

Topological Structural Stability of Attractors

Our goal in this chapter is, loosely speaking, to prove that a semigroup obtained as a perturbation (in some sense) of a *gradient semigroup* is still gradient (a property which will be called *topological structural stability of attractors*). Let us be more specific. We deal with the class of *gradient semigroups relative to an invariant set* (or a family of invariant sets) and having a global attractor. We will see that, under natural constraints, this type of semigroups can be characterized through its dynamical properties, introducing the auxiliary concept of *dynamically gradient semigroups* (those with the dynamical properties of a gradient semigroup but without requiring the existence of a Lyapunov function). Again, under natural assumptions, we obtain that a perturbation of a dynamically gradient semigroup is still a dynamically gradient semigroup. Finally we obtain that dynamically gradient semigroups have an associated Lyapunov function and therefore are gradient.

1. Gradient semigroups

We begin our chapter by considering the *gradient semigroups* relative to an invariant set. These semigroups appear naturally in several applications, and its inherent features allow us to describe with precision the inner structure of its attractors.

DEFINITION 3.1. A semigroup \mathcal{T} in X with a closed \mathcal{T} -invariant set Ξ is said to be **gradient** relative to Ξ if there exists a continuous function $V: X \rightarrow \mathbb{R}$ such that:

- (i) $\mathbb{R}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is nonincreasing for each $x \in X$.
- (ii) If x is such that $V(T(t)x) = V(x)$ for all $t \in \mathbb{R}^+$, then $x \in \Xi$.
- (iii) V is constant in each connected component of Ξ .

A function $V: X \rightarrow \mathbb{R}$ with these properties is called a **Lyapunov function** for \mathcal{T} relative to Ξ .

EXAMPLE 3.2 (Gradient vector fields). The notion of gradient semigroups in differential equations is associated to those vector fields which are the gradient of a potential, that is, if $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^1 potential with $\mathbb{R}^n \ni u \mapsto \nabla\phi(u) \in \mathbb{R}^n$ locally Lipschitz continuous, the system of differential equations

$$(3.1) \quad \begin{cases} \dot{u} = -\nabla\phi(u), & t > 0, \\ u(0) = u_0 \in \mathbb{R}^n \end{cases}$$

is called a *gradient system*.

The potential ϕ has the property that, if $[0, \tau_{u_0}) \ni t \mapsto u(t, u_0)$ is a solution of (3.1), then $[0, \tau_{u_0}) \ni t \mapsto \phi(u(t)) \in \mathbb{R}$ is nonincreasing, and consequently does not blow up as $t \rightarrow \tau_{u_0}^-$. If $\phi(x) \rightarrow \infty$ as $\|x\|_{\mathbb{R}^n} \rightarrow \infty$ we must have $\tau_{u_0} = +\infty$

for all $u_0 \in \mathbb{R}^n$, and we can define the semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ by setting $T(t)u_0 = u(t, u_0)$ for $t \geq 0$ and $u_0 \in \mathbb{R}^n$.

It is easy to see that $\phi(u(\cdot))$ is constant if and only if $u(\cdot)$ is constant and $\nabla\phi(u) = 0$, showing that $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function for \mathcal{T} relative to the set Ξ of equilibria for (3.1), that is, $\Xi = \{u \in \mathbb{R}^n : \nabla\phi(u) = 0\}$.

EXAMPLE 3.3 (A heat equation). Let Ω be a bounded smooth domain in \mathbb{R}^N , $N \geq 1$. Consider the following parabolic initial value problem with Dirichlet boundary condition:

$$(3.2) \quad \begin{cases} u_t = \Delta u + f(u), & x \in \Omega, t > 0, \\ u(x) = 0, & x \in \partial\Omega, t > 0, \\ u(0) = u_0 \in H_0^1(\Omega). \end{cases}$$

Assuming that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function with bounded derivatives up to second order, we can prove that there is a semigroup \mathcal{T} in $H_0^1(\Omega)$ associated to (3.2).

Consider the functional $V: H_0^1(\Omega) \rightarrow \mathbb{R}$ given by

$$V(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(u),$$

where $F(u)(x) = \int_0^{u(x)} f(s)ds$. It is easy to see that V is continuous and if $u: \mathbb{R}^+ \rightarrow H_0^1(\Omega)$ denotes the solution of (3.2), then

$$\frac{d}{dt}V(u(t)) = - \int_{\Omega} |u_t(t)(x)|^2 dx$$

and $V(u(\cdot))$ is constant if and only if $u(\cdot)$ is constant and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a solution of

$$(3.3) \quad \begin{cases} 0 = \Delta u + f(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

showing that $V: H_0^1(\Omega) \rightarrow \mathbb{R}$ is a Lyapunov function for \mathcal{T} relative to the set Ξ of equilibria for (3.2), that is, $\Xi = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : u \text{ satisfies (3.3)}\}$.

EXAMPLE 3.4 (A damped wave equation). Let Ω be a bounded smooth domain in \mathbb{R}^N , $N \geq 1$. For $\beta \geq 0$, consider the damped hyperbolic problem

$$(3.4) \quad \begin{cases} u_{tt} + 2\beta u_t = \Delta u + f(u), & x \in \Omega, t > 0, \\ u(x) = 0, & x \in \partial\Omega, t > 0, \\ u(0) = u_0 \in H_0^1(\Omega), \\ u_t(0) = v_0 \in L^2(\Omega). \end{cases}$$

This boundary-initial value problem can be written as an abstract ordinary differential equation in $H_0^1(\Omega) \times L^2(\Omega)$ in the following way: let $A: D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be defined by $Au = \Delta u$ for all $u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $\Lambda: D(\Lambda) \subset H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ is defined by

$$\Lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & -2\beta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ Au - 2\beta v \end{bmatrix},$$

with $D(\Lambda) = D(A) \times L^2(\Omega)$. We may rewrite (3.4) as

$$(3.5) \quad \begin{cases} \frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \Lambda \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(u) \end{bmatrix}, & t > 0, \\ \begin{bmatrix} u \\ v \end{bmatrix} (0) = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in H_0^1(\Omega) \times L^2(\Omega). \end{cases}$$

Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz function with bounded derivatives up to second order. Then, there is a semigroup \mathcal{T} in $H_0^1(\Omega) \times L^2(\Omega)$ associated to (3.5). Consider now the functional $V: H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ given by

$$V \left(\begin{bmatrix} u \\ v \end{bmatrix} \right) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \beta \int_{\Omega} v^2 - \int_{\Omega} F(u),$$

where $F(u)(x) = \int_0^{u(x)} f(s) ds$.

It is easy to see that $V \left(\begin{bmatrix} u \\ v \end{bmatrix} (\cdot) \right)$ is continuous,

$$\frac{d}{dt} V \left(\begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} (\cdot) \right) = - \int_{\Omega} |u_t(t)(x)|^2 dx$$

(at least if $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in D(A) \times H_0^1(\Omega)$), showing that V is nonincreasing along solutions

and is constant if and only if $\begin{bmatrix} u \\ v \end{bmatrix} (\cdot)$ is constant, $v = 0$ and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (3.3). Thus $V: H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ is a Lyapunov function for \mathcal{T} relative to the set Ξ of equilibria for (3.5), that is,

$$\Xi = \left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} : u \in D(A) \text{ satisfies (3.3)} \right\}.$$

To begin our study of the inner structure of attractors for gradient semigroups relative to an invariant set, we present a result that allows us to characterize the omega and alpha limits of points.

LEMMA 3.5. *Let \mathcal{T} be a gradient semigroup relative to an invariant set Ξ . Then $\omega(x) \subset \Xi$ for each $x \in X$, and if there is a global solution $\phi: \mathbb{R} \rightarrow X$ of \mathcal{T} through x , then $\alpha_{\phi}(x) \subset \Xi$.*

Furthermore, if Ξ is a disjoint union of compact (or closed and positively separated) invariant sets Ξ_1, \dots, Ξ_n , then $\omega(x) \subset \Xi_i$ for some $1 \leq i \leq n$, and if there is a global solution $\phi: \mathbb{R} \rightarrow X$ through x , then $\alpha_{\phi}(x) \subset \Xi_j$ for some $1 \leq j \leq n$.

PROOF. If $y \in \omega(x)$, there is a sequence $\{t_n\}$, with $t_n \rightarrow \infty$, such that $y = \lim_{n \rightarrow \infty} T(t_n)x$. From the continuity of V we have $V(y) = \lim_{n \rightarrow \infty} V(T(t_n)x)$, and since $\mathbb{R}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is nonincreasing we have $V(y) = \lim_{t \rightarrow \infty} V(T(t)x)$. From this and the continuities of \mathcal{T} and V we have

$$V(y) = \lim_{n \rightarrow \infty} V(T(t + t_n)x) = \lim_{n \rightarrow \infty} V(T(t)T(t_n)x) = V(T(t)y)$$

for each $t \geq 0$. From property (ii) in Definition 3.1, we have $y \in \Xi$, and since $y \in \omega(x)$ was arbitrarily chosen, we conclude that $\omega(x) \subset \Xi$.

Now suppose that there exists a global solution $\phi: \mathbb{R} \rightarrow X$ of \mathcal{T} through x . With the same reasoning as above, if $y \in \alpha_{\phi}(x)$ we have $V(y) = \lim_{t \rightarrow \infty} V(\phi(-t))$.

From this and the continuities of \mathcal{T} and V we have $V(T(t)y) = V(y)$ for each $t \geq 0$, therefore $y \in \Xi$ and we conclude that $\alpha_\phi(x) \subset \Xi$.

The last statement of the lemma follows from the fact that $\omega(x)$ and $\alpha_\phi(x)$ are connected (see Lemmas 1.16 and 1.17). \square

Before stating the main result of this section, we will present some results that ensure the existence of global attractors for gradient semigroups.

LEMMA 3.6. *If \mathcal{T} is an eventually bounded and asymptotically compact gradient semigroup relative to a bounded \mathcal{T} -invariant set Ξ , then there exists a positively \mathcal{T} -invariant and bounded set U that \mathcal{T} -absorbs all compact subsets of X .*

PROOF. For each $x \in X$ we have, from Lemma 1.16, that $\omega(x)$ is nonempty, compact, \mathcal{T} -invariant and \mathcal{T} -attracts x . From Lemma 3.5 we have $\omega(x) \subset \Xi$. Since Ξ is \mathcal{T} -invariant and \mathcal{T} is eventually bounded and asymptotically compact we must have $\Xi = \omega(\Xi)$ compact.

It follows that, for any $\epsilon > 0$, $\mathcal{O}_\epsilon(\Xi)$ \mathcal{T} -absorbs points of X . Let $B = \{x \in \mathcal{O}_\epsilon(\Xi) : \gamma^+(x) \subset \mathcal{O}_\epsilon(\Xi)\}$. Since $\mathcal{O}_\epsilon(\Xi)$ \mathcal{T} -absorbs points, we have B nonempty. Clearly $\gamma^+(B) = B$, B is bounded and \mathcal{T} -absorbs points. Also, since \mathcal{T} is eventually bounded and asymptotically compact, Lemma 1.20 implies that $\omega(B)$ is nonempty, compact, \mathcal{T} -invariant and \mathcal{T} -attracts B .

Let V be a neighborhood of $\omega(B)$ and $t_0 \in \mathbb{R}^+$ such that $\gamma_{t_0}^+(V)$ is bounded. Since $\omega(B)$ \mathcal{T} -attracts points of X and $T(t)$ is continuous, for each $x \in X$ there is a neighborhood \mathcal{O}_x of x and $t_x > 0$ such that $T(t)(\mathcal{O}_x) \subset \gamma_{t_0}^+(V)$ for $t \geq t_x$, that is, $\gamma_{t_0}^+(V)$ \mathcal{T} -absorbs a neighborhood of x for each $x \in X$. From this it follows easily that $\gamma_{t_0}^+(V)$ \mathcal{T} -absorbs compact subsets of X . Hence $U = \gamma_{t_0}^+(V)$ has the required properties. \square

LEMMA 3.7. *If \mathcal{T} is an eventually bounded and asymptotically compact gradient semigroup relative to a bounded \mathcal{T} -invariant set Ξ , then \mathcal{T} is bounded dissipative.*

PROOF. By Lemma 3.6, there exists a positively \mathcal{T} -invariant bounded set U that \mathcal{T} -absorbs all compact subsets of X . Since \mathcal{T} is eventually bounded and asymptotically compact, Lemma 1.20 implies that $\omega(U)$ \mathcal{T} -attracts U and therefore it \mathcal{T} -attracts compact subsets of X .

