

## CHAPTER I

# Introduction

The main aim of this monograph is to investigate the *spectral stability*, that is, the absence of eigenvalues with positive real part for the linearization at the solitary wave solutions to the nonlinear Dirac equation

$$(I.1) \quad i\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \quad \psi(t, x) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n,$$

in the case when the nonlinearity is represented by

$$f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad f(\tau) = |\tau|^\kappa + O(|\tau|^K), \quad 0 < \kappa < K.$$

We prove the spectral stability of weakly relativistic solitary wave solutions,  $\omega \lesssim m$ , in the following cases:

$$(I.2) \quad \kappa \lesssim 2/n, \quad K > \kappa, \quad n \in \mathbb{N};$$

$$(I.3) \quad \kappa = 2/n, \quad K > \frac{4}{n}, \quad n \in \mathbb{N}.$$

We use the notation  $\omega \lesssim m$  to indicate that there is  $\varepsilon > 0$  sufficiently small such that  $\omega \in (m - \varepsilon, m)$ .

In particular, we prove the spectral stability for the quintic nonlinear Dirac equation in (1+1)D (that is, in one spatial dimension). We cannot prove the spectral stability for all subcritical values  $\kappa \in (0, 2/n)$ : the point spectrum of the nonlinear Schrödinger equation of order  $1 + 2\kappa$  linearized at a solitary wave becomes rich for small  $\kappa > 0$ ; such cases would require a more detailed analysis. Many of our results for the nonlinear Dirac equation have already appeared in [BC16, BC17, BC18, BC19].

Let us give the plan of the monograph. The introductory chapters are elementary and provide some background on Functional Analysis, Spectral Theory, Quantum Mechanics, and linear stability theory. In particular, we describe the Derrick theorem and the Kolokolov stability criterion. We think that these results already give the feeling of the material and also emphasize the importance of the Spectral Theory approach to the stability of solitary waves. Then we develop the tools needed for the Dirac operator: the Dirac–Pauli Theorem on the choice of Dirac matrices (which we do in arbitrary dimension), the limiting absorption principle, and the Carleman estimates. In the final chapters, we prove the spectral stability of weakly relativistic solitary waves in the charge-subcritical cases and in the “charge-critical” case. More precisely, we show that the presence of eigenvalues with nonzero real part in the spectrum of the linearization at a solitary wave in the nonrelativistic limit,  $\omega \lesssim m$ , is essentially described by the Kolokolov stability criterion [Kol73], which initially appeared in the context of the nonlinear Schrödinger equation. The spectral stability result opens the way to the proofs of asymptotic stability.

**Subjective historical review of the electron theory.** In the laboratory conditions, electrons are observed in London in 1838, when Michael Faraday attaches high voltage to a vacuum tube in his laboratory at the Royal Institution and observes cathode rays. New tubes are being constructed and the vacuum is improving. Further experiments with vacuum tubes are done in Bonn by Heinrich Geissler in 1857 (with the pressure around 100 Pa), Julius Plucker, Johann Wilhelm Hittorf, and then by Sir William Crookes in London and Arthur Schuster in Manchester (now with the pressure below 0.1 Pa), who study the deflection of the *cathode rays* in magnetic and electric fields. The charge-to-mass ratio of the *cathode rays* is estimated. This name, *Kathodenstrahlen*, is coined in 1876 by a German physicist Eugen Goldstein.

In 1874, Irish physicist George Johnstone Stoney, while working on electrolysis, suggests that there is a “single definite quantity of electricity”, estimates its charge, and in 1881 coins the term *electrolion* (and then going to *electron*).

While developing the light bulb, Thomas Edison inserts the third wire into a vacuum tube and files a patent for the “Electrical indicator” (1884), a vacuum tube which is to become a triode in twenty years, several lines below.

