkc^2 = (c^2kz)^* = 0$, and hence $c^2xz = c^2x^*z$ and $zxc^2 = zx^*c^2$ for every $x \in R$. Let $x, y \in R$. Then $c^2\kappa(xzy)c^2 = 0$ since $c^2Kc^2 = 0$. Thus

$$0 = c^2(xzy + y^*zx^*)c^2 = c^2xzyc^2 + c^2y^*zx^*c^2$$
$$= c^2xzyc^2 + c^2yzxc^2 = (c^2xz)y(c^2) + (c^2)y(zxc^2),$$

with $c^2 \neq 0$. By Martindale's Lemma, for every $x \in R$ there is $\lambda_x \in \mathcal{C}$ such that $c^2xz = \lambda_x c^2$. But

$$z(1-e) = (c - e^*c - ce)(1-e) = c - ce - e^*c + e^*ce - ce + ce = c - ce - e^*c = z$$

since $e^*ce \in e^*Ke = 0$. Hence $c^2xz = c^2xz(1-e) = \lambda_x c^2(1-e) = 0$, so $c^2Rz = 0$. This implies $z = 0$ because R is prime and $c^2 \neq 0$. Thus

$$c = e^*c + ce = c^2dc + cdc^2$$

as desired.

(4) By (3), $c = c^2dc + cdc^2 = c^2(dc + cd) + (dc + cd)c^2 \in c^2 \circ K = B$ by (2).

(5) Note that $cd + dc = \kappa(cd) \in K$ and $c(cd + dc)c = c^2dc + cdc^2 = c$ by (3). Hence $\mathcal{C}c \subset cKc$. Conversely, for any $k \in K$ we have

$$ckc = (e^*c + ce)k(e^*c + ce) = e^*cke^*c + cekce,$$

since $eKe^* = 0$ by Proposition 8.84(3) and $ckc \in K$. Now, again by 8.84(3), $ekce = \lambda e$ for some $\lambda \in \mathcal{C}$, and hence $e^*cke^* = (ekce)^* = (\lambda e)^* = \lambda e^*$ because the involution $*$ is of the first kind by Proposition 8.83(4). Then $ckc = \lambda e^*c + \lambda ce = \lambda c$, which completes the proof. ☐

The square root of d. Given a Clifford element c of R and a regular partner d of c^2, put $\sqrt{d} := cd + dc$. As will be seen now, the square-root notation is absolutely justified.

Proposition 8.87. *Let c be a Clifford element of R and let d be a regular partner for c^2. For any $b \in B = \kappa((1-e)Ke)$, we have:*

(1) *$\sqrt{d} \in K_1$ in the 3-grading of Proposition 8.85. In particular, $\sqrt{d}$ is a Jordan element.*
(2) *$(\sqrt{d})^2 = d$.*
(3) *$(\sqrt{d})^3 = 0$.*
(4) *$\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.*
(5) *$\sqrt{d}c\sqrt{d} = \sqrt{d}$.*
(6) *$c\sqrt{d}c = c$.*
(7) *$c^2 \circ \sqrt{d} = c$.*
(8) *$d \circ c = \sqrt{d}$.*
(9) *$[[c, \sqrt{d}], c] = c$.*
(10) *$[[\sqrt{d}, c], \sqrt{d}] = \sqrt{d}$.*
(11) *$[[c, \sqrt{d}], b] = b$.*

PROOF. (1) Note first that $\sqrt{d} = cd + dc \in K$, since $c \in K$ and $d \in H$. Then

$$\begin{aligned}\kappa(e\sqrt{d}(1-e)) &= e(cd+dc)(1-e) + (1-e^*)(dc+cd)e^* \\ &= edc(1-e) + (1-e^*)cde^* = edc - edce + cde^* - e^*cde^* \\ &= (dc^2d)c - e(dcd)c^2 + c(dc^2d) - c^2(dcd)e^* = dc + cd = \sqrt{d},\end{aligned}$$

because $ec = 0$, $e = dc^2$, $dc^2d = d$ and $dcd \in dKd = 0$. We have thus proved (Proposition 8.85) that $\sqrt{d} \in \kappa(eK(1-e)) = K_1$. As K_1 is an abelian inner ideal (because it is an extreme of a finite grading), $\sqrt{d}$ is a Jordan element of K.