Let B be a bounded subset of X . By Lemma 3.6, $\omega(B)$ is nonempty, compact, \mathcal{T} -invariant and \mathcal{T} -attracts B . Since $\omega(B)$ is compact and \mathcal{T} -invariant, and $\omega(U)$ \mathcal{T} -attracts all compact subsets of X , we must have $\omega(B) \subset \omega(U)$. Therefore we conclude that $\omega(U)$ \mathcal{T} -attracts B , which concludes the proof. \square

With these lemmas we are able to state the main result of this section, which gives a characterization for the global attractor of a gradient semigroup.

THEOREM 3.8. *Let \mathcal{T} be an eventually bounded, asymptotically compact, gradient semigroup relative to a bounded \mathcal{T} -invariant set Ξ . Then \mathcal{T} has a global attractor \mathcal{A} . Moreover Ξ is compact, $\Xi \subset \mathcal{A}$ and $\mathcal{A} = W^u(\Xi)$, where*

$$W^u(\Xi) = \{y \in X : \text{there is a global solution } \phi : \mathbb{R} \rightarrow X \\ \text{with } \phi(0) = y \text{ such that } \lim_{t \rightarrow -\infty} \text{dist}_H(\phi(t), \Xi) = 0\}$$

is called the **unstable set** of Ξ . If $\Xi = \bigcup_{i=1}^n \Xi_i$, where $\Xi = \{\Xi_1, \dots, \Xi_n\}$ is a disjoint collection of compact invariant sets, then $\mathcal{A} = \bigcup_{i=1}^n W^u(\Xi_i)$. Finally, if there is a bounded connected set B that contains \mathcal{A} , then \mathcal{A} is connected.

PROOF. From Lemma 3.7 we know that \mathcal{T} is bounded dissipative and since \mathcal{T} is asymptotically compact, Theorem 1.23 implies that \mathcal{T} has a global attractor \mathcal{A} .

Let $x \in \mathcal{A}$. Theorem 1.9 implies that there exists a global solution $\phi: \mathbb{R} \rightarrow X$ of \mathcal{T} through x . Since $\phi(\mathbb{R}) \subset \mathcal{A}$ is relatively compact we have $\alpha_\phi(x) \neq \emptyset$ and, from Lemma 3.5, $\alpha_\phi(x) \subset \Xi$. This shows that $\mathcal{A} \subset W^u(\Xi)$. If $x \in W^u(\Xi)$, there is a global solution $\phi: \mathbb{R} \rightarrow X$ through x and $\lim_{t \rightarrow \pm\infty} \text{dist}_H(\phi(t), \Xi) = 0$, hence $\phi(\mathbb{R})$ is bounded. Since $\phi(\mathbb{R})$ is also \mathcal{T} -invariant we conclude that $\phi(\mathbb{R}) \subset \mathcal{A}$ and consequently $x \in \mathcal{A}$. This shows $\mathcal{A} \supset W^u(\Xi)$ and completes the proof that $\mathcal{A} = W^u(\Xi)$.

If $\Xi = \bigcup_{i=1}^n \Xi_i$ with $\{\Xi_1, \dots, \Xi_n\}$ a disjoint collection of closed invariant sets, the equality $\mathcal{A} = \bigcup_{i=1}^n W^u(\Xi_i)$ follows immediately from Lemma 3.5.

Lastly, if \mathcal{A} is contained in a bounded connected subset B of X , it follows from the \mathcal{T} -invariance of $\omega(B)$, from the fact that $B \supset \mathcal{A}$ and from Lemma 1.17 that $\mathcal{A} = \omega(B)$ is connected. \square

If \mathcal{T} is as in the previous theorem and $\Xi = \bigcup_{i=1}^n \Xi_i$, where $\Xi = \{\Xi_1, \dots, \Xi_n\}$ is a disjoint collection of compact invariant sets, we can use the Lyapunov function of \mathcal{T} relative to Ξ to give a more detailed description of the attractor, in terms of *energy levels*.

To do that, we first need the following lemma, which is an immediate consequence of the continuity of semigroups. This lemma ensures that, given a \mathcal{T} -invariant set Ξ for the semigroup \mathcal{T} and y close to Ξ , $\{T(s)y: 0 \leq s \leq t\}$ remains close to Ξ for large values of t .

LEMMA 3.9. *Let \mathcal{T} be a semigroup and let Ξ be a compact \mathcal{T} -invariant set. Given $t \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$T(s)\mathcal{O}_\delta(\Xi) \subset \mathcal{O}_\epsilon(\Xi) \quad \text{for all } s \in [0, t].$$

PROOF. Suppose that there exist $t_0 \in (0, \infty)$ and $\epsilon_0 > 0$ such that, for each $k \in \mathbb{N}^*$ there are $x_k \in \mathcal{O}_{\frac{1}{k}}(\Xi)$ and $s_k \in [0, t_0]$ with $\text{dist}_H(T(s_k)x_k, \Xi) \geq \epsilon_0$. We may assume that $s_k \rightarrow s_0$ for some $s_0 \in [0, t_0]$ and that $x_k \rightarrow y \in \Xi$. Since $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous, we have $0 = \text{dist}_H(T(s_0)y, \Xi) \geq \epsilon_0$ which is absurd, and concludes the proof. \square

Now we can prove a more general description for attractors of gradient semigroups.

THEOREM 3.10. *Let \mathcal{T} be a gradient semigroup relative to a \mathcal{T} -invariant set Ξ , with a global attractor \mathcal{A} . Suppose that $\Xi = \bigcup_{i=1}^n \Xi_i$ with $\Xi = \{\Xi_1, \dots, \Xi_n\}$ being a disjoint collection of compact invariant sets, $n \in \mathbb{N}^*$, and that the associated Lyapunov function is constant on each Ξ_i , $1 \leq i \leq n$. Let $V(\Xi) = \{\mathbf{n}_1, \dots, \mathbf{n}_p\}$ with $\mathbf{n}_i < \mathbf{n}_{i+1}$ for $1 \leq i \leq p-1$.*

If $1 \leq j \leq p-1$ and $\mathbf{n}_j \leq r < \mathbf{n}_{j+1}$, then $X_r = \{z \in X: V(z) \leq r\}$ is positively \mathcal{T} -invariant and \mathcal{T}_r , the restriction of \mathcal{T} to X_r , has a global attractor $\mathcal{A}^{(j)}$ given by

$$\mathcal{A}^{(j)} = \bigcup \{W^u(\Xi_\ell): V(\Xi_\ell) \leq \mathbf{n}_j\}.$$

In particular, $V(z) \leq \mathbf{n}_j$ for $z \in \mathcal{A}^{(j)}$, $\mathbf{n}_1 = \min\{V(x) : x \in X\}$ and $\mathcal{A}^{(1)} = \bigcup\{\tilde{\Xi} \in \Xi : V(\tilde{\Xi}) = \mathbf{n}_1\}$ consists of all asymptotically stable \mathcal{T} -invariant sets, that is, for each $\tilde{\Xi} \in \Xi$ with $\tilde{\Xi} \subset \mathcal{A}^{(1)}$ there is an $\epsilon > 0$ such that $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \tilde{\Xi}) = 0$ whenever $x \in \mathcal{O}_\epsilon(\tilde{\Xi})$.

PROOF. It is clear from the definition of a Lyapunov function that X_r is positively \mathcal{T} -invariant. To show the existence of a global attractor for \mathcal{T}_r we note that the required properties for the existence of a global attractor are inherited from those of \mathcal{T} , namely, \mathcal{T}_r is bounded dissipative and asymptotically compact. Hence \mathcal{T}_r has a global attractor $\mathcal{A}^{(j)}$. The restriction V_r of V to X_r is a Lyapunov function for \mathcal{T}_r associated with $\{\Xi_\ell : V(\Xi_\ell) \leq \mathbf{n}_j\}$ and the characterization of $\mathcal{A}^{(j)}$ follows.

Let us now prove the last claim. Let $\delta_0 = \frac{1}{2} \min\{d(\tilde{\Xi}, \hat{\Xi})\} > 0$ where the minimum is taken over the sets $\tilde{\Xi}, \hat{\Xi} \in \Xi$ such that $\tilde{\Xi}, \hat{\Xi} \subset \mathcal{A}^{(1)}$ and $\tilde{\Xi} \neq \hat{\Xi}$. Assume by contradiction that there exist $\Xi \ni \tilde{\Xi} \subset \mathcal{A}^{(1)}$, $0 < \delta < \delta_0$ and sequences $\{x_k\}$ in X , $\{t_k\}$ in \mathbb{R}^+ with $x_k \rightarrow \Xi$, $\text{dist}_H(T(t)x_k, \tilde{\Xi}) < \delta$ for $0 \leq t < t_k$ and $\text{dist}_H(T(t_k)x_k, \tilde{\Xi}) = \delta$. From Lemma 3.9 we have $t_k \rightarrow \infty$ and the asymptotic compactness of \mathcal{T} ensures that $\{T(t_k)x_k\}$ has a convergent subsequence, which we denote the same, and let y be its limit. It is simple to see that \mathbf{n}_1 is the minimum value of V in X , and from this it is immediate that

$$\mathbf{n}_1 = V(\tilde{\Xi}) = \lim_{k \rightarrow \infty} V(x_k) \geq \lim_{k \rightarrow \infty} V(T(t_k)x_k) = V(y) \geq \mathbf{n}_1,$$

and

$$\mathbf{n}_1 = V(\tilde{\Xi}) = \lim_{k \rightarrow \infty} V(x_k) \geq \lim_{k \rightarrow \infty} V(T(t + t_k)x_k) = V(T(t)y) \geq \mathbf{n}_1 \text{ for each } t \geq 0.$$

Hence $V(y) = V(T(t)y) = \mathbf{n}_1$ for all $t \in \mathbb{R}^+$ and therefore $y \in \Xi$. Since $\text{dist}_H(y, \tilde{\Xi}) < \delta_0$ we have $y \in \tilde{\Xi}$. But $\text{dist}_H(y, \tilde{\Xi}) = \delta > 0$, which gives us a contradiction. This proves that for each $\tilde{\Xi} \in \mathcal{A}^{(1)}$ and $0 < \delta < \delta_0$ there is a $0 < \delta' < \delta$ such that, for all $x \in \mathcal{O}_{\delta'}(\tilde{\Xi})$, we have $\gamma^*(x) \subset \mathcal{O}_\delta(\tilde{\Xi})$ and proves that $\mathcal{A}^{(1)}$ consists only of stable invariant sets. To conclude, we only need to note that, for each $x \in X$, it follows from Lemma 3.5 that $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \omega(x)) = 0$, and that $\omega(x) \subset \tilde{\Xi}$ for some $\tilde{\Xi} \in \Xi$. \square

2. Dynamically gradient semigroups

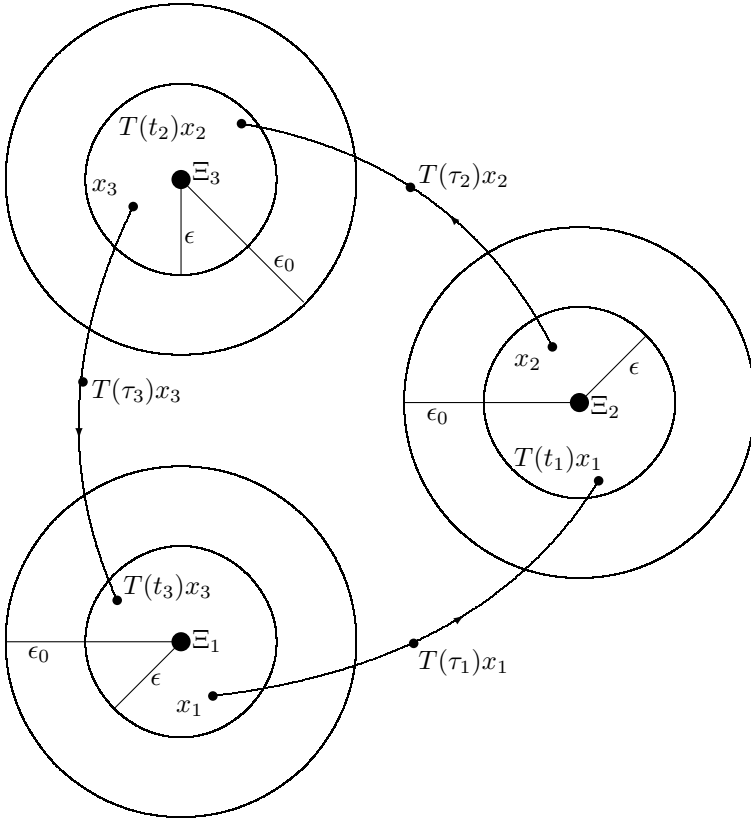
In this section we present the concept of *dynamically gradient semigroups*. These semigroups have the dynamical properties of a gradient semigroup, but their definition does not require the existence of a Lyapunov function.