In 1896, at the Cavendish Laboratory, Cambridge, Joseph John Thomson accurately measures the mass-to-charge ratio of the cathode rays and suggests the existence of the electrons [Tho97]. In [Tho04], taking into account Earnshaw’s theorem on instability of stationary configurations of point charges [Ear42], Thomson comes up with the plum-pudding model, in which point electrons sit (or, rather, move) in the pudding-like atom. Partial differential equations are near: J.J. Thomson was a PhD student of John William Strutt, 3rd Baron Rayleigh, the author of the celebrated two-volume *The Theory of Sound* [Str77, Str78], an outstanding PDE monograph (Richard Courant admired this book and, according to Kurt Friedrichs, strongly recommended to his students). Rayleigh’s ideas were later used by Schrödinger as the basis for what is now the Rayleigh–Schrödinger perturbation theory.

In 1904, John Fleming uses Edison’s “electrical indicator” for amplifying the radio signal for applications to transatlantic communications. This idea is developed by Lee DeForest in 1906 into the “Audion”, or a triode: a vacuum tube with a grid that is used to control the current through the tube, which allows one to amplify the electric signal; the electronic age begins.

J.J. Thomson’s influence on the electron theory continues: one of his PhD students is Ernest Rutherford, under whose direction the Geiger–Marsden gold foil experiment in 1909 leads to the Bohr–Rutherford model of an atom, putting an end to Thomson’s plum-pudding model. In 1911, Niels Bohr receives his doctorate from the University of Copenhagen and sets off to Cambridge, where he works as a postdoc under J.J. Thomson. Two years later, Bohr formulates his famous postulates [Boh13] aimed at describing electron’s behaviour in atoms.

In 1911, Louis de Broglie accompanies his older brother, Maurice, to the First Solvay Conference on Physics. In 1923, following discussions of Planck’s and Einstein’s research on wave-particle duality in the context of photons (the Second Solvay Conference, 1913), he puts forward the idea that the electrons could be described as waves, published in his PhD Thesis [Bro25] and presented at the Fourth Solvay Conference (1924). In November 1925, Peter Debye, a colloquium organizer at ETH, prompts Erwin Schrödinger to give a colloquium on de Broglie’s *phase*

*waves*. Shortly after his talk, again having been prompted by Debye and spending Christmas of 1925 in Arosa, Switzerland, Schrödinger writes down a relativistically invariant equation for the wave function of the electron, now known as the Klein–Gordon equation. Introduction of the Coulomb potential into this equation leads to the energy values that agree—in the leading order—with the electron energy levels  $E_n = -\frac{m e^4}{2\hbar^2 n^2}$ ,  $n \in \mathbb{N}$  (with  $m$  and  $e$  electron’s mass and charge), from the “Old Quantum Theory” by Niels Bohr and Arnold Sommerfeld; that expression explains the empirical formula for the frequencies of Hydrogen spectral lines obtained by Johannes Rydberg back in 1888. To find  $E_n$ , one writes the wave function as  $e^{-ar}F(r)$ , with an appropriate  $a > 0$  and  $F(r) = F_0 + F_1 r + \dots$  a polynomial whose coefficients are computed recurrently; the values of  $E_n$  are to be such that the polynomial  $F(r)$  has finitely many terms, or else the series representing  $F(r)$  could be shown to converge to a function which grows like  $e^{2ar}$  (see, e.g., [Sch49]). Noticing the inconsistency of the higher order “relativistic corrections” to the values of  $E_n$  with the accurately measured wavelengths of the emitted light, Schrödinger shelves his relativistic *H-Atom. Eigenschwingungen* draft and retreats to the non-relativistic limit,

$$E = \sqrt{m^2 c^4 + \mathbf{p}^2 c^2} \approx mc^2 + \frac{\mathbf{p}^2}{2m},$$

arriving at what we now know as the Schrödinger equation [Sch26]; see [Sch49, Moo94, Meh01]. Now of course it is accepted that, under spatial rotations, electron’s wave function transforms according to the four-dimensional “spinorial” representation of  $\mathbf{SO}(3)$ , and thus could not be described by a scalar-valued solution to the relativistic Schrödinger (or Klein–Gordon) equation.

In the summer of 1926, Clinton Davisson attends the Oxford meeting of the British Association for the Advancement of Science on recent advances in quantum mechanics, and learns there at Max Born’s lecture that the plots with peaks from his and Lester Germer’s 1923 experiment are used as a confirmation of the de Broglie hypothesis about the wave nature of electrons. In 1927, diffraction patterns from electron scattering on crystals are independently obtained by George Paget Thomson, J.J. Thomson’s son.

