Introduction

Le Juge: Accusé, vous tâcherez d'être bref.
L'Accusé: Je tâcheré d'être clair.
- G. Courteline

In the theory of *locally compact* hermitian complex symmetric manifolds, finite-dimensional Lie algebras and locally compact Lie groups play a decisive role. The situation is quite different when the assumption on local compactness is dropped, partly because the knowledge of Banach-Lie algebras and Banach-Lie groups is not so detailed in the new framework. However, during the last quarter of the twentieth century, a small branch of functional analysis known as *Hermitian Jordan triple systems* has had a strong and interesting development and, in particular, has become a useful tool for the study of not necessarily locally compact complex symmetric manifolds.

The aim of this book is to provide a reasonably unified, comprehensive, and updated presentation of the theory of Jordan triple systems and its applications in complex and functional analysis. No original contribution to this theory is intended, and all the material included in the book can be found in the sources mentioned in the bibliography. We have only tried to gather this material (which to a large extent is scattered in the original articles) and organise it in a logical way to make its reading easier, a purpose for which we have added, perhaps with some frequency, details that were omitted in the original research articles as they were not addressed to the non-specialised reader.

In Part I of the book we present (within the setting of Banach spaces) the background and previous results on complex analysis and differential geometry that will be needed later. We consider a holomorphic Banach manifold M equipped with a compatible metric d and the group $G := \mathrm{Aut}(M, d)$ of holomorphic automorphisms of M that preserve the metric d. We define in G the topology T_{lu} of local uniform convergence over M and prove that (G, T_{lu}) is a topological group whose left uniform structure is complete. The natural action of G on M is locally uniformly continuous and (G, T_{lu}) enjoys a certain universal property in the category of locally uniform topological groups. We study the set $\mathfrak{g}(M)$ of complete holomorphic vector fields in M, which is a purely real Lie algebra and when endowed with a suitable norm becomes a Banach-Lie algebra. Contrary to the finite-dimensional case, in general (G, T_{lu}) does not admit any Lie group structure. We construct the analytic topology T_a in G, which in general is finer than T_{lu}, and prove that (G, T_a) is a real Banach-Lie group whose Banach-Lie algebra is $\mathfrak{g}(M)$. The natural action of G

in M is analytic (in the real sense) and (G, T_a) enjoys a certain universal property in the category of analytic groups.

After this introductory material we make a detailed study of symmetric complex Banach manifolds. We fix a point $o \in M$, referred to as the base point of M, and consider the associated symmetry $s_o \in G$; its adjoint action $\mathrm{Ad}(s_0)$ on the Lie algebra $\mathfrak{g}(M)$ admits the eigenvalues ± 1 and induces a decomposition into a topologically direct sum $\mathfrak{g}(M) = \mathfrak{k}(M) \oplus \mathfrak{p}(M)$, where $\mathfrak{k}(M)$ and $\mathfrak{p}(M)$ are the corresponding eigenspaces. We construct the canonical local chart of M at o, a special local chart in which the elements of $\mathfrak{k}(M)$ have the expression $X = l(z)\frac{\partial}{\partial z}$, where z is the local coordinate of M at $o \in M$ and $l\colon E \to E$ is a continuous linear mapping such that $\mathrm{i}l$ is a hermitian operator in $E \approx T_o M$, the tangent space to M at o. In the canonical local chart at o, the elements of $\mathfrak{p}(M)$ have the expression $X_c = \big(c - q_c(z)\big)\frac{\partial}{\partial z}$, where $c \in E$ and $q_c\colon E \to E$ is a continuous homogeneous polynomial of degree 2 in z which depends on $c \in E$. We prove that the application $E \to \mathcal{P}(^2E)$ given by $c \mapsto q_c$ is a continuous conjugate linear mapping. With this and a simple change of notation $\{zc^*z\} := q_c(z)$, we get the notion of triple product $\{\cdot\cdot^*\cdot\}$ and, in order to obtain the concept of a hermitian Jordan triple system or J*-triple, we only need the Jordan identity, which simply says that $\mathfrak{g}(M)$ is a Lie algebra. In this way, to each complex symmetric Banach manifold (connected and with a base point) we have associated a J*-triple, and symmetric manifolds that are isomorphic (as symmetric manifolds) have isomorphic J*-triples. We prove that given an arbitrary J*-triple $(E, \|\cdot\|, \{\cdot\cdot^*\cdot\})$ there is a symmetric complex Banach manifold (connected and with a base point) M whose associated J*-triple is the given one. Such a manifold M is not unique, but the set of them contains a privileged element which is uniquely determined up to isomorphisms: the universal covering $\widetilde{M}$ of M, which is simply connected. This establishes that the category of complex symmetric Banach manifolds (connected, simply connected, and with base point) is equivalent to the category of J*-triples, which is the goal of Part I.