Under natural assumptions we show that dynamically gradient semigroups are gradient and that dynamically gradient semigroups are stable under perturbations. This stability of the gradient structure under perturbations (which we call *topological structural stability*) is a global property of attractors that is entirely obtained from the continuity properties of the semigroups. Our presentation is based on [74].

Let \mathcal{T} be a semigroup with a global attractor \mathcal{A} , which contains a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Let

$$(3.6) \quad \epsilon_0 = \epsilon_0(\Xi) = \frac{1}{2} \min_{1 \leq i < j \leq n} \text{dist}(\Xi_i, \Xi_j) > 0.$$

We make the following definition.

FIGURE 3.1. Example of an ϵ -chain.

DEFINITION 3.11. Let $\Xi^* \in \Xi$ and $0 < \epsilon < \epsilon_0$. An ϵ -**chain**¹ at Ξ^* consists of

- (a) a subcollection $\{\Xi_{\ell_1}, \Xi_{\ell_2}, \dots, \Xi_{\ell_k}\}$ of Ξ with $\Xi^* = \Xi_{\ell_1} =: \Xi_{\ell_{k+1}}$ and

$$\Xi_{\ell_i} \cap \Xi_{\ell_j} = \emptyset \quad \text{for } 1 \leq i < j \leq k;$$

- (b) a finite collection of points $\{x_1, \dots, x_k\}$ in X with

$$\text{dist}_H(x_i, \Xi_{\ell_i}) < \epsilon \quad \text{for } 1 \leq i \leq k;$$

- (c) finite sequences of nonnegative real numbers $\{t_1, \dots, t_k\}$ and $\{\tau_1, \dots, \tau_k\}$ with $\tau_i < t_i$, for all $i = 1, \dots, k$, such that

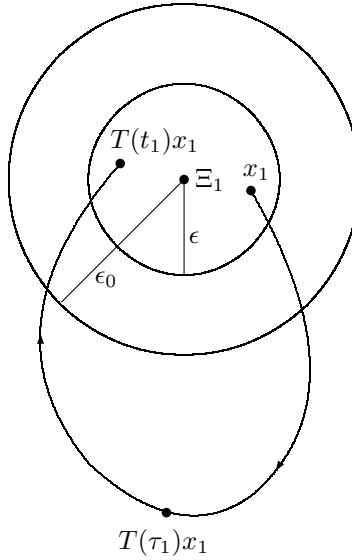
$$\text{dist}_H(T(t_i)x_i, \Xi_{\ell_{i+1}}) < \epsilon \quad \text{for } 1 \leq i \leq k,$$

and also

$$\text{dist}_H\left(T(\tau_i)x_i, \bigcup_{j=1}^k \mathcal{O}_{\epsilon_0}(\Xi_{\ell_j})\right) > 0 \quad \text{for } 1 \leq i \leq k.$$

An isolated invariant set $\Xi^* \in \Xi$ is said to be **chain recurrent** if there exists $\delta \in (0, \epsilon_0)$ and ϵ -chains at Ξ^* , for each $\epsilon \in (0, \delta)$.

¹See Figures 3.1 and 3.2.

FIGURE 3.2. Example of an ϵ -chain when $k = 1$.

We are now ready to define the main concept of this section, the *dynamically gradient semigroups*.

DEFINITION 3.12. Let \mathcal{T} be a semigroup in X with a global attractor \mathcal{A} and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$ in \mathcal{A} . We say that \mathcal{T} is a **dynamically gradient semigroup** relative to Ξ if the following two conditions are satisfied:

(G1) Any global solution $\xi: \mathbb{R} \rightarrow X$ of \mathcal{T} in \mathcal{A} satisfies

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi(t), \Xi_i) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\xi(t), \Xi_j) = 0,$$

for some $1 \leq i, j \leq n$.

(G2) No set of Ξ is chain recurrent.

We note that condition (G1) is imposed only for solutions in the global attractor. However, as for gradient semigroups, we are able to prove that for a dynamically gradient semigroup the ω -limit set of each point is contained in one of the isolated invariant sets of Ξ .

The following lemmas will play an important role in the remaining results of the chapter. Our first lemma establishes that semigroups possessing a global attractor with a disjoint collection of isolated invariant sets and satisfying (G1) have the property that: given any suitably small neighborhood V of the union of the isolated invariant sets and any bounded set $B \subset X$, there is a time $t(V, B) > 0$ before which any solution starting at a point of B must visit the neighborhood V . The result also holds for families of semigroups.

LEMMA 3.13. Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups, continuous and asymptotically compact at $\eta = 0$, with a global attractor \mathcal{A}_η and a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1,\eta}, \dots, \Xi_{n,\eta}\}$, for each $\eta \in [0, 1]$, such that \mathcal{T}_0 satisfies (G1). Given $0 < \delta < \epsilon_0(\Xi_0)$, with $\epsilon_0(\Xi_0)$ as in (3.6), and a bounded

set $B \subset X$, there exist positive numbers $t_0 = t_0(\delta, B)$ and $\eta_0 > 0$ such that $\{T_\eta(t)u_0 : 0 \leq t \leq t_0\} \cap \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0}) \neq \emptyset$ for all $u_0 \in B$ and $\eta \in [0, \eta_0]$.

Moreover, if $\max_{j=1, \dots, n} \text{dist}_H(\Xi_{j,\eta}, \Xi_{j,0}) \rightarrow 0$, as $\eta \rightarrow 0$, we can replace $\Xi_{i,0}$ in the previous conclusion by $\Xi_{i,\eta}$.

PROOF. We argue by contradiction. Assume that there exist sequences $\{u_k\}$ in B , $\{\eta_k\}$ in $[0, 1]$, with $\eta_k \rightarrow 0$, and a sequence of positive numbers $\{t_k\}$ with $t_k \rightarrow \infty$ such that $\{T_{\eta_k}(t)u_k : 0 \leq t \leq t_k\} \cap \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0}) = \emptyset$. Extracting subsequences, from Lemma 2.4 with $a_n = 0$, $b_n = \frac{t_n}{2}$ and $c_n = t_n$, we have a bounded global solution $\xi: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 such that $T_{\eta_k}(t + \frac{t_k}{2})u_k \rightarrow \xi(t)$ uniformly in compact subsets of \mathbb{R} . Hence, $\xi(t) \notin \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_{i,0})$ for all $t \in \mathbb{R}$ and this contradicts (G1). \square

Our next lemma establishes that dynamically gradient semigroups have the property that, given a neighborhood U of an isolated \mathcal{T} -invariant set Ξ , there is a smaller neighborhood V of Ξ such that, if a solution starts in V and, after some time, leaves U , then it never returns to V . The result also holds for families of semigroups.

LEMMA 3.14. *Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups, continuous and collectively asymptotically compact at $\eta = 0$. Assume that for each $\eta \in [0, 1]$ there exist a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1,\eta}, \dots, \Xi_{n,\eta}\}$ and a global attractor \mathcal{A}_η , and assume that $\max_{1 \leq i \leq n} \text{dist}_H(\Xi_{i,\eta}, \Xi_{i,0}) \rightarrow 0$ as $\eta \rightarrow 0$ and let $0 < \delta_0 < \epsilon_0(\Xi_0)$. If \mathcal{T}_0 is dynamically gradient, then given $0 < \delta < \delta_0$, there exist $\eta_0 > 0$ and $\delta' > 0$ (independent from $\eta \in [0, \eta_0]$) such that, if $\eta \in [0, \eta_0]$, $\text{dist}_H(z_0, \Xi_{i,\eta}) < \delta'$, $1 \leq i \leq n$, and for some $t_1 > 0$ we have $\text{dist}_H(T_\eta(t_1)z_0, \Xi_{i,\eta}) \geq \delta$, then $\text{dist}_H(T_\eta(t)z_0, \Xi_{i,\eta}) > \delta'$ for all $t \geq t_1$.*

PROOF. Assume that, for some $1 \leq i \leq n$, there exist a sequence $\{z_k\}$ in X , $\{\eta_k\} \subset [0, 1]$ with $\eta_k \rightarrow 0$, $\text{dist}_H(z_k, \Xi_{i,\eta_k}) < \frac{1}{k}$ and sequences $\sigma_k < \tau_k$ in \mathbb{R}^+ such that $\text{dist}_H(T_{\eta_k}(\sigma_k)z_k, \Xi_{i,\eta_k}) \geq \delta$ and $\text{dist}_H(T_{\eta_k}(\tau_k)z_k, \Xi_{i,\eta_k}) < \frac{1}{k}$. We claim that this contradicts property (G2) of \mathcal{T}_0 .

First note that $\{\sigma_k\}$ can be chosen such that $T_{\eta_k}(s)u_k \in \mathcal{O}_\delta(\Xi_{i,\eta_k})$ for all $s < \sigma_k$ and, from Lemma 2.4, $\sigma_k \rightarrow \infty$. Taking subsequences we can construct a global solution $\xi_0: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T}_0 such that $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_0(t), \Xi_{i,0}) = 0$. From the fact that \mathcal{T}_0 is dynamically gradient there exists $j \neq i$ such that $\lim_{t \rightarrow \infty} \text{dist}_H(\xi_0(t), \Xi_{j,0}) = 0$.

From the fact that $\lim_{t \rightarrow \infty} \text{dist}_H(\xi_0(t), \Xi_{j,0}) = 0$ and from the construction of ξ_0 , there exist a subsequence $\{\eta_{k_m}\}$ of $\{\eta_k\}$ and subsequences $\{\tau_{k_m}\}$, $\{t_{k_m}\}$ in \mathbb{R}^+ , $\tau_{k_m} < \sigma_{k_m}$, such that

$$\text{dist}_H(T_{\eta_{k_m}}(\tau_{k_m})z_{k_m}, \Xi_{j,\eta_{k_m}}) < \frac{1}{m} \quad \text{and} \quad \text{dist}_H(T_{\eta_{k_m}}(\sigma_{k_m})z_{k_m}, \Xi_{j,\eta_{k_m}}) \geq \delta.$$

Proceeding exactly as in the previous step, we obtain a solution $\xi_1: \mathbb{R} \rightarrow \mathcal{A}$ such that $\lim_{s \rightarrow -\infty} \text{dist}_H(\xi_1(s), \Xi_{j,0}) = 0$. From the fact that \mathcal{T}_0 is dynamically gradient there exists $\Xi_{\ell,0} \in \Xi_0$, $\Xi_{\ell,0} \notin \{\Xi_{i,0}, \Xi_{j,0}\}$, such that $\lim_{s \rightarrow \infty} \text{dist}_H(\xi_1(s), \Xi_{\ell,0}) = 0$. In a finite number of steps, we construct a chain recurrent set in Ξ_0 , which leads to a contradiction with (G2). \square

A first consequence of Lemmas 3.14 and 3.13 is given in the next result, which establishes that any solution of a dynamically gradient semigroup must converge to an isolated invariant set.

LEMMA 3.15. *Suppose that \mathcal{T} is a dynamically gradient semigroup with global attractor \mathcal{A} and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Given $x \in X$ there exists $1 \leq j \leq n$ such that*

$$\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \Xi_j) = 0,$$

that is, $\omega(x) \subset \Xi_j$.