The relativistically invariant equation for electrons is invented in December 1927, in Cambridge, by Paul Dirac, who notices that

$$(\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)^2 = p_1^2 + p_2^2 + p_3^2, \quad \forall \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3,$$

with  $\sigma_i$  the Pauli matrices, doubles the matrix size to be able to extract the square root of the energy-momentum relation  $E^2 = m^2 c^4 + \mathbf{p}^2 c^2$ , arriving at the first-order relation  $E = c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2$  where  $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ , with  $\alpha^i$ ,  $1 \leq i \leq 3$ , and  $\beta$  the  $4 \times 4$  Dirac matrices, and then uses Schrödinger’s substitution of  $E$  and  $\mathbf{p}$  by the operators  $i\hbar\partial_t$  and  $-i\hbar\nabla$  to write down the evolution equation for the electron wave function [Dir28] (for the historical details, see [Meh01]). The matrix form of the Dirac equation results in the spinor-valued wave function and triumphally yields the accurate values of the relativistic corrections to the energy levels of the Hydrogen atom.

**Self-interacting spinor fields.** Let us point out that the Lamb shift [LR47] could not be explained in the framework of the linear Dirac equation in the external potential, thus illustrating the need for a more accurate description of the nonlinear effects in the Dirac–Maxwell system, which are known to physicists as the

interaction of an electron with the *vacuum energy fluctuations*. As a matter of fact, neither can the linear theory explain electron’s need for “quantum jumps” from one state to another, postulated by Bohr, since in a linear system any superposition of solutions is also a solution. Speaking of the *Copenhagen interpretation of quantum mechanics*, Steven Weinberg says: “According to Bohr, in a measurement the state of a system such as a spin collapses to one result or another in a way that cannot itself be described by quantum mechanics, and is truly unpredictable. This answer is now widely felt to be unacceptable” [Wei17]. At the same time, *quantum jumps* can be rigorously explained in the framework of the global attractors of *nonlinear dispersive Hamiltonian systems* [Kom03, KK07, Kom12]: the energy leaks via a weak, higher order self-interaction (caused by a nonlinear nature of the system) into the essential spectrum and disperses, until the system arrives at a one-frequency state (*Bohr’s quantum orbit*) which no longer radiates the energy. To summarize, it is possible that, in the words of the Russian writer Sergey Dovlatov, the nonlinear effects in Quantum Physics could be *nice, but small; or, rather, small, but nice* (he was speaking about his salary).

The idea to employ nonlinear models in Quantum Physics could be traced back to the last paragraph in the article of Dmitri Ivanenko [Iva38] (remarkably, in the context of the nonlinear Dirac equation). It is followed up in particular in [FLR51, FFK56, Hei57]. Widely known models of self-interacting spinor fields are the massive Thirring model [Thi58] (spinor field with the vector self-interaction) and the Soler model [Iva38, Sol70] (spinor field with the scalar self-interaction); the one-dimensional analogue of the Soler model is known as the (massive) Gross–Neveu model of quark confinement [GN74, LG75]. Let us also mention that nonlinear equations of Dirac type also appear under the name of coupled-mode equations in nonlinear optics [SS94] and in the description of matter-wave Bose–Einstein condensates trapped in an optical lattice [PSK04].

In the Klein–Gordon context, self-interacting fields appear in Leonard Schiff’s nonlinear meson theory [Sch51a, Sch51b] and start receiving mathematical attention since the articles by Konrad Jörgens and Irving Segal [Jör61, Seg63] in the study of the well-posedness of the nonlinear Klein–Gordon equation in the energy space. This is followed by the research on nonlinear scattering by Irving Segal, Walter Strauss, and Cathleen Morawetz [Seg66, Str68, MS72], the work on the existence of solitary waves [Str77a, BL79, BL83a], and an extensive research on their linear and orbital stability, in the nonlinear Schrödinger and Klein–Gordon context; see, e.g., [Kol73, BC81, CL82, Sha83, Wei85, Sha85, GSS87, Gri88, DBGRN15, DBRN19], which is followed by the research on the asymptotic stability of solitary waves by Avy Soffer and Michael Weinstein [SW90, SW92], Vladimir Buslaev and Galina Perelman [BP92b, BP95], which is further developed in [PW97, SW99, Cuc01, BS03, Cuc03]. The next aim of the theory could be the Soliton Resolution Conjecture [Kom03, Sof06, Tao07, KK07, Tao09, KLLS15, DJKM17].