(2) $(\sqrt{d})^2 = (cd+dc)(cd+dc) = c(dcd) + cd^2c + dc^2d + (dcd)c = dc^2d = d$.

(3) $(\sqrt{d})^3 = (\sqrt{d})^2\sqrt{d} = d(cd+dc) = dcd + d^2c = 0$.

(4) If follows from (1), (2) and (3) that $\sqrt{d}$ is a Clifford element of R. Hence, by Proposition 8.86(5), $\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.

(5) It follows directly from the definition of regular partner:

$$\begin{aligned}&\sqrt{d}c\sqrt{d} = (cd+dc)c(cd+dc) \\ &= c(dc^2d) + c(dcd)c + dc^3d + (dc^2d)c = cd + dc = \sqrt{d}.\end{aligned}$$

(6) $c\sqrt{d}c = c(cd+dc)c = c^2dc + cdc^2 = c$, by 8.86(3).

(7) $c^2 \circ \sqrt{d} = c^2(cd+dc) + (cd+dc)c^2 = c^2dc + cdc^2 = c$.

(8) $d \circ c = dc + cd = \sqrt{d}$.

(9) By (6) and (7),

$$[[c,\sqrt{d}],c] = 2c\sqrt{d}c - c^2 \circ \sqrt{d} = 2c - c = c.$$

(10) It follows from (2), (5) and (8):

$$[[\sqrt{d},c],\sqrt{d}] = 2\sqrt{d}c\sqrt{d} - (\sqrt{d})^2 \circ c = 2\sqrt{d} - \sqrt{d} = \sqrt{d}.$$

(11) By 8.86(1),

$$[[c,\sqrt{d}],b] = [[c,cd+dc],b] = [c^2d - dc^2, b] = [e^* - e, b] = e^*b + be = b.$$

□

Trace and bilinear form. Statement 8.86(5) can be rephrased by saying that there exists a linear form tr on the $\mathcal{C}$-vector space K such that $\mathrm{tr}(k)c = ckc$ for all $k \in K$. Since $c\sqrt{d}c = c$ by Proposition 8.87(6), we have $\mathrm{tr}(\sqrt{d}) = 1$ and hence

$$K = \mathcal{C}\sqrt{d} \oplus \ker(\mathrm{tr}).$$

Moreover, it follows from Proposition 8.83 that there exists a symmetric bilinear form $\langle\,,\rangle$ on the $\mathcal{C}$-vector space K such that $\langle k_1, k_2\rangle c^2 = c^2k_1k_2c^2$ for all $k_1, k_2 \in K$.

The theorem. Everything is now ready to prove the main result of the section: If c is a Clifford element of R, then the abelian inner ideal $c^2 \circ K = \kappa((1-e)Ke)$ (see Proposition 8.86) can be endowed with a structure of Jordan algebra of Clifford type (Example 8.4) and that this Jordan algebra is isomorphic to K_c.

PROPOSITION 8.88. *Let c be a Clifford element of R and set $B = c^2 \circ K$. We have:*

(1) $B = \mathcal{C}c \oplus X$, *where* $X := \{c^2 \circ k :\ k \in \ker(\mathrm{tr})\}$.
(2) $B = \mathrm{ad}_c^2 K$.

PROOF. (1) We have seen that $K = \ker(\mathrm{tr}) \oplus \mathcal{C}\sqrt{d}$. Hence

$$B = c^2 \circ K = c^2 \circ \ker(\mathrm{tr}) + \mathcal{C}c^2 \circ \sqrt{d} = c^2 \circ \ker(\mathrm{tr}) + \mathcal{C}c$$

because $c^2 \circ \sqrt{d} = c$ by Proposition 8.87(7), where the sum $c^2 \circ \ker(\mathrm{tr}) + \mathcal{C}c$ is direct since $c^2 \circ k_0 = \alpha c$, with $\mathrm{tr}(k_0) = 0$ and $\alpha \in \mathcal{C}$, implies

$$\alpha c^2 = c(c^2 k_0 + k_0 c^2) = (ck_0c)c = \mathrm{tr}(k_0)c = 0.$$

and hence $\alpha = 0$ because $c^2 \neq 0$ by the very definition of Clifford element.