The equivalence of the above-mentioned categories justifies the algebraic study of J*-triples, which is the purpose of Part II. A J*-triple is a complex Banach space with a ternary law of composition $E \times E \times E \to E$, $(x, y, z) \mapsto \{xy^*z\}$, referred to as the triple product, satisfying certain *algebraic* properties (linearity and symmetry in the external variables x, z, conjugate linearity in y, and the Jordan identity), certain *topological* properties (continuity of the triple product), and some *metric* properties (the hermitian character of the operators $x\Box x^* \in \mathcal{L}(E)$, $x \in E$). Ternary laws of composition are not frequently considered in mathematics, hence a study of J*-triples may be convenient since the structure of a J*-triple is not a familiar one. Tripotents of E (the elements of E that satisfy the algebraic equation $\{ee^*e\} = e$) and the relations of compatibility and orthogonality play a relevant role in that study. If $e \in E$ is a tripotent, then the spectrum of the operator $e\Box e^* \in \mathcal{L}(E)$ is contained in the set $\{0, 1/2, 1\}$ and each point in the spectrum is an eigenvalue. This decomposes E into the topologically direct sum of the corresponding eigensubspaces, each of which is a J*-subtriple of E and the triple product satisfies certain additional properties called the Peirce rules. In the set of tripotents of E one can introduce the notions of minimality and maximality, which are defined in purely algebraic terms though they allude to a (not yet introduced) order relation. Of particular interest are those J*-triples E that admit a maximal family of minimal pairwise orthogonal tripotents, referred to as atomic

J*-triples. The cardinality of such families (if they exist), called the rank of E, is a J*-invariant. Finite-dimensional J*-triples always have finite rank but the converse is not true and we make a detailed study of finite rank J*-triples. In particular, any J*-triple in a reflexive Banach space has finite rank, which allows us to make a complete holomorphic classification of symmetric Banach manifolds in reflexive Banach spaces. This part ends with a study of those J*-triples E such that E is a Hilbert space and a classification of the symmetric hermitian manifolds.

Part III of this book begins with the problem of characterising those J*-triples E whose associated symmetric manifold M can be identified with a bounded symmetric domain in E. After all, the theory of J*-triples has its origins in the study of bounded symmetric domains. Recall briefly from Part I the construction of the manifold M: The orbit $\mathcal{O}$ of the origin $0 \in E$ under the set of transformations $z \mapsto T_c(z) := (\exp X_c)z$ where $c \in E$ and $X_c \in \mathfrak{p}(M)$ is the vector field $X_c := (c - \{zc^*z\}\frac{\partial}{\partial z}$, that is, the set

$$\mathcal{O} := \big\{(\exp X_c)0 \colon c \in E\big\} \tag{0.1}$$

is a connected manifold with $0 \in \mathcal{O}$. Its universal cover $\mathcal{O}^{\sim}$ is a connected, simply connected manifold with base point 0 which, endowed with a suitable metric $\delta_{\mathcal{O}}$, becomes the manifold associated with the given J*-triple E. Recall also that $(\exp X_c)0$ is the value at $t = 1$ of the solution $y(t, c)$ of the Ricatti differential equation

$$\frac{d}{dt}y(t,c) = c - \{y(t,c)\, c^*\, y(t,c)\}, \qquad y(0,c) = 0. \tag{0.2}$$