Although the research on the Dirac-based systems follows much slower, there is an increasing interest to this subject. In particular, the existence of standing waves in the nonlinear Dirac equation is studied in [Sol70, CV86, Mer88, ES95]. The discussion of the applications of classical self-interacting spinor fields in Quantum Theory is in [Rañ83a]. The relation of nonlinear theory and the Pauli exclusion

principle is considered in [Rañ83b]. Local and global well-posedness of the nonlinear Dirac equation is further addressed in [EV97] (semilinear Dirac equation in (3+1)D) and in [MNNO05] (nonlinear Dirac equation in (3+1)D). There are many results on the local and global well-posedness: in one spatial dimension, we mention [ST10, MNT10, Huh11, Can11, Pel11, Huh13a, Huh13b, HM15, ZZ15]; the higher-dimensional setting is considered in [Bou04, Bou08a, BC14, Huh14, BH15, BH16].

**Stability.** Now we move to the question of stability of solitary waves. The stability is understood in many cases for the nonlinear Schrödinger, Klein–Gordon, and Korteweg–de Vries equations (see, e.g., the review [Str89]). In these systems, at the points of the functional space corresponding to solitary waves, the Hamiltonian function is of finite Morse index. In simpler cases, the Morse index is equal to one, and the perturbations in the corresponding direction are prohibited by one of the conservation laws when the Kolokolov stability condition [Kol73] is satisfied (also known as Vakhitov–Kolokolov stability criterion). In other words, the solitary waves could be demonstrated to correspond to conditional minimizers of the energy under the charge constraint; this results not only in spectral stability but also in orbital stability [CL82, GSS87]. The nature of stability of solitary wave solutions of the nonlinear Dirac equation, observed in the early numerical studies of dynamics [AC81], seems different from this picture [Rañ83a, Section V]. The Hamiltonian function is not bounded from below, and is of infinite Morse index at the points corresponding to solitary waves; the NLS-style approach to stability fails. As a consequence, we do not know how to prove the *orbital stability* [CL82, GSS87] except via proving the asymptotic stability first. The only known exception is the completely integrable massive Thirring model in (1+1)D, where the orbital stability was proved by means of a coercive conservation law [PS14, CPS16] coming from higher order integrals of motion.

On the way to proving the asymptotic stability, one starts with the linear (also known as *spectral*) stability. The purely imaginary essential spectrum of the linearization operator is readily available via Weyl’s theorem on the essential spectrum; the discrete spectrum is more delicate. While the linear instability of ground states in the nonlinear Schrödinger equation can only come from a positive eigenvalue, whose presence is conveniently controlled by the Kolokolov stability criterion [Kol73], the Dirac equation presents a less comfortable situation: the linearization at the solitary waves in the nonlinear Dirac equation can possess eigenvalues anywhere in the complex plane. In particular, the linear instability in nonlinear equations of Dirac type may develop due to the bifurcations from the embedded thresholds at  $\pm i(m + |\omega|)$  as in [BPZ98], in the context of the one-dimensional coupled-mode equation; from the collision of the thresholds  $\pm i(m \pm |\omega|)$  at  $z = \pm mi$  when  $\omega = 0$ , as in [KS02], in the context of the massive Thirring model; “in the nonrelativistic limit”  $\omega \lesssim m$  (by this we mean that there is  $\varepsilon > 0$  small enough so that  $\omega \in (m - \varepsilon, m)$ ), when linear instability of Schrödinger equation is inherited by weakly relativistic solitary waves in the nonlinear Dirac equation with supercritical nonlinearity [CGG14]. (It was discovered numerically that this latter instability disappears when  $\omega \in (0, m)$  becomes sufficiently small [CMKS<sup>+</sup>16], although it later reappears for  $\omega \gtrsim 0$ ; that is, for  $\omega \in (0, \varepsilon)$  with  $\varepsilon > 0$  small enough.) The birth of “unstable” eigenvalues with positive real part can also happen from the collision

of purely imaginary eigenvalues in the spectral gap (away from the origin), like in the Soler model in two spatial dimensions [CMKS<sup>+</sup>16].