(2) For any $k \in K$ we have $\mathrm{ad}_c^2 k = c^2 k - 2ckc + kc^2 = c^2 \circ k - 2\,\mathrm{tr}(k)c \in B$. Conversely, let $c^2 \circ k_0 + \alpha c \in B$, with $k_0 \in \ker(\mathrm{tr})$ and $\alpha \in \mathcal{C}$. Then

$$c^2 \circ k_0 + \alpha c = \mathrm{ad}_c^2 k_0 - \alpha\, \mathrm{ad}_c^2 \sqrt{d} = \mathrm{ad}_c^2(k_0 - \alpha\sqrt{d}),$$

since $ck_0c = 0$ and $\mathrm{ad}_c^2 \sqrt{d} = -c$ by Proposition 8.87(9). □

LEMMA 8.89. *The formula* $\langle c^2 \circ k, c^2 \circ k' \rangle_0 := -\langle k, k' \rangle$ *defines a symmetric* $\mathcal{C}$*-bilinear form on the vector space* X.

PROOF. Suppose that $c^2 \circ k_1 = c^2 \circ k_1'$. By multiplying the two members of this equality on the right by $k_2 c^2$ we obtain $c^2 k_1 k_2 c^2 = c^2 k_1' k_2 c^2$ since $c^2 K c^2 = 0$. This proves that $\langle\, ,\rangle_0$ is well defined. □

REMARK 8.90. Consider the 3-grading $K = K_{-1} \oplus K_0 \oplus K_1$ yielded by the idempotent $e = dc^2$ (Proposition 8.85(3)), with

$$K_{-1} = B,\ K_0 = \kappa(eKe) \oplus gKg, \ \text{ and }\ K_1 = \kappa(eKg).$$

(1) Since the pair $(d, \sqrt{d})$ plays a role symmetric to that played by (c^2, c), we also have that $K_1 = d \circ K = \{d \circ k : \ k \in K, \sqrt{d}k\sqrt{d} = 0\} \oplus \mathcal{C}\sqrt{d} = ad^2_{\sqrt{d}} K$. (2) X could be zero in Proposition 8.88; in this case, we would have $B = \mathcal{C}c$. But this case can only happen if R is 9-dimensional over $\mathcal{C}$. Let $X := H \oplus \mathbb{F}z$ be the orthogonal sum of a hyperbolic plane $H = \mathbb{F}x \oplus \mathbb{F}y$ and the line $\mathbb{F}z = H^\perp$, with z being an anisotropic vector, and let R be the simple ring $\mathcal{L}(X)$ with the adjoint as involution. Following the notation of Example 8.81, the linear map $c = x^*z - z^*x$ is a Clifford element of R such that $\mathrm{ad}_c^2 K = \mathbb{F}c$.

THEOREM 8.91. *Let R be a centrally closed ring with involution of characteristic different from $2, 3$ and let c be a Clifford element of R. Then K_c is a Clifford Jordan algebra.*

PROOF. Since $c = [[c, \sqrt{d}], c]$ (8.87(9)), by Proposition 8.62, $K_c \cong J(c, \sqrt{d})$ is the Jordan algebra defined on the $\mathcal{C}$-vector space $\mathrm{ad}_c^2 K = c^2 \circ K = \mathcal{C}c \oplus X$ (Proposition 8.88) by the product

$$(\alpha_1 c + c^2 \circ k_1) \bullet (\alpha_2 c + c^2 \circ k_2) = [[\alpha_1 c + c^2 \circ k_1, \sqrt{d}], \alpha_2 c + c^2 \circ k_2],$$