Solving the equation (0.2) is a hard problem and we are led to consider it within the more restrictive setting of the *monogeneous* J*-triples, that is, subtriples $E^c \subset E$ that are generated by one single of its elements $c \in E$. Actually, equation (0.2) only involves the monogeneous J*-subtriple E^c. Thus we begin with a detailed study of this type of J*-triples. Under certain restrictions on the generating element c [the algebraic condition $\{cc^*c\} = 0 \Rightarrow c = 0$ (anisotropy) and spectral positivity of the operator $c\Box c^*|_{\mathcal{L}(E^c)}$] we get a *continuous injective* J*-homomorphism $\mathcal{J} \colon E^c \to \mathcal{C}_0(\Omega)$ for a locally compact space Ω which depends[1] on c. Recall that $\mathcal{J}$ in general is *neither isometric nor surjective*, though the image $\mathcal{J}(E^c)$ is a dense subset in $\mathcal{C}_0(\Omega)$. With the help of $\mathcal{J}$ the initial value problem (0.2) is converted into an initial value problem in $\mathcal{C}_0(\Omega)$ where it can easily be solved. In this way we manage to prove that the *partial orbit* (0.1), now denoted $\mathcal{O}^c$, is the open unit (semi)-ball B^c_∞ of E^c relative to the *spectral seminorm* $\|x\|_\infty := \rho_c(x\Box x^*|_{\mathcal{L}(E^c)})^{1/2}$, $x \in E^c$, (here ρ_c stands for the spectral radius in $\mathcal{L}(E^c)$, which might depend on c)

$$\mathcal{O}^c = B^c_\infty := \big\{x \in E^c \colon \rho_c(x\Box x^*|_{\mathcal{L}(E^c)}) < 1\big\}.$$

The spectral seminorm is continuous in E^c and, as the orbit $\mathcal{O}^c$ now is a (semi)-ball, it is automatically an open connected and simply connected neighbourhood of the origin. Hence $\mathcal{O}^c$ is a bounded domain in E^c if and only if $\|\cdot\|_\infty$ and the norm $\|\cdot\|$ that E induces on E^c are topologically equivalent in E^c, that is, if and only if there are constants (possibly depending on c) $0 < k_c \le K_c < \infty$ such that

$$k_c\|x\| \le \|x\|_\infty \le K_c\|x\|$$

[1] Most of the elements that appear in the following considerations depend on c. As long as c is fixed we may omit the reference to it and so avoid heavy notation. But later on, various generating elements c will have to be considered and to avoid confusion a reference to c is convenient.

holds for all $x \in E^c$. If that is the case, then the pair $(B^c_\infty, \delta_{B^c_\infty})$, where $\delta_{B^c_\infty}$ is the Carathéodory metric in B^c_∞, is the symmetric manifold associated with E^c. No need of considering the universal cover $(B^c_\infty)^\sim$ is now required since B^c_∞ already is simply connected. At this point what we have proved might be summarized as follows:

$$\begin{gathered}\text{In the class of monogeneous J}^*\text{-triples}\, E^c:\\ \big(E^c\,\text{anisotropic} + \text{positive} + \|\cdot\| \approx \|\cdot\|_\infty\big) \Leftrightarrow M^c\,\text{is a bounded domain,}\end{gathered} \tag{0.3}$$

where M^c stands for the symmetric connected simply connected manifold with base point associated with E^c and $\approx$ means topological equivalence of norms. Some additional information is gathered on the way: E^c endowed with the norm $\|\cdot\|_\infty$ is a hermitian triple and $\|\cdot\|_\infty$ is the only equivalent norm in E^c which satisfies $\|x\|^2_\infty = \|x \Box x^*\|_\infty$ for all $x \in E^c$. Moreover, by changing the norm $\|\cdot\|$ of E^c into its equivalent $\|\cdot\|_\infty$, the J*-isomorphism $\mathcal{J}\colon E^c \to \mathcal{C}_0(\Omega)$ becomes a *surjective linear isometry* of E^c onto the abelian C*-algebra $\mathcal{C}_0(\Omega)$.

Now we go back to the case of a general J*-triple E and try to gather the information previously obtained on the various sections $E^c \subset E$ for $c \in E$ in order to drop the assumption on the monogeneous character of E. The result (0.3) suggests that the goal now is the proof of

$$\begin{gathered}\text{For an arbitrary J}^*\text{-triple}\,\text{E:}\\ \big(E\,\text{anisotropic} + \text{positive} + \|\cdot\| \approx \|\cdot\|_\infty\big) \Leftrightarrow M\,\text{is a bounded domain.}\end{gathered} \tag{0.4}$$

Let us sketch the proof of the implication " $\Leftarrow$ ". From Part I we know that the vector fields $X_c = (c - \{zc^*z\})\frac{\partial}{\partial z}$ are complete in the manifold M associated to E, hence if M is a bounded domain in E, then E has the following property (referred to as boundedness of E):

(a) *There is a bounded neighbourhood W of the origin $0 \in E$ (actually we may take $W := M$) such that all polynomial vector fields $X_c := (c - \{zc^*z\})\frac{\partial}{\partial z}$ with $c \in E$ are complete in W.*