The spectral stability of solitary waves to the cubic nonlinear Dirac equation in (1+1)D (the Gross–Neveu model) was demonstrated in [BC12a], where the spectrum of the linearization at solitary waves was computed via the Evans function technique; no nonzero-real-part eigenvalues have been detected. This (1+1)D stability result was later confirmed in [Lak18] via numerical simulations of the evolution. Subsequent numerical computations of the spectral stability in the nonlinear Dirac in one and two spatial dimensions were performed in [CMKS<sup>+</sup>16].

Our main goal is to prove the asymptotic stability in the general situation, in systems with the translational invariance, without any simplifying restrictions (like “radially symmetric”) on the perturbations. This is to be achieved by the combination of the description of the structure of the solitary wave manifold and the spectral stability results (which are the main subject of this monograph) with the decay estimates. In particular, we mention the local decay estimates,  $L^p$  decay estimates, and the Strichartz estimates. In the context of the Schrödinger operator, the classical results are [JK79, KY89, KY89, JSS91, GJY04, RS04, Bec11] and [Str77b, Yaj87, GV92, KT98]. A concise exposition of the stationary scattering theory of Agmon–Jensen–Kato can be found in [KK12]. In the context of the Dirac operator, the related results are in [MNO03, MNNO05, Bou06, DF07, Bou08b]. Given the amount of the recent material and the rate of the progress in the field, we choose not to include the description of these results in the present monograph.

Some results on asymptotic stability in the context of the nonlinear Dirac equation were obtained in the three-dimensional case with the external potential [Bou06, Bou08b]. There, the spectrum of the linear part of the operator  $D_m + V$ , besides the essential spectrum  $\mathbb{R} \setminus (-m, m)$ , is assumed to contain two simple eigenvalues; let us denote them by  $\lambda_0$  and  $\lambda_1$ , with  $\lambda_0 < \lambda_1$ . There is a bifurcation of small amplitude solitary waves for the nonlinear equation from the associated eigenspaces. The corresponding linearized operators are small exponentially localized perturbations of  $D_m + V$ , so that the perturbation theory allows for a precise knowledge of the resulting spectral stability. Depending on the distance from  $\lambda_0$  to  $\lambda_1$  compared to the distance from  $\lambda_0$  to the essential spectrum, the resulting point spectrum for the linearized operator may have nonzero-real-part eigenvalues, or instead it may be discrete and purely imaginary and hence spectrally stable if a “Fermi Golden Rule” assumption, at the linearization level, is satisfied (similarly to the Schrödinger case; we refer to [Sig93, BP95, SW90, SW92, TY02a, TY02c, TY02d, TY02b, Miz08, GNT04, CM08]). In the former case, linear and dynamical instabilities take place. In the latter case, the linear stability follows from the spectral one via the perturbation theory. Using the dispersive properties of perturbations of  $D_m$ , one concludes that there is a stable manifold of real codimension 2. Due to the presence of nonzero discrete modes, even in the linearly stable case, the dynamical stability is not guaranteed. Before considering the results on the dynamics outside this manifold, for perturbations along the remaining two real directions, one could ask what might happen if  $D_m + V$  had only one eigenvalue. The answer follows quite immediately with the ideas from [Bou06, Bou08b]. In this case, there is only one family of solitary waves and it is asymptotically stable. Notice that the asymptotic profile is possibly another solitary wave which is close to the perturbed one. In the one-dimensional case, this was studied properly in [PS14].