for all $\alpha_1, \alpha_2 \in \mathcal{C}$ and $k_1, k_2 \in K$ such that $ck_1c = ck_2c = 0$. Endow the $\mathcal{C}$-vector space X with the symmetric bilinear form $\langle\, ,\rangle_0$ defined in Lemma 8.89, and consider the Clifford Jordan algebra $C \oplus X$ defined by $\langle\, ,\rangle_0$ (Example 8.4). We claim that the linear isomorphism $(\alpha c + c^2 \circ k) \mapsto (\alpha, c^2 \circ k)$ of $J(c, \sqrt{d})$ onto $\mathcal{C} \oplus X$ is actually an isomorphism of Jordan algebras. Since $\frac{1}{2} \in \Phi$, it suffices to check the identity

$$(\alpha c + c^2 \circ k)^2 = [[\alpha c + c^2 \circ k, \sqrt{d}], \alpha c + c^2 \circ k] = \alpha^2 c + \langle c^2 \circ k, c^2 \circ k \rangle_0 + 2\alpha(c^2 \circ k).$$

Using the bilinearity of the Lie product reduces the checking to three products: (i) scalar by scalar, (ii) scalar by vector, and (iii) vector by vector.

(i) $[[\alpha c, \sqrt{d}], \alpha c] = \alpha^2[[c, \sqrt{d}], c] = \alpha^2 c$, by Proposition 8.87(9).

(ii) $[[\alpha c, \sqrt{d}], c^2 \circ k] = \alpha[[c, cd+dc], c^2k+kc^2] = \alpha[c^2d-dc^2, c^2k+kc^2] = \alpha(c^2 \circ k)$, where we have used $c^2dc^2 = c^2$, $c^4 = 0$ and $c^2kc^2 = c^2(dk + kd)c^2 = 0$, the latter because $c^2Kc^2 = 0$ and $(dk + kd)^* = -(kd + dk)$, since $d^* = d$ and $k^* = -k$.

(iii) $[[c^2 \circ k, \sqrt{d}], c^2 \circ k] = 2(c^2 \circ k)\sqrt{d}(c^2 \circ k) - (c^2 \circ k)^2 \circ \sqrt{d}$, with

$$(c^2 \circ k)\sqrt{d}(c^2 \circ k) = (c^2k + kc^2)(cd + dc)(c^2k + kc^2) = (c^2kdc + kc^2dc)(c^2k + kc^2) = 0$$

since $c^3 = 0$ and $ckc = 0$ ($\mathrm{tr}(k) = 0$), and

$$\begin{aligned}(c^2 \circ k)^2 \circ \sqrt{d} = c^2k^2c^2(cd + dc) + (cd + dc)c^2k^2c^2 = c^2k^2c^2dc \\ + cdc^2k^2c^2 = \langle k, k\rangle(c^2dc + cdc^2) = \langle k, k\rangle c\end{aligned}$$

since $c = c^2dc + cdc^2$ by Proposition 8.86(1). Therefore,

$$(c^2 \circ k) \bullet (c^2 \circ k) = -\langle k, k\rangle c = \langle c^2 \circ k, c^2 \circ k\rangle_0 c,$$

which completes the proof. □

REMARK 8.92. As $\sqrt{d}$ is a Clifford element of R (Proposition 8.87(1)), the above theorem also proves that $K_{\sqrt{d}}$ is a Clifford Jordan algebra.

8.5. The Kurosh problem for Lie algebras

Kurosh's problem for associative algebras. In 1941 A. G. Kurosh formulated the following Burnside-type problem for algebras: *Is any finitely generated associative nil algebra nilpotent?*

E. S. Golod showed that this is not always the case. However, the Kurosh problem has a positive solution in the class of PI-algebras. In fact, one of the high points of the theory of PI-algebras was the solution of the Kurosh problem given by I. Kaplansky [**Kap48b**], J. Levitzki [**Lev53**], and A. I. Shirshov [**Shi57**] in the following form:

THEOREM 8.93. *Let R be an associative PI-algebra generated by the elements $a_1, \ldots, a_m$. Let S be the multiplicative semigroup generated by $a_1, \ldots, a_m$. Suppose that every element of S is nilpotent. Then R is nilpotent.*

Kurosh's problem for Jordan algebras. In 1971, Shirshov posed the Jordan version of the Kurosh problem: *Must a Jordan nil algebra of bounded index be locally nilpotent?* This problem was solved affirmatively in the case of linear Jordan algebras by E. Zelmanov in [**Zel82**]. An extension of this solution to Jordan systems over an arbitrary ring of scalars has been recently obtained in [**ACZ15**].