Thus we begin with a study of bounded J*-triples and prove that boundedness is a hereditary property that implies anisotropy and spectral positivity. Moreover, in a bounded J*-triple E the mapping $x \mapsto \rho(x\Box x^*)^{1/2}$, $x \in E$, is a continuous norm (the spectral norm in E, denoted by $\|\cdot\|_\infty$) and we establish the following key point:

(b) *The spectral norm $\|x\|_\infty$ of an element $x \in E$ depends only on the monogeneous subtriple E^x generated by x in E, therefore we can compute it in E^x.*

Thus, up to now we have established that if M is a bounded domain in E, then E is anisotropic, positive, and, by the continuity of the spectral norm, we have $\|x\|_\infty \le K\|x\|$ for all $x \in E$. Hence to complete the proof of (0.4), it remains to see that

$$k\|x\| \le \|x\|_\infty \tag{0.5}$$

holds for some constant $k > 0$ and all $x \in E$. To establish the latter inequality we use the following key observation:

(c) *Each of the initial value problems (0.2) with $c \in E$ that we have to solve in order to compute the orbit $\mathcal{O}$ involves only the monogeneous subtriple E^c.*

Since we have already proved that E is anisotropic and positive and that these two qualities are hereditary, each monogeneous section E^c is also anisotropic and positive, and, as it has already been established, we have $\mathcal{O}^c = B^c_\infty$. Therefore, the orbit of the origin under the set of transformations $z \mapsto T_c(z) := \exp(c - c^*)0$ for $c \in E$ is given by

$$\{\exp(c - c^*)0\colon c \in E\} = \bigcup_{c\in E} \mathcal{O}^c = \bigcup_{c\in E} B^c_\infty = \bigcup_{c\in E} B_\infty \cap E^c = B_\infty. \tag{0.6}$$

In particular, $\mathcal{O} = B_\infty$ is connected and simply connected (as it is a ball) and no mention of the universal cover is needed. The pair $(\mathcal{O}, \delta_{\mathcal{O}})$, where $\delta_{\mathcal{O}}$ is the Carathéodory metric in $\mathcal{O}$, is the manifold M associated with E.

Now we proceed to establish the implication " $\Rightarrow$ ". Assume that E is anisotropic and positive. Since these qualities are hereditary, each monogeneous section E^c is also anisotropic and positive, and therefore by what we have already established for monogeneous J*-triples we have $\mathcal{O}^c = B^c_\infty$ for all $c \in E$. Then from the key statements (b) and (c) we derive

$$\mathcal{O} = \bigcup_{c\in E} \mathcal{O}^c = \bigcup_{c\in E} B^c_\infty = B_\infty.$$

In particular, as $\mathcal{O}$ is a ball, it is connected and simply connected and hence it coincides with the manifold associated to E, that is, we have $M = \mathcal{O} = B_\infty$, a set which due to the assumption $\|\cdot\| \approx \|\cdot\|_\infty$ is a bounded domain in E. This completes the proof of (0.4).

Some additional information is gathered on the way: E endowed with the norm $\|\cdot\|_\infty$ is a hermitian triple and $\|\cdot\|_\infty$ is the only equivalent norm in E that satisfies $\|x\|^2_\infty = \|x\square x^*\|_\infty$ for all $x \in E$. If the manifold M associated with a J*-triple E is a bounded symmetric domain, then also the manifold M^c associated with any monogeneous section E^c of E is a bounded domain, but the converse is not true. Indeed, as we have seen above, under such an assumption we know that for each $c \in E$ there are constants (possibly depending on $c \in E$) such that

$$k_c\|x\| \le \|x\|_\infty \le K_c\|x\| \tag{0.7}$$

hold for all $x \in E^c$. However, there is nothing to ensure that $\inf_{c\in E} k_c > 0$ and $\sup_{c\in E} K_c < \infty$. Even if we had $\|\cdot\| \approx \|\cdot\|_\infty$ in E, there is nothing to ensure that these two norms coincide. The study of necessary and sufficient conditions for the coincidence of the norm and the spectral norm in E leads us to another class of J*-triples known as JB*-triples.