PROOF. It follows from Lemma 3.14 that given $\delta \in (0, \delta_0)$ there is a $\delta' \in (0, \delta)$ such that if $\text{dist}_H(y, \Xi_i) < \delta'$ and for some $t_{y, \delta} > 0$, $\text{dist}_H(T(t_{y, \delta})y, \Xi_i) \geq \delta$, then $\text{dist}_H(T(t)v, \Xi_i) > \delta'$ for all $t \geq t_{y, \delta}$. On the other hand, using that $\gamma^+(x)$ is bounded, it follows from Lemma 3.13 that, given δ' there exists $t_{\delta'} = t_{\delta'}(\gamma^+(x)) \geq 0$ such that, for each $y \in \gamma^+(x)$, we have

$$\{T(t)y : 0 \leq t \leq t_{\delta'}\} \cap \bigcup_{i=1}^n \mathcal{O}_{\delta'}(\Xi_i) \neq \emptyset.$$

From the fact that Ξ is finite, there exist $\Xi_j \in \Xi$ and, for each $\delta \in (0, \delta_0)$, a $s_\delta \geq 0$ such that $T(s)x \in \mathcal{O}_\delta(\Xi_j)$ for all $s \geq s_\delta$. This, together with another application of Lemma 3.14, completes the proof. \square

Later in this chapter, in Section 4, we will prove the topological structural stability of dynamically gradient semigroups, that is, suitably small autonomous perturbations of dynamically gradient semigroups are still dynamically gradient semigroups. To that end, the following notion is, sometimes, a convenient replacement for condition (G2) of Definition 3.12.

DEFINITION 3.16. Let \mathcal{T} be a semigroup with global attractor \mathcal{A} and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Let $0 < \delta < \epsilon_0$, where ϵ_0 is given in (3.6). A **homoclinic structure** in \mathcal{A} is a disjoint subcollection $\{\Xi_{\ell_1}, \dots, \Xi_{\ell_k}\}$ of Ξ and a finite number of bounded global solutions $\xi_i : \mathbb{R} \rightarrow X$, for $1 \leq i \leq k$, such that, with $\Xi_{\ell_{k+1}} = \Xi_{\ell_1}$,

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_i(t), \Xi_{\ell_i}) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \text{dist}_H(\xi_i(t), \Xi_{\ell_{i+1}}) = 0$$

for $1 \leq i \leq k$, and

$$(3.7) \quad \min_{1 \leq i \leq k} \sup_{t \in \mathbb{R}} \text{dist}_H\left(\xi_i(t), \bigcup_{i=1}^n \mathcal{O}_\delta(\Xi_i)\right) > 0.$$

REMARK 3.17. Condition (3.7) has a technical nature and it is used only when $k = 1$ to ensure that the global solution $\xi : \mathbb{R} \rightarrow \mathcal{A}$ is not entirely contained in the isolated invariant set, that is, $\xi(t) \in \Xi_i$ for all $t \in \mathbb{R}$ and some $i \in \{1, \dots, n\}$ does not make a homoclinic structure. A pictorial example of a homoclinic structure can be seen in Figure 3.3.

In what follows we prove that, if (G1) is satisfied, we can replace (G2) in Definition 3.12 by the nonexistence of homoclinic structures.

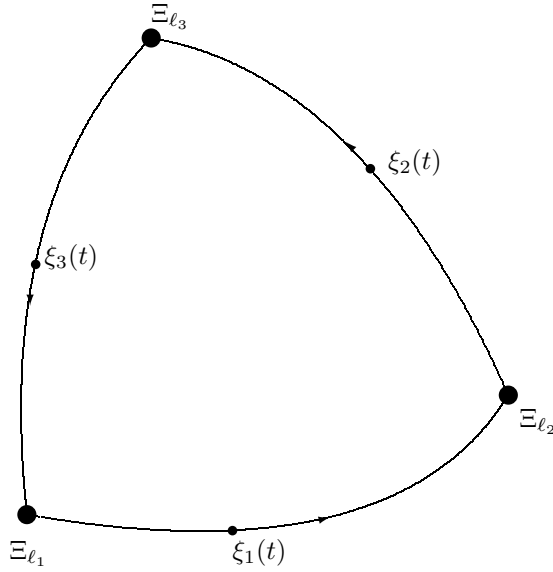


FIGURE 3.3. Example of a homoclinic structure.

PROPOSITION 3.18. *If \mathcal{T} is a semigroup with a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$ which has a global attractor \mathcal{A} and satisfies (G1), then (G2) is satisfied if and only if \mathcal{A} does not have any homoclinic structure.*

PROOF. Assuming that \mathcal{A} has a homoclinic structure it is easy to see that any of the isolated invariant sets in it is chain recurrent. On the other hand, if one of the isolated invariant sets $\Xi \in \Xi$ is chain recurrent, there exist a subset $\{\Xi_{k_1}, \dots, \Xi_{k_\ell}\}$, $\Xi = \Xi_{k_1} =: \Xi_{k_{\ell+1}}$ which we will denote (after a reordering of Ξ) by $\{\Xi_1, \dots, \Xi_\ell\}$ and, for each positive integer k , points x_1^k, \dots, x_ℓ^k and positive numbers t_1^k, \dots, t_ℓ^k such that

$$\text{dist}_H(x_i^k, \Xi_i) < \frac{1}{k} \quad \text{and} \quad \text{dist}_H(T(t_i^k)x_i^k, \Xi_{i+1}) < \frac{1}{k}, \quad 1 \leq i \leq \ell,$$

and we may assume that $\overline{\bigcup_{k=1}^{\infty} \bigcup_{t \in [0, t_i^k]} T(t)x_i^k} \cap \Xi_j \neq \emptyset$ if j is different from i and $i+1$ (adding more isolated invariant sets to the $\frac{1}{k}$ chains if needed).

Consider ϵ_0 as in (3.6), and choose $0 < \delta < \epsilon_0$ and $\tau_i^k > 0$ such that $\text{dist}_H(T(t)x_i^k, \Xi_i) < \delta$ for all $0 \leq t < \tau_i^k$ and $\text{dist}_H(T(\tau_i^k)x_i^k, \Xi_i) = \delta$. It is clear from the continuity of \mathcal{T} and from the invariance of Ξ_i that $\tau_i^k \rightarrow \infty$ as $k \rightarrow \infty$. For $t \in [-\tau_i^k, \infty)$ let $\xi_i^k(t) = T(\tau_i^k + t)x_i^k$. Taking subsequences we define the global solutions $\xi_i: \mathbb{R} \rightarrow X$ by $\xi_i(t) = \lim_{k \rightarrow \infty} \xi_i^k(t)$. Since each $\xi_i(t)$ must converge to an isolated invariant set as $t \rightarrow +\infty$ and as $t \rightarrow -\infty$ and since $\xi_i(t) \in \mathcal{O}_\delta(\Xi_i)$ for all $t \leq 0$ we have that $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_i(t), \Xi_i) = 0$. Since $\overline{\bigcup_{k=1}^{\infty} \bigcup_{t \in [0, t_i^k]} T(t)x_i^k} \cap \Xi_j \neq \emptyset$, j different from i and $i+1$, we must have $\lim_{t \rightarrow \infty} \text{dist}_H(\xi_i(t), \Xi_{i+1}) = 0$.

The collection of isolated invariant sets $\{\Xi_1, \dots, \Xi_\ell\} \subset \Xi$ and the set of global solutions $\{\xi_i: \mathbb{R} \rightarrow X, 1 \leq i \leq \ell\}$ are such that $\Xi = \Xi_1 = \Xi_{\ell+1}$ and

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_i(t), \Xi_i) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\xi_i(t), \Xi_{i+1}) = 0,$$

for $1 \leq i \leq \ell$. Hence \mathcal{A} has a homoclinic structure. \square

COROLLARY 3.19. *If \mathcal{T} is a dynamically gradient semigroup relative to a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$ and \mathcal{A} is its global attractor, then there are isolated invariant sets Ξ_α and Ξ_ω such that Ξ_α has trivial stable set in \mathcal{A} , that is, $W^s(\Xi_\alpha) \cap \mathcal{A} = \Xi_\alpha$, and Ξ_ω has trivial unstable set, that is, $W^u(\Xi_\omega) = \Xi_\omega$.*

PROOF. Let us prove the existence of at least one isolated invariant set Ξ_ω with a trivial unstable set. If such a set does not exist, then for each Ξ_i , $1 \leq i \leq n$, there is a global solution $\xi_i: \mathbb{R} \rightarrow \mathcal{A}$ such that $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_i(t), \Xi_i) = 0$ and $\xi_i(\mathbb{R}) \not\subset \Xi_i$. Choose $\xi_1: \mathbb{R} \rightarrow \mathcal{A}$ such that $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_1(t), \Xi_1) = 0$ and $\xi_1(\mathbb{R}) \not\subset \Xi_1$. Set $\Xi_{\ell_1} = \Xi_1$. From (G1), there exists $\Xi_{\ell_2} \in \Xi$ with $\ell_2 \neq \ell_1$ such that $\lim_{t \rightarrow \infty} \text{dist}_H(\xi_1(t), \Xi_{\ell_2}) = 0$. Choose $\xi_2: \mathbb{R} \rightarrow X$ such that $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_2(t), \Xi_{\ell_2}) = 0$ and $\xi_2(\mathbb{R}) \not\subset \Xi_{\ell_2}$. Let $\ell_3 \notin \{\ell_1, \ell_2\}$ be such that $\lim_{t \rightarrow \infty} \text{dist}_H(\xi_2(t), \Xi_{\ell_3}) = 0$. In a finite number of steps we arrive at a contradiction. This proves the existence of an isolated invariant set with trivial unstable set.

The proof of the existence of an isolated invariant set Ξ_α with trivial stable set in \mathcal{A} is similar and it is left to the reader. \square

REMARK 3.20. We point out that in the next two chapters we will deal with *discrete semigroups*. But for a dynamically gradient semigroup \mathcal{T} with a global attractor \mathcal{A} , if we consider the associated discrete semigroup $\{T^n: n \in \mathbb{N}\}$, where $T = T(1)$, then it is easy to see that \mathcal{A} is also the global attractor of $\{T^n: n \in \mathbb{N}\}$ (see [28, Remark 3.3] for details).

2.1. Examples.

EXAMPLE 3.21. If \mathcal{T} is a gradient semigroup relative to a finite set of equilibria $\Xi = \{x_1^*, \dots, x_n^*\}$ and has a global attractor, it is not difficult to prove that \mathcal{T} is dynamically gradient relatively to Ξ , using Lemma 3.5.

EXAMPLE 3.22. Consider the semigroup in \mathbb{R}^2 which is pictorially represented by Figure 3.4.

If x_1^* is the stable node, x_2^* is the saddle and x_3^* is the unstable focus, choose $\Xi_1 = \{x_1^*\}$, $\Xi_2 = W^u(x_2^*) \cup \{x_2^*\}$ and $\Xi_3 = \{x_3^*\}$. It follows that the semigroup \mathcal{T} associated to Figure 3.4 is dynamically gradient relative to the disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \Xi_3, \Xi_3\}$.

EXAMPLE 3.23. Consider the following autonomous equation in polar coordinates:

$$(3.8) \quad \begin{cases} \dot{r} = -r(r-1)(r-2), \\ \dot{\theta} = 1, \end{cases}$$

which has, in the (x, y) -plane, $\sigma_0 = (0, 0)$ as a fixed point and two periodic orbits

$$P_{r_0} = \{(r_0 \cos \theta, r_0 \sin \theta) : \theta \in [0, 2\pi]\},$$

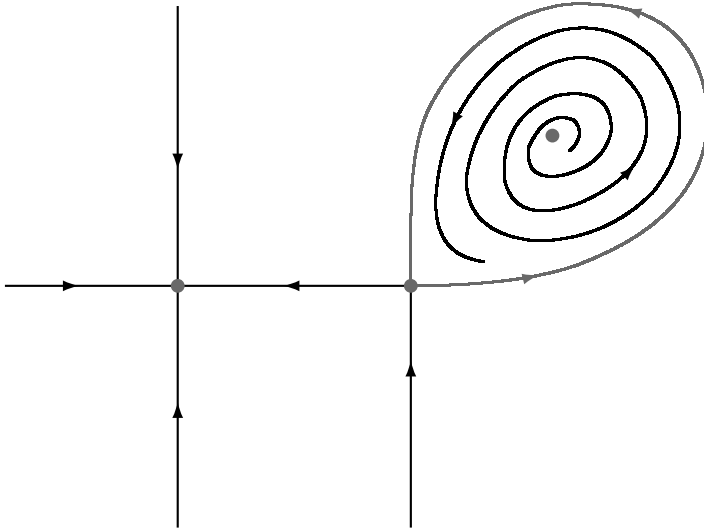


FIGURE 3.4. Isolated invariant sets: the connected sets formed by the three equilibria and the homoclinic.