Kurosh's problem for Lie algebras. Let L be a Lie algebra over a field $\mathbb{F}$. Recall that a subset $S \subset L$ is called a Lie set if $[a, b] \in S$ for any $a, b \in S$. For a subset $X \subset L$, the Lie set of L generated by X is denoted by $S\langle X\rangle$. The following Kurosh-type theorem for Lie algebras can be regarded as an infinite-dimensional extension of the Engel–Jacobson Theorem (2.54).

THEOREM 8.94. [**Zel17**, Theorem 1.1] *Let L be a Lie PI-algebra over an arbitrary field $\mathbb{F}$ generated by elements $a_1, \ldots, a_m$. If every element of the Lie set $S\langle a_1, \cdots, a_m\rangle$ is ad-nilpotent, then L is nilpotent.*

This Zelmanov's theorem has important implications in group theory (see [**Zel17**, Theorems 1.2 and 1.3]) extending significatively his positive solution of the Restricted Burnside Problem [**Zel90**, **Zel91**], and his work [**Zel92a**] on compact torsion groups.

The Jordan approach. As another example of the use of Jordan techniques in Lie theory, we give the proof (due also to E. Zelmanov) of Theorem 8.94 in the particular case that $\mathrm{char}(\mathbb{F}) = 0$.

Let L be a Lie algebra as in 8.94, $\mathrm{char}(\mathbb{F}) = 0$. Our first aim is to show that L contains a nonzero absolute zero divisor. Choose a nonzero element s in $\langle a_1, \cdots, a_m\rangle$ and suppose that the index of ad_s is $n > 2$. By Theorem 4.27, any $x \in \mathrm{ad}_s^{n-1} L$ is a Jordan element. Since $S\langle a_1, \cdots, a_m\rangle$ spans L, we can assume that s is a Jordan element of L.

Lemma 8.95. *The Jordan algebra L_s is McCrimmon radical, i.e. $L_s = \mathrm{Mc}(L_s)$.*

Proof. Since L is PI, the Jordan algebra L_s is also PI (Proposition 8.57). Moreover, for any $t \in S$, $[s, t]$ is ad-nilpotent. Hence, by Proposition 8.51(4), L_s is spanned by nilpotent elements. Then $L_s = \mathrm{Mc}(L_s)$ by Lemma 8.31. □

Lemma 8.96. *The Lie algebra L contains a nonzero absolute zero divisor.*

Proof. Let s be a nonzero Jordan element of S. If s is not an absolute zero divisor, then L_s is nonzero and hence, by Lemma 8.95, it contains a nonzero absolute zero divisor. By Proposition 8.51(1), L itself contains a nonzero absolute zero divisor. □

Just infinite Lie algebras. In this subsection we will assume that L is a Lie algebra over an arbitrary field $\mathbb{F}$.

Lemma 8.97. *Let L be finitely generated by elements $a_1, \ldots, a_m$ such that every $s \in S\langle a_1, \cdots, a_m\rangle$ is ad-nilpotent, and let I be an ideal of L of finite codimension, then I is finitely generated as an algebra.*

Proof. The finite-dimensional Lie algebra L/I is spanned by a Lie set for which every element is ad-nilpotent. By the Engel-Jacobson theorem (2.54), L/I is nilpotent. In other words, there exists $k \geq 1$ such that $L^k \subset I$.

