Introduction

The algebraic notion of a groupoid was distilled over time as it was realized that varied situations could be usefully studied by taking advantage of the underlying algebraic structure. Ronald Brown's survey article [**Bro87**] is an excellent overview of both the history and the myriad areas in which groupoids play a role. There Brown credits Brandt [**Bra27**] for starting the party in 1927. Good bibliographies for topological groupoids appear in [**BDNH76**] and [**BH76**]. For the smooth setting—involving Lie groupoids, Lie Algebroids, and their connection to foliation theory—see [**MM03**] and [**Mac05**].

However the focus in this book is strictly on applications to operator algebras and their representations. Hence a reasonable starting point for our story would be Mackey's virtual groups [**Mac68**, **Mac66**] in the mid 1960s. Full fledged groupoids appear in Westman's work [**Wes67**, **Wes68**], Peter Hahn's work [**Hah78**] as well as Arlan Ramsay's [**Ram82**]. However, the tale told here has its origins in Jean Renault's [**Ren80**] which initiated the idea of associating a C^*-algebra to a locally compact groupoid in a manner exactly paralleling the situation for a locally compact group. One of Renault's primary motivations was to find a C^*-algebra analogue of the work of Feldman and Moore in [**FM75**, **FM77a**, **FM77b**] associating a von-Neumann algebra to a twisted equivalence relation. Renault's work also inspired Kumjian's [**Kum86**] which is also a foundational work in the subject.

As the name suggests, a groupoid is a generalization of a group. The primary difference is that multiplication is not everywhere defined. As a result, there will be multiple identities or units. The elements of the groupoid can be thought of as "arrows" between units and multiplication as composition of the arrows. Since groups and transformation groups (arising from a group action on a space) are fundamental examples, we expect to construct a C^*-algebra from continuous, compactly supported functions on the underlying topological space of the groupoid. To ensure that there are sufficiently many such functions, we need the groupoid to have a locally compact topology such that the groupoid operations are continuous. Then we can give $C_c(G)$ a $*$-algebra structure and we realize $C^*(G)$ as an appropriate completion exactly as in the case of a group C^*-algebra or a transformation group C^*-algebra. Renault laid all this out very carefully in Chapter II of [**Ren80**].

The business of associating interesting C^*-algebras to locally compact groupoids has been very active since the publication of Renault's thesis in 1980. Naturally, there have been a number of important developments and refinements since that time such as groupoid equivalence, the disintegration theorem, and great advancements in our understanding of amenability of groupoids. Some of these are covered in Paterson's book [**Pat99**], Muhly's unpublished CBMS lectures notes [**Muh99**], and of course Anantharaman-Delaroche and Renault's [**ADR00**]. The purpose of this work is to both highlight and summarize these advancements, as well as exhibit some sharpening of the original ideas and techniques from the literature. In addition to reviewing the basic constructions, the new work we want to include is as follows.

We need to see that bounded $*$-representations of $C_c(G)$ are actually the integrated forms of a suitably defined unitary representations of G in analogy with the case for groups—this is often referred to as the "Disintegration Theorem". Renault had a basic version of this result in [**Ren80**] and provided a very general version later in [**Ren87**] with additional details appearing in the unpublished notes by Muhly [**Muh99**]. The techniques demand a heavy dose of measure theory and involve direct integrals. As a result, separability is required. One of our goals here is to describe this theory in detail.

Another important development was the notion of equivalence for groupoids and Renault's Equivalence Theorem implying that equivalent groupoids have Morita equivalent C^*-algebras. The proof in [**MRW87**] relied heavily on the Disintegration Theorem and required considerable overhead to construct the requisite bimodules. Recently, a more direct proof using linking algebras and linking groupoids appeared in [**SW12**], and we use that approach here.

The Morita equivalence provided by the Equivalence Theorem also allows a sharpening of the notion of induced representations for groupoids and their C^*-algebras using the Rieffel theory from [**RW98**, §2.4]. This was done on an *ad hoc* basis for some years and was formalized in [**Goe09**, §6.1] and [**IW09b**, §2].

Because amenability plays such an important role for groups, it is natural to try to formalize a notion of amenability for groupoids. Although Renault stressed in [**Ren80**] that his first steps in defining such a notion there were only meant as first approximations, the fact is that his initial definitions have stood up well. In fact, groupoid amenability has become important in many applications. Nevertheless, the situation is still in flux and far from satisfactory. There are three distinct flavors—if not more—to deal with. The primary notion is topological amenability—called simply amenability here—and it is based on the existence of well-behaved functions of positive type on G. There is a second notion called measurewise amenability which is based on a generalized notion of an invariant mean on $L^\infty(G)$ for suitable measures on G. Then there is simply the question of whether the universal norm and reduced norm agree on $C_c(G)$. I call this property metric amenability although the term is not in general use. We know that amenability implies measurewise amenability and that measurewise amenability implies metric amenability. For groups, these three notions coincide. Until recently, it seemed possible that they were the same for all locally compact groupoids. But recent work starting with Willett's example in [**Wil15**] has revealed that metric amenability is not equivalent to measurewise amenability. It is still possible that measurewise amenability and amenability are equivalent, but we currently only know this in special cases. The general theory took a huge jump forward with the *tour de force* [**ADR00**] published in 2000 by Anantharaman-Delaroche and Renault. However, the focus in [**ADR00**] was on Borel groupoids rather than locally compact ones. Furthermore, Renault's recent work in [**Ren15**] has allowed some dramatic improvements and new applications. Hence it seemed wise to include an extensive summary of their theory here specialized to locally compact groupoids.