With the help of the results previously obtained for the monogeneous subtriples E^c, we prove that, for any hermitian J*-triple E, the following statements are equivalent:

- (i) for each $c \in E$ one has $\sigma(c\square c^*) \subset [0, \infty)$ and $\|c\square c^*\| = \|c\|^2$,
- (ii) for each $c \in E$, E^c is isometrically isomorphic to a commutative C*-algebra,
- (iii) for each $c \in E$ we have $\sigma(c\square c^*) \subset [0, \infty)$ and $\|\{cc^*c\}\| = \|c\|^3$,

where $\sigma(c\square c^*) \subset \mathbb{C}$ stands for the spectrum of $c\square c^*$ in $\mathcal{L}(E)$.

Hermitian J*-triples E satisfying properties (i) to (iii) are called JB*-triples. A J*-triple E is a JB*-triple if and only if E is bounded and equipped with its spectral norm. Clearly condition (ii) above involves only the monogeneous sections E^c of E and, in the notation we have used, means that the constants k_c and K_c in (0.7) do not depend on c and actually they are $k_c = K_c = 1$.

Let D be a bounded symmetric domain in a complex Banach space E and fix any point $o \in D$. Then we may view D as a symmetric Banach manifold with base point o and if we identify the tangent space T_oD to D at o with E, then E supports the structure of a hermitian J*-triple which is bounded and after a change of equivalent norm in E (that is, after a biholomorphic linear transformation, if needed), D is converted into the open unit ball B_∞ of E relative to the spectral norm. In this way, the category of bounded symmetric domains (with base point) is equivalent to the category of JB*-triples and we get an infinite-dimensional version of the Riemann mapping theorem. In particular, every bounded symmetric domain in a complex Banach space is convex and simply connected, hence we also get an infinite-dimensional version of the Harish-Chandra realisation of bounded symmetric domains in $\mathbb{C}^n$. We have decided to include Chapter 13 (as it provides us with a deep insight into the theory) though it clearly overlaps with Chapter 15, where a study of the group $\mathrm{Aut}(D)$ of holomorphic automorphisms of the unit ball D of an arbitrary JB*-triple is made. After so much abstract theory, specific examples are welcomed and we present the unit ball of classical Cartan factors as examples of bounded symmetric domains whose symmetric dual manifolds are described as certain Grassmann manifolds.

In Part IV we make a study of JBW*-triples, that is, JB*-triples $(E, \|\cdot\|, \{\cdot\cdot^*\cdot\})$ such that E is a dual Banach space. In general, a JB*-triple may have no tripotents except for $e = 0$, therefore it is convenient to avoid this situation. Complete or maximal tripotents in E are, whenever they exist, precisely the extreme points of the closed unit ball of E. Hence if E is a dual Banach space, then the Krein-Milman theorem assures that the set of tripotents is plentiful. This justifies the intoduction of the JBW*-triples. Each W*-algebra and each JBW*-algebra is a JBW*-triple in a natural way, and the role that projections play in the study of these algebras is now played by tripotents. We study the elementary properties of JBW*-triples and establish several characterisations of the predual space E_* of E. We introduce several notions of uniqueness of the predual X_* of a Banach space X and discuss the relations between them showing that in case X is JBW*-triple all these notions coincide. This allows us to define without any ambiguity the weak* topology on E. We prove that in every JBW*-triple the triple product is separately w*-w*-continuous and study the relations between separate w*-w*-continuity of the triple product and the uniqueness of the predual space. The bidual space E^{**} of a JB*-triple E is a JBW*-triple and the triple product of E^{**} extends the one of E. We show that isomorphisms between two JBW*-triples are w*-w*-continuous; however, holomorphic automorphisms of the unit ball B_∞ in general are not w*-w*-continuous, which gives rise to the introduction of new subcategories of JBW*-triples. We make a detailed study of the set of w*-closed ideals in a JBW*-triple E, whose structure and representations are discussed in depth. In particular, we establish the Gelfand-Naimark theorem for JBW*-triples. The analogy between JBW*-triples and W*-algebras or JBW*-algebras leads us to the introduction of a functional calculus for JBW*-triples which turns out to be a useful tool; it also leads to the introduction of various classes of tripotents (open, closed, compact, etc., tripotents) and to the classification of JBW*-triples of type I which, in case they are irreducible, are precisely the Cartan factors. Broadly speaking, this part deals with the study of the structure and the geometry of JBW*-triples considered as Banach spaces, which is used to further the study of complex analysis.

Contrary to the case of JBW*-triples, one can also consider the class of JB*-triples (referred to as continuous JB*-triples) which have no non-zero tripotents. Their study requires a different approach and we do not consider them here.

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