Suppose that every commutator ρ in $a_1, \ldots, a_m$ of length less than k is ad-nilpotent of index at most t. Let $N = ktm^k$. It follows from Lemma 2.63 that every product $\mathrm{ad}(a_{i_1}) \cdots \mathrm{ad}(a_{i_N})$, $1 \leq i_1, \cdots, i_N \leq m$, can be represented as

$$\mathrm{ad}(a_{i_1}) \cdots \mathrm{ad}(a_{i_N}) = \sum_j v_j\ \mathrm{ad}(\rho_j),$$

where the v_j are (possibly empty) products of the $\mathrm{ad}(a_i)$'s and the ρ_j's are commutators in $a_1, \ldots, a_m$ of length greater than or equal to k. Furthermore, each summand on the right hand side has the same degree in each a_i as the left hand side. It follows that the algebra L^k is generated by commutators ρ in $a_1, \ldots, a_m$ such that $k < \mathrm{length}(\rho) < 2N$.

Since L is finitely generated, $\dim_{\mathbb{F}}(L/L^k) < \infty$. Let $b_1, \ldots, b_r \in I$ be a basis of the vector space I/L^k. Then the Lie algebra I is generated by $b_1, \ldots, b_r$ and all commutators ρ such that $k < \mathrm{length}(\rho) < 2N$, which proves the lemma. □

DEFINITION 8.98. A Lie algebra L over a field $\mathbb{F}$ is called *just infinite* if it is infinite-dimensional and every nonzero ideal of L has finite codimension.

LEMMA 8.99. *Let L be an infinite-dimensional Lie algebra over a field $\mathbb{F}$ generated by elements $a_1, \dots, a_m$ such that every element of $S\langle a_1, \cdots, a_m \rangle$ is ad-nilpotent. Then L has a just infinite homomorphic image.*

PROOF. Let $I_1 \subset I_2 \subset \cdots$ be an ascending chain of ideals of infinite codimension. We claim that the union $I = \bigcup_i I_i$ has also infinite codimension. Indeed, if $\dim_{\mathbb{F}}(L/I) < \infty$, then we have by Lemma 8.97 that I is finitely generated, hence I is equal to one the terms of the ascending chain, a contradiction. By Zorn's Lemma, the algebra L has a maximal ideal M of infinite codimension. Then L/M is just infinite, which proves the lemma. $\square$

Proof of Theorem 8.94 (char($\mathbb{F}$) = 0**).** Suppose that L is not nilpotent. Then it is necessarily infinite-dimensional and by Lemma 8.99 we may also suppose that it is just infinite. Then (8.96) L contains a nonzero absolute zero divisor and hence, by Theorem 3.20, a nonzero locally nilpotent ideal. Let I be a nonzero locally nilpotent ideal of L. Since L is just infinite, I has finite codimension. Hence I is finitely generated as an algebra by Lemma 8.97, and therefore nilpotent. This implies that I is finite-dimensional, which contradicts the assumption that L is infinite-dimensional and proves that L is nilpotent. $\square$

E. Zelmanov proved in [**Zel83a**, Proposition 1] the following early version of Theorem 8.94.

THEOREM 8.100. *Any nil Lie PI-algebra L over a field of characteristic* 0 *is locally nilpotent.*

PROOF. Suppose that L is not locally nilpotent. By factorizing L by its locally nilpotent radical (Proposition 2.66), we may assume that L is nonzero and has no nonzero locally nilpotent ideals. Then L is nondegenerate (Theorem 3.20). Let $a \in L$ be a nonzero Jordan element (whose existence is guaranteed by Kostrikin's descent lemma). Then the Jordan algebra L_a is nondegenerate (Proposition 8.51(1)), PI (Proposition 8.57), and nil (Proposition 8.51(4)). Thus, by Lemma 8.31, $L_a = \mathrm{Mc}(L_a) = 0$, and hence $\mathrm{ad}_a^2 L = 0$, which is a contradiction, so L is locally nilpotent. $\square$

8.6. Nil Lie algebras of finite width

As a further application of the Jordan techniques in Lie theory, we give in this section an outline of the proof of the following theorem due to C. Martínez and E. Zelmanov. All the algebras considered here are over a field $\mathbb{F}$ of characteristic 0.