One of the important results for transformation group C^*-algebras is the Gootman-Rosenberg-Sauvageot result verifying the Effros-Hahn conjecture that asserts every primitive ideal of a transformation group C^*-algebra is induced from an isotropy group provided that the group acting is amenable [**Sau77**, **Sau79**, **GR79**]. In [**Ren91**], Renault proved some closely related results for groupoids which lead to some powerful characterizations of simplicity for certain groupoid C^*-algebras. So it also seemed worthwhile to revisit some of these simplicity results from [**Ren91**].

Reader's Guide. I have tried to keep the required prerequisites for reading this book to a minimum with the goal that a reasonably well-prepared graduate student working in functional analysis can work through the material here without spending significant time with other sources. Here "reasonably well-prepared" has to be defined to include familiarity with Morita theory as exposed in Chapters 2 and 3 of [**RW98**]. Some familiarity with with the group C^*-algebra construction and a basic knowledge of crossed products would be especially helpful, and I have not hesitated to make frequent use of [**Wil07**] for background material. At the very least, it will be necessary to learn a bit about Borel Hilbert bundles and direct integrals as in [**Wil07**, Appendix F].

In Chapter 1, we begin with the necessary definitions and constructions to build a C^*-algebra, $C^*(G)$, out of a locally compact Hausdorff groupoid G. To do this we need an analogue of Haar measure on a group and define a Haar system for this purpose. Then we define a $*$-algebra structure on $C_c(G)$ and complete with respect to the universal norm arising from suitably bounded $*$-representations of $C_c(G)$ as bounded operators on a Hilbert space. Although producing interesting examples is difficult without more technology in place, we at least describe how to show that our C^*-algebra constructions generalize both the group C^*-algebra construction as well as the transformation group C^*-algebra construction.

In Chapter 2 we define groupoid equivalence and state the Equivalence Theorem. For this we need to take an in depth look at groupoid actions on spaces which is a interesting and important topic in its own right. Since the concept will be crucial to the proof of the Equivalence Theorem, we also introduce linking groupoids. Then we state the various parts of the Equivalence Theorem and explore some immediate consequences in order to provide some interesting examples.

Chapter 3 is reserved for some measure theoretic results needed in the sequel. It is just a fact of life that it is hard to avoid measure theory when studying the C^*-algebras of locally compact groupoids. It is certainly not an unreasonable option to skip this chapter and come back to the appropriate section only as needed.

Chapter 4 is devoted to the proofs of the various parts of the Equivalence Theorem.

In Chapter 5, we consider the representation theory of $C^*(G)$. This chapter avoids mention of unitary representations and hence the associated measure theory and direct integral theory. This still allows us to develop a full Rieffel theory of induced representations. We also examine the basic structure of $C^*(G)$, especially in the case that the action of G on its unit space is well-behaved.

In Chapter 6, we consider the question of when a given locally compact groupoid has a Haar system and if it does, to what extent is the Haar system unique. Unlike the case of a locally compact groups and Haar measure, the situation appears very subtle and a number of interesting open questions persist.

In Chapter 7 we start working seriously with unitary representations of a groupoid and the associated measure theory and direct integrals. We need this background to finally tackle the Disintegration Theory in Chapter 8. While unitary representations themselves do not play ubiquitous role in the theory, they are crucial to establishing certain continuity criteria on representations that are in turn fundamental. In fact, we need this material to fully justify assertions made in the proof of the Equivalence Theorem in Chapter 4.

Chapters 9 and 10 are devoted to distilling out a thorough summary of Anantharaman-Delaroche and Renault's theory of amenability from [**ADR00**] and [**Ren15**] specialized to the setting of locally compact groupoids.

Finally, in Chapter 11, we look at the Renault's Effros-Hahn results and the symplicity of $C^*(G)$ following Renault's [**Ren91**].