Note that the one-dimensional framework suffers from relatively weak dispersion which makes the analysis of the stabilization process rather delicate. As for the dynamics outside the above-mentioned stable manifold, the techniques rely on the analysis of nonlinear resonances between discrete isolated modes and the essential spectrum where the dispersion takes place. This requires the normal form analysis in order to isolate the leading resonant interactions. The former is possible due to the “Fermi Golden Rule” assumption. Such an analysis was done in [BC12b] but in a slightly different framework: instead of considering the perturbative framework, the authors chose the translation-invariant case, imposing a series of assumptions that led to the spectral stability of solitary waves. These assumptions are verified in some perturbative context with  $V \neq 0$ ; this case is analyzed in [CT16].

The asymptotic stability approach from [BC12b, PS12, CPS17] is developed under important restrictions on the types of admissible perturbations. Such restrictions are needed to avoid the translational invariance and, most importantly, to prohibit the perturbations in the direction of exceptional eigenvalues  $\pm 2\omega i$  of the linearization operator at a solitary wave  $\phi_\omega(x)e^{-i\omega t}$ . These eigenvalues are a feature of the Soler model and are present in the spectrum for any nonlinearity  $f$  in the Soler model (I.1); see [BC12a, Gal77, DR79]. These eigenvalues violate the “Fermi Golden Rule”: they do not “interact” (that is, do not resonate) with the essential spectrum; the energy from the corresponding modes does not disperse to infinity. To proceed with the proof of the asymptotic stability for this case, one would need to consider the set of bi-frequency solitary waves and to include modulation equations for the corresponding parameters.

There are several open questions which we would like to mention.

(1) Formally, the Hamiltonian of the Soler model (nonlinear Dirac equation with the scalar self-interaction) has the “infinite Morse index” at the points of the functional space corresponding to the solitary wave solutions. At the same time, this model seems to be even “more stable” than the nonlinear Schrödinger equation. For example, in dimension two, the cubic nonlinear Schrödinger equation is critical (the solitary waves  $\phi_\omega(x)e^{-i\omega t}$  with  $\phi_\omega(x) = |\omega|^{1/2}\Phi(|\omega|^{1/2}x)$  have the same charge for all  $\omega < 0$ ), and as a consequence the linearization at any solitary wave has a larger size Jordan block at  $z = 0$ , resulting in an instability of solitary waves [CP03]. On the contrary, by [CMKS<sup>+</sup>16, BC19], the nonlinear Dirac equation with cubic nonlinearity in two dimensions is spectrally stable, without a higher order Jordan block at  $z = 0$ . We also mention that there is a blow-up phenomenon in the charge-critical as well as in the charge-supercritical nonlinear Schrödinger equation, see in particular [ZSS71, ZS75, Gla77, Wei83, Mer90]; at the same time, we are not aware of the blow-up results in the models based on the nonlinear Dirac equation with  $U(1)$ -invariance. Overall, the nonlinear Dirac equation seems surprisingly stable. Does it possess a stability mechanism which we do not know yet?

We point out that the stability in the nonlinear Dirac equation *does not* need the standard *Dirac sea* assumption that all the negative energy states are filled and that the Pauli principle prohibits more than one particle in a state.

(2) While the nonlinear Dirac equation is only a simplified model of self-interacting spinor fields, we expect that our approach can be modified to tackle the spectral stability of solitary waves in the Dirac–Maxwell system [Gro66, Wak66] and in similar Dirac-based models. (Let us mention that the local well-posedness of the

Dirac–Maxwell system was proved in [Bou96]. The existence of standing waves in the Dirac–Maxwell system is proved in [EGS96] (for  $\omega \in (-m, 0)$ ) and [Abe98] (for  $\omega \in (-m, m)$ ); see also the review [ES02].) Are there stable localized solutions in Dirac–Maxwell and similar systems? Are there any interesting implications in Physics?

(3) Presently, the Dirac equation gives a consistent description of the electron in the Hydrogen atom, but even for a two-electron Helium one faces problems [KOY15]. What is an appropriate model for the many-body problem based on the nonlinear Dirac equation? Are there localized solutions? How does one approach the linear stability in such models? Let us mention [BES05, ES05, Der12, Lev14].

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