THEOREM 8.101. [**MZ99**, Theorem 1] *Let $L = \bigoplus_{\alpha \in \Gamma} L_\alpha$ be a Lie algebra graded by an abelian group Λ over a field $\mathbb{F}$ of characteristic* 0*. Suppose that*

(i) *there exists $d > 0$ such that $\dim_{\mathbb{F}} L_\alpha \leq d$ for all $\alpha \in \Gamma$,*
(ii) *every homogeneous element $a \in L_\alpha$ is ad-nilpotent.*

Then the Lie algebra L is locally nilpotent.

Let A be an arbitrary $\mathbb{F}$-algebra generated by a subset X, let $x_1, \ldots, x_k$ be elements of X, and let $l_1 \geq 1, \ldots, l_k \geq 1$ be natural numbers. Denote by

$$\binom{x_1, \ldots, x_k}{l_1, \ldots, l_k}$$

the $\mathbb{F}$-linear span of all products in $x_1, \ldots, x_k$ (with all possible arrangements of brackets) involving l_1 elements x_1, ..., l_k elements x_k.

Following [**MZ99**], we say that the pair (A, X), satisfies the condition C_d if for arbitrary elements $x_1, \ldots, x_k \in X$, $k \geq 1$, and arbitrary natural numbers $l_1, \ldots, l_k$, we have:

(i) $\dim_{\mathbb{F}} \binom{x_1, \ldots, x_k}{l_1, \ldots, l_k} \leq d$,
(ii) every element of $\binom{x_1, \ldots, x_k}{l_1, \ldots, l_k}$ is nilpotent.

If A is associative or Jordan, the nilpotency of an element is understood in the usual sense. If A is Lie, we are meaning ad-nilpotency.

THEOREM 8.102. *If L is a Lie algebra generated by a subset X and the pair (L, X) satisfies C_d, then L is locally nilpotent.*

PROOF. As in the original proof [**MZ99**, Lemma 1.9], we can assume that L is nondegenerate, X is closed under commutators, and then to prove that $L = 0$. If L were nonzero, then by Kostrikin's descent lemma X would contain a nonzero Jordan element. Let $a \in X$ be a nonzero Jordan element of L. It is clear that the Jordan algebra L_a has $\overline{X}$ as a set of generators and that the pair $(L_a, \overline{X})$ satisfies the condition C_d. Hence, by [**MZ99**, Lemma 1.4], L_a coincides with its McCrimmon radical, but L_a is nondegenerate, a contradiction. □

Now we return to the proof of Theorem 8.101. Clearly, the pair (L, X), where $X = \bigcup_{\alpha \in \Gamma} L_\alpha$, satisfies the condition C_d, so Theorem 8.102 applies. □

8.7. Exercises

EXERCISE 8.103. Let A be an alternative algebra over a ring of scalars Φ containing $\frac{1}{2}$. Show that A with the bullet product $x \bullet y = \frac{1}{2}(xy + yx)$ becomes a Jordan algebra.

EXERCISE 8.104. Let J be a Jordan algebra. Verify that the operator $[l_a, l_b]$, $a, b \in J$, is a derivation, assuming that J is special (as quoted in (8.1) this is true in general).

EXERCISE 8.105. Let J be a Jordan algebra and let I be a Φ-submodule of J such that $U_x I \subset I$ for all $x \in \hat{J}$. Show that I is an ideal of J.

EXERCISE 8.106. Let R be an associative Φ-algebra and let $a \in R$. Define in R a new product by $x \cdot_a y = xay$, and set $\mathrm{Ker}\{a\} = \{z \in R : aza = 0\}$. Show:

(1) R with the product $\cdot_a$ is an associative algebra, denoted by $R^{(a)}$, and $\mathrm{Ker}\{a\} = \{z \in R : aza = 0\}$ is an ideal of $R^{(a)}$.
(2) Set $R_a = R^{(a)}/\,\mathrm{Ker}\{a\}$. If R is semiprime, prime, left (right) primitive, or simple, then so is R_a.
(3) $R_a^+ = (R^+)_a$.
(4) If R has an involution $*$ and $a^* = \pm a$, then the map $x \mapsto \pm x^*$ induces an involution in R_a. If $a^* = -a$, then $\mathrm{Skew}(R, *)_a \cong \mathrm{Sym}(R_a, -*)$.