Further Reading. There is vast and ever-growing literature on groupoids and their C^*-algebras, and I am not going to attempt a thorough review here. As a "Took kit", this book is meant to be a start-up tool and a point of access to the literature. Nevertheless, there are some omissions in this book that should be mentioned. An important application of groupoid theory is to associate a C^*-algebra to a foliation. However the groupoids associated to foliations, while locally compact, are not necessarily Hausdorff—only locally Hausdorff.[1] If G is locally compact, locally Hausdorff, then there may be very few functions in $C_c(G)$. Connes was able to find a suitable replacement $\mathscr{C}(G)$ for $C_c(G)$ and show how to build a suitable C^*-algebra, still denoted $C^*(G)$, out of $\mathscr{C}(G)$ which reduces to our constructions when G is Hausdorff [**Con79**, **Con82**]. To reduce the overhead and clarify the main issues, I have opted not to include the locally compact, locally Hausdorff case here. A summary of the general case with further references can be found in [**MW08b**, §2]. As shown in [**MW08b**, Theorem 7.8], there is a full version of the Disintegration Theorem in this generality. Furthermore, the Equivalence Theorems (via linking groupoids) can also be proved for locally compact, locally Hausdorff groupoids (see [**SW12**, §§4–5]). Since Renault and Anantharaman-Delaroche's [**ADR00**] is written for Borel groupoids, much of their work is valid for locally compact, locally Hausdorff groupoids as well and can be extracted from [**ADR00**] by the dedicated reader. In particular, the results in [**Ren15**] are valid in the locally Hausdorff setting as are those in [**Ren91**] which spawned our simplicity results in Chapter 11.

C^*-dyanamical systems have a straightforward generalization to a groupoid acting on a C^*-algebra which fibres suitably over the unit space of G—that is, a $C_0(G^{(0)})$-algebra. The details can be found in [**MW08b**] where we also generalize the Equivalence Theorem to groupoid C^*-dynamical systems with G locally compact, locally Hausdorff. For the sake of exposition, I have not included a continuous 2-cocycle $\sigma : G^{(2)} \to \mathbf{T}$ in the construction of $C^*(G, \sigma)$ as in [**Ren80**]. In fact, $C^*(G, \sigma)$ is a special case of the C^*-algebra associated to a Twist as defined by

[1]For those bound by the Bourbaki tradition, a topological space is called "compact" only when it satisfies the Heine-Borel property *and* is Hausdorff. Spaces that satisfy the Heine-Borel property and are not necessarily Hausdorff are called "quasi-compact". In the Bourbaki world, a locally compact space is necessarily locally Hausdorff. While this has the advantage of never having to write "locally compact, locally Hausdorff", it does obscure the rather dramatic fact that the underlying spaces need not be Hausdorff so I prefer the messier term—and never having to say quasi-compact.

Kumjian (see [**Kum86**, **Kum85**]). Such constructions and many more can be subsumed under the umbrella of C^*-algebras built from a Fell bundle over a groupoid G. As described in [**MW08a**, §2], the Fell bundle C^*-algebra construction subsumes not only the construction of C^*-algebras associated to Kumjian's twists but also groupoid crossed product variations paralleling such as Green twisted systems as defined in [**Ren87**] as well as Busby-Smith twisted systems for group C^*-dynamical systems as in [**PR89**]. The theory and details (for a locally compact second countable Hausdorff groupoid G) can be found in [**MW08a**] where we again extend the Equivalence Theorem and the Disintegration Theorem to this general setting.

Using Renault's work in [**Ren91**], we were able to prove in [**IW09a**], in analogy with the Gootman-Rosenberg-Sauvageot Theorem for transformation group C^*-algebras, that every primitive ideal of $C^*(G)$ is induced from an isotropy group provided G is amenable. However, the proof requires a dose of groupoid crossed products, so it was not possible to include it here.

Even though this book is meant to be a start-up tool, space considerations made it impossible to include many important recent examples and applications of groupoid theory. Some samples include the following and the references therein. There is a Baum-Connes conjecture for groupoids [**Tu00**] which has generated important work [**Tu12**, **STY02**]. Other examples and applications in noncommutative geometry can be found in [**vEY19**, **vEY17**, **LGTX07**, **TXLG04**]. Very important examples of groupoids arise in the study of graph C^*-algebras [**KPRR97**, **Pat02**] and higher-rank graph C^*-algebras and further generalizations [**Dea95**, **RW17**, **KL17**, **Yee07**, **FMY05**]. The classification program means that questions about purely infinite and simple groupoid C^*-algebras generate considerable interest [**BCSS16**, **BCS15**, **BCFS14**].

Errata. As I become aware of mistakes and typos in this book, I will add them to a list available at `http://math.dartmouth.edu/groupoids/`. If you find a mistake, typo, or obscurity that is not listed there, I would very much appreciate your emailing me via the link provided there so that I can add your contribution to the list.

Assumptions and Conventions. Since some of basic underlying results in the theory—such as the Disintegration Theorem and the Equivalence Theorem—require separability, almost all the proofs in this book assume that the topological spaces in play are second countable. In particular, "groupoid" almost always means a second countable, locally compact Hausdorff groupoid. I use the usual convention that homomorphisms between C^*-algebras are $*$-preserving. Representations of C^*-algebras are presumed to be nondegenerate unless stated otherwise. Ideals in C^*-algebras are always closed and two-sided.

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