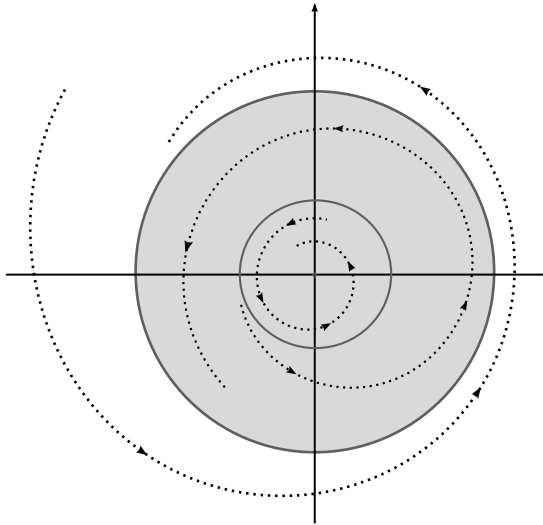


FIGURE 3.5. Global attractor.

with $r_0 = 1$ or 2 . We can see that $(0, 0)$ is stable, the periodic orbit P_1 is unstable and P_2 is stable.

It is easy to see that (3.8) has a global attractor $\mathcal{A}_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$ and that (3.8) generates a dynamically gradient semigroup with the family of isolated invariant sets $\mathbf{P} = \{\sigma_0, P_1, P_2\}$. Its internal dynamics can be seen in Figure 3.5.

EXAMPLE 3.24. Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuously differentiable functions, and for $h(\cdot) = f(\cdot)$ or $h(\cdot) = g(u, \cdot)$, for each $u \in \mathbb{R}$, we

have

$$(3.9) \quad \limsup_{|s| \rightarrow \infty} \frac{h(s)}{s} \leq -\delta < 0,$$

for some $\delta > 0$.

Consider the initial boundary value problem

$$(3.10) \quad \begin{cases} \dot{u} = f(u), \\ \dot{v} = g(u, v), \\ u(0) = u_0 \in \mathbb{R}, \\ v(0) = v_0 \in \mathbb{R}, \end{cases}$$

which defines a semigroup \mathcal{T} in \mathbb{R}^2 with a global attractor \mathcal{A} .

Assume that for all u^* such that $f(u^*) = 0$ we have $f'(u^*) \neq 0$ and $g'(u^*, v^*) \neq 0$ for those v^* such that $g(u^*, v^*) = 0$. It is clear that, if $(u(\cdot), v(\cdot)): \mathbb{R} \rightarrow \mathbb{R}^2$ is a global solution of (3.10) its first coordinate converges (backwards and forwards; see Lemma 3.5) to u_{\pm}^* such that $f(u_{\pm}^*) = 0$. It follows that $v(\cdot): \mathbb{R} \rightarrow \mathbb{R}$ is a solution of the asymptotically autonomous (backwards and forwards) problem $\dot{v} = g(u(t), v)$ and (as a consequence of the results in Chapter 8) it converges (backwards and forwards) to v_{\pm}^* satisfying $g(u_{\pm}^*, v_{\pm}^*) = 0$. This reasoning ensures that the global solutions of (3.10) satisfy (G1). The fact that (G2) is satisfied is immediate, since it has to be satisfied for the first coordinate.

REMARK 3.25. We are not able to exhibit a Lyapunov function for (3.10), but we will show in Theorem 3.41 that such a function exists.

The same reasoning above can be applied to general cascade examples in \mathbb{R}^n or to some cascade-like examples that arise in partial differential equations such as the following: let Ω be a bounded smooth domain in \mathbb{R}^n and $P_0, P_1 \in \overline{\Omega}$. Assume that $f, g \in C^2(\mathbb{R}, \mathbb{R})$ and satisfy (3.9). Consider the initial boundary value problem

$$(3.11) \quad \begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega, \\ u(0) = u_0 \in C(\overline{\Omega}), & \\ v_t = v_{xx} + g(v) & \text{in } (0, 1), \\ v(t, 0) = u(t, P_0), \quad v(t, 1) = u(t, P_1), & \\ v(0, \cdot) = v_0 \in H^1(0, 1), & \end{cases}$$

which defines a semigroup \mathcal{T} in $W^{1,p}(\Omega) \times W^{1,p}(0, 1)$ with a global attractor \mathcal{A} (see, for example, [12]).

Assume that (3.11) has a finite number of equilibria and each of them is hyperbolic. It is clear that (G2) is satisfied and, as before, the solutions in \mathcal{A} are backwards and forwards asymptotic to equilibria (satisfy (G1)). Consequently, \mathcal{T} is dynamically gradient.

3. Dynamically gradient semigroups are gradient

In this section we exhibit a Morse decomposition for a dynamically gradient semigroup and, using that, prove the existence of a Lyapunov function for it. With this we will have proved that a dynamically gradient semigroup is, in fact, gradient. This result allows us to conclude that the gradient structure of semigroups is robust under autonomous perturbations, that is, a *small perturbation of a gradient*

semigroup is still a gradient semigroup, by a simple application of Theorem 3.42. The material presented here is adapted from [5].

3.1. Morse decomposition for dynamically gradient semigroups. Now we will introduce the notion of a *Morse decomposition* for a global attractor \mathcal{A} of a dynamically gradient semigroup \mathcal{T} . We will define firstly the notion of *attractor-repeller pairs*.

DEFINITION 3.26. Let \mathcal{T} be a semigroup in a metric space X with a global attractor \mathcal{A} . We say that a nonempty subset Ξ of \mathcal{A} is a **local attractor** if there exists an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. The **repeller** Ξ^* associated to a local attractor Ξ is the set defined by

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \emptyset\}.$$

The pair (Ξ, Ξ^*) is called an **attractor-repeller pair** for \mathcal{T} .

LEMMA 3.27. *Let \mathcal{T} be a semigroup in a metric space X with a global attractor \mathcal{A} . If Ξ is a local attractor, then there exists $\epsilon > 0$ such that $\Xi^* \cap \mathcal{O}_\epsilon(\Xi) = \emptyset$. Moreover, Ξ^* is closed and \mathcal{T} -invariant.*

PROOF. Since Ξ is a local attractor, by definition there exists $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. Hence if $x \in \mathcal{O}_\epsilon(\Xi)$, then $\omega(x) \cap \Xi \neq \emptyset$ and therefore $x \notin \Xi^*$. Now, since $\omega(T(t)x) = \omega(x)$ for each $t \geq 0$ we have $x \in \Xi^*$ if and only if $T(t)x \in \Xi^*$ and the \mathcal{T} -invariance follows. To see that Ξ^* is closed note that, since the closure of a \mathcal{T} -invariant is \mathcal{T} -invariant and since Ξ^* does not intersect $\mathcal{O}_\epsilon(\Xi)$, all points in the closure of Ξ^* are also in Ξ^* . \square

DEFINITION 3.28. An ordered n -tuple $\Xi = \{\Xi_1, \dots, \Xi_n\}$ of isolated invariant sets is called a **Morse decomposition** of \mathcal{A} if, whenever $\phi: \mathbb{R} \rightarrow \mathcal{A}$ is a global solution of \mathcal{T} , then either

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\phi(t), \Xi_j) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\phi(t), \Xi_i) = 0 \quad \text{with } i < j,$$

or $\phi(\mathbb{R}) \subset \Xi_i$ for some $1 \leq i \leq n$.

Observe that, from Lemma 1.20, Ξ is a local attractor if and only if it is compact, \mathcal{T} -invariant and \mathcal{T} -attracts $\mathcal{O}_\epsilon(\Xi)$ for some $\epsilon > 0$.

REMARK 3.29. We stress the fact that the above definition differs slightly from the usual definition since the local attractor is required to \mathcal{T} -attract a neighborhood of Ξ in X and not in \mathcal{A} as in [105, 192]. We will prove next that both definitions coincide.

LEMMA 3.30. *Let \mathcal{T} be a semigroup in a metric space X with a global attractor \mathcal{A} . If Ξ is compact, \mathcal{T} -invariant and there is an $\epsilon > 0$ such that Ξ \mathcal{T} -attracts $\mathcal{O}_\epsilon(\Xi) \cap \mathcal{A}$, then given $\delta > 0$ there is a $\delta' > 0$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, where $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) = \bigcup_{x \in \mathcal{O}_{\delta'}(\Xi)} \bigcup_{t \geq 0} T(t)x$.*

PROOF. Given $0 < \delta < \epsilon$ if there is no $\delta' > 0$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, there are $x \in \Xi$, $X \ni x_n \rightarrow x$ and, from Lemma 3.9, $\mathbb{R} \ni t_n \rightarrow \infty$ such that $\text{dist}_H(T(t_n)x_n, \Xi) = \delta$ and $T(t)x_n \in \mathcal{O}_\delta(\Xi)$, $t \in [0, t_n]$. Since \mathcal{T} has a global attractor it is not difficult to see that there is a global solution $\xi: \mathbb{R} \rightarrow X$ of \mathcal{T} such that $\xi_{n_k}: [-t_{n_k}, \infty) \rightarrow X$ given by $\xi_{n_k}(t) = T(t_{n_k} + t)x_{n_k}$ satisfies $\frac{\xi_{n_k}(t)}{t} \rightarrow \xi(t)$ for each $t \in \mathbb{R}$ and some subsequence $n_k \rightarrow \infty$. Clearly $\xi(t) \in \mathcal{O}_\delta(\Xi) \cap \mathcal{A} \subset$

$\mathcal{O}_\epsilon(\Xi) \cap \mathcal{A}$ for all $t \leq 0$, and $\text{dist}_H(\xi(0), \Xi) = \delta$, and consequently Ξ cannot \mathcal{T} -attract $\mathcal{O}_\epsilon(\Xi) \cap \mathcal{A}$. \square

LEMMA 3.31. *If \mathcal{T} is a semigroup in X with a global attractor \mathcal{A} and $S(t) = T(t)|_{\mathcal{A}}$, clearly \mathcal{S} is a semigroup in the metric space \mathcal{A} . If Ξ is a local attractor for \mathcal{S} in \mathcal{A} and K is a compact subset of \mathcal{A} such that $K \cap \Xi^* = \emptyset$, then Ξ attracts K . Furthermore Ξ is a local attractor for \mathcal{T} in X .*

PROOF. Let K be a compact subset of \mathcal{A} such that $K \cap \Xi^* = \emptyset$. If Ξ does not \mathcal{T} -attract K , there exist a $\delta > 0$, $t_n \rightarrow \infty$, $x \in K$ and $K \ni x_n \rightarrow x$ such that $\text{dist}_H(T(t_n)x_n, \Xi) \geq \delta$. From Lemma 3.30 there exists $0 < \delta' < \delta$ such that $\text{dist}_H(T(t)x_n, \Xi) \geq \delta'$ for all $t \in [0, t_n]$. This implies that $\text{dist}_H(T(t)x, \Xi) \geq \delta'$ for all $t \geq 0$ and $\omega(x) \cap \Xi = \emptyset$, which is a contradiction with $K \ni x \notin \Xi^*$, proving the first part of the result.