(5) Suppose that a is von Neumann regular, $a = aba$ for some $b \in R$. In the Φ-module aRa, define a product by $axa \cdot_b aya := axabaya = axaya$. Then the map $\overline{x} \mapsto axa$ is an isomorphism of R_a onto $(aRa, \cdot_b)$. Furthermore, R_a is unital with $1_{R_a} = \overline{b}$.
(6) If e is an idempotent of R, then $R_e \cong eRe$.
(7) If R is unital and $u \in R$ invertible, then $R_u \cong R$.

EXERCISE 8.107. Prove (J3) and the following three identities using Macdonald principle:

(i) $l_x U_x = U_x l_x = \frac{1}{2} U_{x,x^2}$,
(ii) $(U_x y)^2 = U_x U_y x^2$,
(iii) $\{x, y, U_x z\} = U_z\{x, y, z\}$,
(iv) $V(U_x y, y) = V(x, U_y x)$.

EXERCISE 8.108. Let J be unital, $x \in J$, and y in the subalgebra of J generated by $\{1, x\}$. Use Shirshov-Cohn principle to prove that $U_{x \bullet y} = U_x U_y$.

EXERCISE 8.109. Let $e \in J$ be an idempotent of a Jordan algebra J. Show:

(1) $U_e J$ is a unital Jordan algebra with e as unit element.
(2) $J_e \cong U_e J$.

EXERCISE 8.110. Let I be an ideal and e an idempotent of a nondegenerate Jordan algebra J. Using (J3) show that the Jordan algebras I and $U_e J$ are nondegenerate.

EXERCISE 8.111. Let e be an idempotent of a Jordan algebra J, and let M be an inner ideal of $U_e J$. Show that M is an inner ideal of J.

EXERCISE 8.112. Let R be a unital associative Φ-algebra, $2 \in \Phi^*$. Show:

(1) u is invertible in R if and only if it is invertible in the Jordan algebra R^+.
(2) R is a division (associative) algebra if and only if R^+ is a division Jordan algebra.
(3) R^+ is a division Jordan algebra if and only if for any nonzero elements $a, b \in R$ there exists $x \in R$ such that $axa = b$.
(4) If R is a division algebra with involution, then $\mathrm{Sym}(R, *)$ is a division Jordan algebra.

EXERCISE 8.113. Show that the Clifford Jordan algebra $J(X, \langle\, , \rangle)$, defined by a symmetric bilinear form $\langle\, , \rangle$ on a vector space X over a field $\mathbb{F}$ of characteristic not 2, is a division Jordan algebra if and only if $\langle x, x\rangle$ is not a square in $\mathbb{F}$ for any nonzero vector $x \in X$.

EXERCISE 8.114. Let J be a division Jordan algebra. Show that for any nonzero element $u \in J$, the isotope $J^{(u)}$ is a division Jordan algebra.

EXERCISE 8.115. Using just the definition, show that the U-operator of L_a satisfies the fundamental Jordan identity (J3).

EXERCISE 8.116. [**CGGL12**] Let L be a G-graded Lie algebra and let $a \in L$ be a homogeneous Jordan element. Show that the Jordan algebra L_a is G-graded, and give an example where the grading induced in L_a by a nontrivial grading of L is trivial.

Exercise 8.117. Let $L = L_{-n} \oplus \cdots \oplus L_n$ be a $2n+1$-graded Lie Φ-algebra, $\frac{1}{6} \in \Phi$, and let $a \in L_{-n}$. Show that the Φ-submodule L_n, regarded as a subalgebra of $L^{(a)}$, is a Jordan algebra, and that this Jordan algebra $L_n^{(a)}$ factored out the ideal $\mathrm{Ker}_L\{a\} \cap L_n$ is isomorphic to L_a.