For the remaining part note that, from Lemma 3.30, there exists $\delta' > 0$ such that $\omega(\mathcal{O}_{\delta''}(\Xi)) \cap \Xi^* = \emptyset$ for all $\delta'' < \delta'$. From the \mathcal{S} -invariance of $\omega(\mathcal{O}_{\delta'}(\Xi))$ and the property that Ξ \mathcal{S} -attracts compact subsets of \mathcal{A} that do not intersect Ξ^* , we must have that $\omega(\mathcal{O}_{\delta''}(\Xi)) \subset \Xi$. Since $\omega(\mathcal{O}_{\delta''}(\Xi))$ \mathcal{T} -attracts $\mathcal{O}_{\delta''}(\Xi)$, we have the result. \square

LEMMA 3.32. *Let \mathcal{T} be a semigroup in X with a global attractor \mathcal{A} and an attractor-repeller pair (Ξ, Ξ^*) . A global solution $\xi: \mathbb{R} \rightarrow X$ of \mathcal{T} with the property that, for some $\delta > 0$ such that $\mathcal{O}_\delta(\Xi^*) \cap \Xi = \emptyset$, $\xi(t) \in \mathcal{O}_\delta(\Xi^*)$ for all $t \leq 0$ must satisfy $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi(t), \Xi^*) = 0$.*

PROOF. If the conclusion is false there exist a $\delta' > 0$ and a sequence $t_n \rightarrow \infty$ such that $\text{dist}_H(\xi(-t_n), \Xi^*) \geq \delta'$. This leads to a contradiction with the fact that Ξ must attract $K = \{z \in \mathcal{A}: \text{dist}_H(z, \Xi^*) \geq \delta'\}$. \square

LEMMA 3.33. *Let \mathcal{T} be a semigroup in X with a global attractor \mathcal{A} and an attractor-repeller pair (Ξ, Ξ^*) . If $\xi: \mathbb{R} \rightarrow X$ is a global bounded solution for \mathcal{T} through $x \notin \Xi \cup \Xi^*$, then $\lim_{t \rightarrow \infty} \text{dist}_H(\xi(t), \Xi) = 0$ and $\lim_{t \rightarrow -\infty} \text{dist}_H(\xi(t), \Xi^*) = 0$. Furthermore, if $x \in X \setminus \mathcal{A}$, then $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \Xi \cup \Xi^*) = 0$.*

PROOF. If $\xi: \mathbb{R} \rightarrow X$ is a global bounded solution through $x \notin \Xi \cup \Xi^*$ it follows from Lemma 3.31 that $\lim_{t \rightarrow \infty} \text{dist}_H(\xi(t), \Xi) = 0$.

Assume now that $\xi(t)$ does not converge to Ξ^* as $t \rightarrow -\infty$. We divide this part of the proof of the result into two cases:

Case 1: $\overline{\xi(\mathbb{R})} \cap \Xi^* = \emptyset$. In this case, from Lemma 3.31, $\overline{\xi(\mathbb{R})}$ is attracted by Ξ . The invariance of $\overline{\xi(\mathbb{R})}$ implies that $\overline{\xi(\mathbb{R})} \subset \Xi$ which is a contradiction with $\xi(0) = x \notin \Xi \cup \Xi^*$.

Case 2: $\overline{\xi(\mathbb{R})} \cap \Xi^* \neq \emptyset$. In this case, there exist $\delta > 0$ and sequences $t_n, \tau_n \rightarrow \infty$ such that $\text{dist}_H(\xi(-t_n), \Xi^*) = \delta$ and $\xi(-t_n + \tau_n) \rightarrow z \in \Xi^*$ and $\text{dist}_H(\xi(-t_n + t), \Xi) \leq \delta$ for all $0 \leq t \leq \tau_n$. From this we obtain a global solution $\zeta: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} with the property that $\text{dist}_H(\zeta(t), \Xi^*) \leq \delta$ for all $t \geq 0$. Therefore we have $\omega(\zeta(0)) \notin \Xi$, which implies that $\zeta(0) \in \Xi^*$, leading to a contradiction.

For the last part of the result, let $x \in X \setminus \mathcal{A}$. If $\overline{\gamma^+(x)} \cap \Xi \neq \emptyset$ we have $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \Xi) = 0$. Otherwise, there exists $\delta > 0$ with $\gamma^+(z) \cap \mathcal{O}_\delta(\Xi) = \emptyset$, and in this case we claim that $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x, \Xi^*) = 0$. If the claim is false, there

are $\epsilon > 0$ and a sequence $t_n \rightarrow \infty$ such that $\text{dist}_H(T(t_n)x, \Xi^*) \geq \epsilon$. Considering the sequence of functions $\xi_n: [-t_n, \infty) \rightarrow X$ defined by $\xi_n(t) = T(t+t_n)x$ for $t \geq -t_n$, we construct a global solution $\xi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} such that $\text{dist}_H(\xi(0), \Xi^*) \geq \epsilon$ and $\text{dist}_H(\xi(t), \Xi) \geq \delta$ for all $t \in \mathbb{R}$. Hence $\omega(\xi(0)) \cap \Xi = \emptyset$ and $\xi(0) \notin \Xi^*$ which is a contradiction. \square

LEMMA 3.34. *Let \mathcal{T} be a dynamically gradient semigroup in a metric space X with a global attractor \mathcal{A} and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then there exists $1 \leq i \leq n$ such that Ξ_i is a local attractor.*

PROOF. Assume that there are no local attractors in Ξ . Choose $0 < \epsilon < \epsilon_0(\Xi)$. It follows from Lemma 3.14 that, for each suitably small $\delta \in (0, \epsilon)$ there exist an $x_{i,\delta} \in \mathcal{O}_\delta(\Xi_i)$ and a $\tau_{i,\delta} > 0$ such that $\text{dist}_H(T(\tau_{i,\delta})x_{i,\delta}, \Xi_i) \geq \epsilon$ and $\text{dist}_H(T(t + \tau_{i,\delta})x_{i,\delta}, \Xi_i) \geq \delta$ for all $t \geq 0$. It follows that $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)x_{i,\delta}, \Xi_j) = 0$, for some $j \neq i$. That produces a chain recurrent isolated invariant set and leads to a contradiction. \square

Next we describe the construction of a Morse decomposition of the attractor of a dynamically gradient semigroup. Let \mathcal{T} be a dynamically gradient semigroup with associated disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. If (after possible reordering) Ξ_1 is a local attractor and

$$\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi_1 = \emptyset\},$$

then each Ξ_i , $i > 1$, is contained in Ξ_1^* and that for any $x \notin \mathcal{A} \setminus \{\Xi_1 \cup \Xi_1^*\}$ and global solution $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} through x we have

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\phi(t), \Xi_1^*) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\phi(t), \Xi_1) = 0.$$

We already know that Ξ_1^* is \mathcal{T} -invariant and, hence, we can consider the restriction \mathcal{T}_1 of $T(t)$ to $X_{1,0} := \Xi_1^*$. Note that, making this restriction, we have removed all possible global solutions $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} such that $\phi(t) \rightarrow \Xi_1$ as $t \rightarrow \infty$, that is, if $\phi_1: \mathbb{R} \rightarrow \mathcal{A}$ is a global solution of \mathcal{T}_1 and $\phi_1(t) \rightarrow \Xi_i$, then $i > 1$. Also, it is simple to see that \mathcal{T}_1 is a dynamically gradient semigroup in $X_{1,0}$ with disjoint collection of isolated invariant sets $\{\Xi_2, \dots, \Xi_n\}$, and this implies that if a global solution $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} satisfies $\Xi_j \xrightarrow{t \rightarrow -\infty} \phi(t) \xrightarrow{t \rightarrow \infty} \Xi_1$, then we must have $j \geq 1$.

We may assume, without loss of generality, that Ξ_2 is a local attractor for the semigroup \mathcal{T}_1 in $X_{1,0}$. If $X_{2,1}$ is the repeller associated to the isolated invariant set Ξ_2 for \mathcal{T}_1 in $X_{1,0}$ we may proceed and consider the restriction \mathcal{T}_2 of the semigroup \mathcal{T}_1 to $X_{2,1}$ and \mathcal{T}_2 is a dynamically gradient semigroup in $X_{2,1}$ with associated disjoint collection of isolated invariant sets $\{\Xi_3, \dots, \Xi_n\}$. Note that in this case, we have removed all the global solutions $\phi: \mathbb{R} \rightarrow X$ of \mathcal{T} such that $\phi(t) \rightarrow \Xi_2$ as $t \rightarrow \infty$ and therefore, if $\Xi_j \xrightarrow{t \rightarrow -\infty} \phi(t) \xrightarrow{t \rightarrow \infty} \Xi_2$, then $j \geq 2$.

Proceeding with this until all isolated invariant sets are exhausted we obtain a reordering of $\{\Xi_1, \dots, \Xi_n\}$ in such a way that Ξ_j is a local attractor for the restriction of \mathcal{T} to $X_{j-1, j-2}$, where $X_{0, -1} = \mathcal{A}$.

With the construction above, if a global solution $\phi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} satisfies

$$(3.12) \quad \Xi_\ell \xrightarrow{t \rightarrow -\infty} \xi(t) \xrightarrow{t \rightarrow \infty} \Xi_k,$$

then $\ell \geq k$. This proves that this reordering of $\Xi = \{\Xi_1, \dots, \Xi_n\}$ (which we denote the same) is a Morse decomposition to \mathcal{A} .

This proves the following result.

THEOREM 3.35. *If \mathcal{T} is a dynamically gradient semigroup with disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$, then there is a reordering of Ξ that it is a Morse decomposition to \mathcal{A} .*

Now we will prove that, for a suitably chosen sequence of local attractors $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n = \mathcal{A}$, we have that $\Xi_j = A_j \cap A_{j-1}^*$, $1 \leq j \leq n$, and that we have a Morse decomposition in the sense of [105, 192]. To this end, using

$$W^u(\Xi_i) = \{x \in X : \text{there exists a global solution } \xi : \mathbb{R} \rightarrow X \text{ of } \mathcal{T} \\ \text{through } x \text{ such that } \xi(t) \xrightarrow{t \rightarrow -\infty} \Xi_i\},$$

we can define $A_0 = \emptyset$, $A_1 = \Xi_1$ and for $j = 2, 3, \dots, n$

$$(3.13) \quad A_j = \bigcup_{i=1}^j W^u(\Xi_i).$$

It is clear that $A_n = \mathcal{A}$.

With this construction, we can prove that each A_j is a local attractor, $j = 0, \dots, n$, and if we consider the correspondent repellers A_j^* , then the sequence of attractor-repeller pairs $\{(A_j, A_j^*)\}_{j=0, \dots, n}$ generates exactly the Morse decomposition given by Ξ ; that is, $\Xi_j = A_j \cap A_{j-1}^*$ for all $j = 1, \dots, n$ and $\bigcap_{j=0}^n (A_j \cup A_j^*) = \bigcup_{i=1}^n \Xi_i$.

LEMMA 3.36. *The A_j defined in (3.13) is compact, for each $j = 0, \dots, n$.*

PROOF. We prove this result using a finite induction argument. Clearly the result is true for $j = 0$ and $j = 1$. Now assume that A_j is compact for $1 \leq j < n$ and we prove that A_{j+1} is also compact. Since $A_{j+1} \subset \mathcal{A}$, it suffices to show that A_{j+1} is closed. To this end let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in A_{j+1} with $x_k \rightarrow x \in \mathcal{A} \setminus A_{j+1}$. If there exists a subsequence of $\{x_k\}_{k \in \mathbb{N}}$ in $A_j \cup \Xi_{j+1}$, the compactness of $A_j \cup \Xi_{j+1}$ completes the proof. If this is not the case, then there exists $k_0 \in \mathbb{N}$ such that $x_k \in W^u(\Xi_{j+1}) \setminus \Xi_{j+1}$ for all $k \geq k_0$. Thus, for each such k , there exists a global solution $\xi_k : \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} through x_k such that $\text{dist}_H(\xi_k(t), \Xi_{j+1}) \rightarrow 0$ as $t \rightarrow -\infty$.

From these, we can construct a sequence of global solutions $\{\zeta_1, \dots, \zeta_{k-1}\}$ connecting isolated invariant sets $\{\Xi_{\ell_1}, \dots, \Xi_{\ell_p}\}$ with $\Xi_{\ell_1} = \Xi_{j+1}$,

$$\Xi_{\ell_i} \xrightarrow{t \rightarrow -\infty} \zeta_i(t) \xrightarrow{t \rightarrow \infty} \Xi_{\ell_{i+1}}, \quad 1 \leq i \leq p-1,$$

and $\zeta_{i_0}(0) = x$ for some $1 \leq i_0 \leq p-1$. Since x is not in the unstable manifolds of Ξ_ℓ , $1 \leq \ell \leq j+1$, we have $\text{dist}_H(\zeta_{i_0}(t), \Xi_m) \xrightarrow{t \rightarrow -\infty} 0$ for some $j+1 < m \leq n$, which contradicts (3.12). \square

THEOREM 3.37. *Let \mathcal{T} be a dynamically gradient semigroup with disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$, reordered in such a way that it is a Morse decomposition. Then A_j defined in (3.13) is a local attractor for \mathcal{T} in X and*

$$\Xi_j = A_j \cap A_{j-1}^* \quad \text{for } j = 1, \dots, n.$$

PROOF. From Lemma 3.31, it is sufficient to prove that $A_j = A_{j-1} \cup W^u(\Xi_j)$ is a local attractor for \mathcal{T} restricted to \mathcal{A} .

Claim: there exists $\delta_0 > 0$ such that $\mathcal{O}_{\delta_0}(A_j) \cap (\bigcup_{i=j+1}^n \Xi_i) = \emptyset$.

Indeed, assume that there exists a sequence $\{x_k\}$ with $x_k \in \bigcup_{i=j+1}^n \Xi_i$ and $x_k \rightarrow A_j$, as $k \rightarrow \infty$. We can assume that $x_k \in \Xi_{i_0}$, for all $k \in \mathbb{N}$, for some fixed

$j + 1 \leq i_0 \leq n$. By Lemma 3.36, we can assume that $x_k \rightarrow x \in A_j \cap \Xi_{i_0} = \emptyset$, which is a contradiction and proves our claim.

If there are $\delta < \delta_0$ and $\delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j)) \subset \mathcal{O}_\delta(A_j)$, then $\omega(\mathcal{O}_{\delta'}(A_j))$ \mathcal{T} -attracts $\mathcal{O}_{\delta'}(A_j)$ and (as $\omega(\mathcal{O}_{\delta'}(A_j))$ is \mathcal{T} -invariant) is contained in A_j proving that A_j is a local attractor. If that is not the case, for $\delta < \delta_0$, there is a sequence $\{x_k\}$ in $\mathcal{A} \setminus A_j$ with $\text{dist}_H(x_k, A_j) \rightarrow 0$, through each x_k a global solution $\xi_k: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} and a sequence $t_k \rightarrow \infty$, such that $\text{dist}_H(\xi_k(t), A_j) \leq \delta$ for all $t \in [0, t_k]$. In this way, using Lemma 2.4, we construct a global solution $\xi: \mathbb{R} \rightarrow \mathcal{A}$ such that $\text{dist}_H(\xi_k(t_k), A_j) = \delta$ and $\xi(0) \notin A_j$ and $\text{dist}_H(\xi(t), A_j) \leq \delta$ for all $t \leq 0$. This and the fact that \mathcal{T} is a dynamically gradient semigroup gives us a contradiction, since $\xi(t) \rightarrow \Xi_k$ as $t \rightarrow -\infty$ for some $1 \leq k \leq j$, which implies that $\xi(0) \in A_j$.

To prove that $\Xi_j = A_j \cap A_{j-1}^*$ note that

$$A_j = \bigcup_{i=1}^j W^u(\Xi_i)$$

and $A_{j-1}^* = \{z \in \mathcal{A}: \omega(z) \cap A_{j-1} = \emptyset\}$. Hence, given $z \in A_j \cap A_{j-1}^*$ we have that the global solution $\xi: \mathbb{R} \rightarrow \mathcal{A}$ through z must satisfy that

$$\{\Xi_i: 1 \leq i \leq j\} \xrightarrow{t \rightarrow -\infty} \xi(t) \xrightarrow{t \rightarrow \infty} \{\Xi_i: j \leq i \leq n\}.$$

As a consequence, together with the fact that \mathcal{T} is a dynamically gradient semigroup with disjoint collection of isolated invariant sets $\{\Xi_1, \dots, \Xi_n\}$, for which any global solution $\xi: \mathbb{R} \rightarrow \mathcal{A}$ of \mathcal{T} satisfies $\Xi_\ell \xrightarrow{t \rightarrow -\infty} \xi(t) \xrightarrow{t \rightarrow \infty} \Xi_k$ with $\ell \geq k$, we obtain that $z \in \Xi_j$. This shows that $A_j \cap A_{j-1}^* \subset \Xi_j$. The other inclusion is immediate from the definition of A_j and A_{j-1}^* . \square

PROPOSITION 3.38. *Let \mathcal{T} be a dynamically gradient semigroup with associated disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$ reordered in such a way that it is a Morse decomposition. Then,*

$$\bigcap_{j=0}^n (A_j \cup A_j^*) = \bigcup_{i=1}^n \Xi_i.$$

PROOF. If $z \in \bigcup_{i=1}^n \Xi_i$, let $k \in \{1, 2, \dots, n\}$ be such that $z \in \Xi_k = A_k \cap A_{k-1}^*$. Hence $z \in A_k \subset A_{k+1} \subset \dots \subset A_n$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \dots \subset A_0^*$. Thus

$$z \in \bigcap_{j=k}^n A_j \cap \bigcap_{j=1}^{k-1} A_j^* \subset \bigcap_{j=k}^n (A_j \cup A_j^*) \cap \bigcap_{j=0}^{k-1} (A_j \cup A_j^*) = \bigcap_{j=0}^n (A_j \cup A_j^*),$$

proving the inclusion $\bigcup_{i=1}^n \Xi_i \subset \bigcap_{j=0}^n (A_j \cup A_j^*)$.

Now, let $z \in \bigcap_{j=0}^n (A_j \cup A_j^*)$ and $I := \{i_1, i_2, \dots, i_k\}$, $J := \{j_1, j_2, \dots, j_l\}$ such that $I \cup J = \{0, 1, \dots, n\}$ with $I \cap J = \emptyset$ and $z \in A_i$ for all $i \in I$ and $z \in A_j^*$ for all $j \in J$. Clearly, if $i := \min I$, necessarily $I = \{i, i+1, i+2, \dots, n\}$ and $J = \{0, 1, \dots, i-1\}$, consequently $z \in A_i$ and $z \in A_{i-1}^*$. So, $z \in A_i \cap A_{i-1}^* = \Xi_i$, from which we conclude that $\bigcap_{j=0}^n (A_j \cup A_j^*) \subset \bigcup_{i=1}^n \Xi_i$ and the proof is completed. \square

3.2. Lyapunov functions for dynamically gradient semigroups. Inspired by the results in the work of Conley [105] (see also the work of Ribakowski [192]) we present in this section the equivalence between gradient semigroups and dynamically gradient semigroups. The approach used here is that of [5, 78], which does not assume backwards uniqueness of solutions when backwards solutions exist.

The key result in this section is Theorem 3.41, which states the equivalence between the concepts of dynamically gradient semigroups and gradient semigroups. The main task is to construct a Lyapunov function for a dynamically gradient semigroup, and to give an idea of such construction, we will state two auxiliary results, which will play a crucial role in the proof of such equivalence.

LEMMA 3.39. *If \mathcal{T} is a semigroup with global attractor \mathcal{A} , the map $h: X \rightarrow \mathbb{R}$ given by*

$$h(x) = \sup_{t \geq 0} \text{dist}_H(T(t)x, \mathcal{A}), \quad \text{for each } x \in X$$

is well defined, continuous, nonincreasing along solutions of \mathcal{T} and $h^{-1}(0) = \mathcal{A}$.

PROOF. Indeed, by Lemma 3.31, given $\varepsilon > 0$ let $0 < \varepsilon' < \varepsilon$ such that $\gamma^+(\mathcal{O}_{\varepsilon'}(\mathcal{A})) \subset \mathcal{O}_{\varepsilon}(\mathcal{A})$. This and the \mathcal{T} -invariance of \mathcal{A} show the continuity of h on each $x \in \mathcal{A}$.

Let $x_0 \in X \setminus \mathcal{A}$ be given. We have $h(x_0) \geq \text{dist}_H(x_0, \mathcal{A}) > 0$ and we consider $0 < \mu < \text{dist}_H(x_0, \mathcal{A})$. From the continuity of the map $X \ni x \mapsto \text{dist}_H(x, \mathcal{A}) \in [0, \infty)$, we can choose a bounded neighborhood V of x_0 in X such that $\text{dist}_H(z, \mathcal{A}) > \mu$ if $z \in V$. Now, let $\tau > 0$ be such that $\gamma^+(T(\tau)V) \subset \mathcal{O}_{\mu}(\mathcal{A})$. Hence,

$$h(z) = \sup_{0 \leq s \leq \tau} \text{dist}_H(T(s)z, \mathcal{A}), \quad \text{for each } z \in V,$$

and from the continuity properties of the \mathcal{T} , it follows that h is continuous in x_0 .

To see that h is nonincreasing along solutions of \mathcal{T} note that, if $z \in X$ and $t_1 > 0$, then

$$\begin{aligned} h(T(t_1)z) &= \sup_{t \geq 0} \text{dist}_H(T(t)T(t_1)z, \mathcal{A}) = \sup_{t \geq 0} \text{dist}_H(T(t+t_1)z, \mathcal{A}) \\ &= \sup_{t \geq t_1} \text{dist}_H(T(t)z, \mathcal{A}) \leq \sup_{t \geq 0} \text{dist}_H(T(t)z, \mathcal{A}) = h(z). \end{aligned}$$

□

PROPOSITION 3.40. *Let \mathcal{T} be a semigroup in a metric space X with global attractor \mathcal{A} and an attractor-repeller pair (Ξ, Ξ^*) . Then there exists a function $f: X \rightarrow \mathbb{R}$ satisfying the following:*

- (i) *f is continuous in X .*
- (ii) *f is nonincreasing along solutions of \mathcal{T} .*
- (iii) *$f^{-1}(0) = \Xi$ and $f^{-1}(1) \cap \mathcal{A} = \Xi^*$.*
- (iv) *Given $z \in X$, if $f(T(t)z) = f(z)$ for all $t \geq 0$, then $z \in \Xi \cup \Xi^*$.*

PROOF. First note that Ξ and Ξ^* are disjoint compact subsets of X . With the convention that $\text{dist}_H(z, \emptyset) = 1$ for each $z \in X$, define the function $l: X \rightarrow [0, 1]$ associated to (Ξ, Ξ^*) by

$$l(z) = \frac{\text{dist}_H(z, \Xi)}{\text{dist}_H(z, \Xi) + \text{dist}_H(z, \Xi^*)} \quad \text{for each } z \in X,$$

which is the canonical Urysohn function if both Ξ and Ξ^* are nonempty. Clearly l is well defined. Setting $d_0 = \text{dist}(\Xi, \Xi^*) > 0$, it holds that $|l(z) - l(w)| \leq \frac{2}{d_0} d(z, w)$ for $z, w \in X$, that is, l is uniformly continuous. Moreover, $l^{-1}(0) = \Xi$ and $l^{-1}(1) = \Xi^*$.

Define also the function $k: X \rightarrow \mathbb{R}$ by

$$(3.14) \quad k(z) = \sup_{t \geq 0} l(T(t)z) \quad \text{for each } z \in X.$$

We now show that k is continuous and nonincreasing along solutions of \mathcal{T} , $k(X) \subset [0, 1]$ (with equality when X is connected and Ξ and Ξ^* are nonempty), $k^{-1}(0) = \Xi$ and $k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

Clearly $k(X) \subset [0, 1]$ follows and $t \geq 0$. If X is connected and Ξ, Ξ^* are nonempty, due to the fact that \mathcal{A} is connected and l is continuous, we have $k(X) = [0, 1]$.

To prove that $[0, \infty) \ni t \mapsto k(T(t)z) \in [0, 1]$ is nonincreasing for each $z \in X$ note that, if $0 \leq t_1 \leq t_2$ we have

$$\begin{aligned} k(T(t_1)z) &= \sup_{t \geq 0} l(T(t)T(t_1)z) = \sup_{t \geq 0} l(T(t+t_1)z) = \sup_{t \geq t_1} l(T(t)z) \\ &\geq \sup_{t \geq t_2} l(T(t)z) = \sup_{t \geq 0} l(T(t+t_2)z) = k(T(t_2)z). \end{aligned}$$

It is clear from the definition of k and from the \mathcal{T} -invariance of Ξ and Ξ^* that $k(\Xi) = \{0\}$ and $k(\Xi^*) = \{1\}$. Now, if $z \in X$ is such that $k(z) = 0$, then $l(T(t)z) = 0$ for all $t \geq 0$. In particular, $0 = l(T(0)z) = l(z)$, and so, $z \in \Xi$, that is, $k^{-1}(0) \subset \Xi$ which shows that $k^{-1}(0) = \Xi$. On the other hand, if $z \in \mathcal{A}$ is such that $k(z) = 1$ and $z \notin \Xi^*$, then $\omega(z) \subset \Xi$. From the continuity of l and the fact that $\omega(z)$ \mathcal{T} -attracts z , we obtain $\lim_{t \rightarrow \infty} l(T(t)z) = 0$. So there exists $t_0 > 0$ such that $1 = k(z) = \sup_{0 \leq t \leq t_0} l(T(t)z)$. This implies the existence of $t' \in [0, t_0]$ such that

$l(T(t')z) = 1$, that is, $T(t')z \in \Xi^*$. Consequently $\omega(z) = \omega(T(t')z) \subset \Xi^*$, which contradicts the fact that $\omega(z) \subset \Xi$ and so, if $k(z) = 1$ for some $z \in \mathcal{A}$ we must have that $z \in \Xi^*$. From this we conclude that $k^{-1}(1) \cap \mathcal{A} \subset \Xi^*$ and so $k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

We now prove that, if $z \in \mathcal{A}$ and $k(T(t)z) = k(z)$ for all $t \geq 0$, then $z \in \Xi \cup \Xi^*$. If $z \in \mathcal{A} \setminus (\Xi \cup \Xi^*)$, $\omega(z) \subset \Xi$ and from the definition of k and the fact that $\omega(z)$ \mathcal{T} -attracts z we have $k(z) = \lim_{t \rightarrow \infty} k(T(t)z) = 0$. Since $k^{-1}(0) = \Xi$, z must belong to Ξ , which is a contradiction.

Next we prove the continuity of k . We split the proof into three cases:

Case 1: Continuity of k in Ξ^ .*

Since $l(z) \leq k(z) \leq 1$, for all $z \in X$, given $z_0 \in \Xi^*$ and $z \in X$ we have

$$|k(z) - k(z_0)| = 1 - k(z) \leq 1 - l(z).$$

This and the continuity of l imply the continuity of k in z_0 .

Case 2: Continuity of k in Ξ .

From the continuity of l in Ξ , given $\varepsilon > 0$, there is a $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \varepsilon)$. Hence Lemma 3.30 implies that there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, from which we conclude that $k(\mathcal{O}_{\delta'}(\Xi)) \subset [0, \varepsilon]$.

Case 3: Continuity of k in $X \setminus (\Xi \cup \Xi^)$.*

Given $z_0 \in X \setminus (\Xi \cup \Xi^*)$, from Lemma 3.33, we have either

$$\lim_{t \rightarrow \infty} \text{dist}_H(T(t)z_0, \Xi) = 0 \quad \text{or} \quad \lim_{t \rightarrow \infty} \text{dist}_H(T(t)z_0, \Xi^*) = 0.$$

Assume that $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)z_0, \Xi^*) = 0$ and note firstly that $k(z_0) = 1$. Now, given $\varepsilon > 0$, from the continuity of l in Ξ^* there is a neighborhood V of Ξ^* in X such that $l(V) \subset (1 - \varepsilon, 1]$. If $t_0 > 0$ is such that $T(t_0)z_0 \in V$, from the continuity of $T(t_0): X \rightarrow X$, let U be a neighborhood of z_0 such that $T(t_0)U \subset V$, from which it follows that $k(z) > 1 - \varepsilon$ for all $z \in U$ (for $T(t_0)z \in V$ and then

$1 - \varepsilon < l(T(t_0)z) \leq k(z)$). This proves the continuity of k in points z_0 of $X \setminus (\Xi \cup \Xi^*)$ for which $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)z_0, \Xi^*) = 0$.

Now, if $z_0 \in X \setminus (\Xi \cup \Xi^*)$ and $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)z_0, \Xi) = 0$, it holds that $l(z_0) > 0$.

Choose $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \frac{l(z_0)}{2})$ and, from Lemma 3.30, there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$. From this, there is a $t_0 > 0$ with the property that $T(t_0)z_0 \in \mathcal{O}_{\delta'}(\Xi)$ and therefore $T(t)z_0 \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. From the continuity of $T(t_0): X \rightarrow X$, there is a neighborhood U_1 of z_0 in X such that $T(t_0)U_1 \subset \mathcal{O}_{\delta'}(\Xi)$. Then, for all $z \in U_1$ we have that $T(t)z \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. Finally, from the continuity of l , let U_2 be a neighborhood of z_0 in X such that $l(z) > \frac{l(z_0)}{2}$ for all $z \in U_2$ and write $U := U_1 \cap U_2$, so that for all $z \in U$ it holds that $k(z) = \sup_{0 \leq t \leq t_0} l(T(t)z)$. Reasoning as before we obtain the continuity of k in points z_0 of $X \setminus (\Xi \cup \Xi^*)$ for which $\lim_{t \rightarrow \infty} \text{dist}_H(T(t)\Xi_0, \Xi) = 0$.

Let h be the function defined in Lemma 3.39, that is, $h(z) = \sup_{t \geq 0} \text{dist}_H(T(t)z, \mathcal{A})$ for $z \in X$, and define $f: X \rightarrow \mathbb{R}$ by

$$f(z) = k(z) + h(z) \quad \text{for each } z \in X.$$

The continuity of f follows from the continuity of k (proved above) and h (proved in Lemma 3.39). Since k and h are nonincreasing along solutions of \mathcal{T} (see above and Lemma 3.39), so is f .

Clearly $f(\Xi) = \{0\}$. If $f(z) = 0$ for some $z \in X$, then $h(z) = k(z) = 0$ and we must have $z \in \Xi$. This shows that $f^{-1}(0) = \Xi$. Also, since $f|_{\mathcal{A}} = k|_{\mathcal{A}}$ we have that $f^{-1}(1) \cap \mathcal{A} = k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

Finally, to prove (iv), let $z \in X$ such that $f(T(t)z) = f(z)$ for all $t \geq 0$. Since h and k are nonincreasing along solutions, we can see that $h(T(t)z) = h(z)$ and $k(T(t)z) = k(z)$ for all $t \geq 0$. Moreover, since $\text{dist}_H(T(t)z, \mathcal{A}) \rightarrow 0$ as $t \rightarrow \infty$, we have $h(z) = 0$, and hence $z \in \mathcal{A}$. Then the equality $k(T(t)z) = k(z)$ for all $t \geq 0$ shows us that $z \in \Xi \cup \Xi^*$ and concludes the proof. \square

We are able now to state and prove the main result of this section.

THEOREM 3.41. *Let \mathcal{T} be a semigroup in a metric space X with global attractor \mathcal{A} and a disjoint collection of isolated invariant sets $\Xi = \{\Xi_1, \dots, \Xi_n\}$. Then \mathcal{T} is a gradient semigroup with respect to Ξ if and only if it is a dynamically gradient semigroup with respect to Ξ . In addition, the Lyapunov function $V: X \rightarrow \mathbb{R}$ of a dynamically gradient semigroup may be chosen in such a way that $V(\Xi_k) = k - 1$, for each $k = 1, \dots, n$.*

PROOF. It is clear that a gradient semigroup with respect to Ξ is a dynamically gradient semigroup with respect to Ξ . Suppose now that \mathcal{T} is a dynamically gradient semigroup with respect to Ξ reordered in such a way that it is a Morse decomposition for \mathcal{A} . Let $\emptyset = A_0 \subset A_1 \subset \dots \subset A_n = \mathcal{A}$ be the sequence of local attractors defined in (3.13) and $\emptyset = A_n^* \subset A_{n-1}^* \subset \dots \subset A_0^* = \mathcal{A}$ their corresponding repellers such that for each $j = 1, 2, \dots, n$, we have $\Xi_j = A_j \cap A_{j-1}^*$.

Let h be the function defined in Lemma 3.39 and let k_j be the function constructed in (3.14) for the attractor-repeller pair (A_j, A_j^*) , for each $j = 1, \dots, n$.

Define the continuous function $V: X \rightarrow \mathbb{R}$ by

$$V(z) = h(z) + \sum_{j=1}^n k_j(z) \quad \text{for each } z \in X.$$

Then V is a Lyapunov function for the dynamically gradient semigroup \mathcal{T} with respect to Ξ . Indeed, since h and each k_j , $1 \leq j \leq n$, are nonincreasing along solutions of \mathcal{T} , so is V .

If $z \in X$ is such that $V(T(t)z) = V(z)$ for all $t \geq 0$, then arguing as in the proof of item (iv) of Proposition 3.40 we have $z \in A_j \cup A_j^*$, for each $j = 0, 1, \dots, n$. Therefore from Lemma 3.38 we have

$$z \in \bigcap_{j=0}^n (A_j \cup A_j^*) = \bigcup_{i=1}^n \Xi_i.$$

For each $0 \leq j \leq n$, $k_j: \mathcal{A} \rightarrow [0, 1]$, $k_j^{-1}(0) = A_j$ and $k_j^{-1}(1) \cap \mathcal{A} = A_j^*$. If $k \in \{1, \dots, n\}$ and $z \in \Xi_k = A_k \cap A_{k-1}^*$, it follows that $z \in A_k \subset A_{k+1} \subset \dots \subset A_n = \mathcal{A}$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \dots \subset A_0^* = \mathcal{A}$. Hence $k_j(z) = 0$ if $k \leq j \leq n$ and $k_j(z) = 1$ if $1 \leq j \leq k-1$, and so

$$V(z) = \sum_{j=1}^n k_j(z) = \sum_{j=0}^{k-1} k_j(z) + \sum_{j=k}^n k_j(z) = \sum_{j=0}^{k-1} 1 + \sum_{j=k}^n 0 = k - 1.$$

□

4. Topological structural stability

In this section, we present the results on *topological structural stability* of attractors of gradient semigroups, which is the permanence of the gradient structure of the global attractors for small perturbations of a semigroup. To that end we prove that small perturbations of a dynamically gradient semigroup are dynamically gradient semigroups, that is, for a dynamically gradient semigroup, conditions (G1) and (G2) are stable under perturbations. This result can be found in [74, 78].

THEOREM 3.42. *Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups, continuous and collectively asymptotically compact at $\eta = 0$ (see Definition 2.3). Assume that*

- (a) \mathcal{T}_η has a global attractor \mathcal{A}_η , for each $\eta \in [0, 1]$, and $\bigcup_{\eta \in [0,1]} \mathcal{A}_\eta$ is bounded;
- (b) for each $\eta \in [0, 1]$, \mathcal{A}_η contains a disjoint collection of isolated invariant sets $\Xi_\eta = \{\Xi_{1,\eta}, \dots, \Xi_{n,\eta}\}$ such that $\lim_{\eta \rightarrow 0} \text{dist}_H(\Xi_{i,\eta}, \Xi_{i,0}) = 0$, for all $1 \leq i \leq n$;
- (c) there exist $\delta > 0$ and $\eta_0 \in (0, 1]$ such that $\Xi_{i,\eta}$ is the maximal invariant set in $\mathcal{O}_\delta(\Xi_{i,\eta})$, $1 \leq i \leq n$ and $0 \leq \eta \leq \eta_0$;
- (d) \mathcal{T}_0 is a dynamically gradient semigroup with global attractor \mathcal{A}_0 and disjoint collection of isolated invariant sets $\Xi_0 = \{\Xi_{1,0}, \dots, \Xi_{n,0}\}$.

Then there exists an $\eta_1 > 0$ such that, for all $\eta \in (0, \eta_1]$, \mathcal{T}_η is a dynamically gradient semigroup with respect to Ξ_η .

PROOF. Note firstly that (b) and (c) imply that there exists $\delta > 0$ such that, for suitably small η , if a solution ξ_η satisfies $\text{dist}_H(\xi_\eta(t), \Xi_{i,0}) \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \rightarrow \Xi_{i,\eta}$ as $t \rightarrow \infty$.

We argue by contradiction to prove that for all suitably small η , \mathcal{T}_η satisfies (G1). Assume that there is a sequence $\eta_k \rightarrow 0$ and corresponding global solutions ξ_k of \mathcal{T}_{η_k} in \mathcal{A}_{η_k} such that

$$(3.15) \quad \limsup_{t \rightarrow \infty} \text{dist}_H \left(\xi_k(t), \bigcup_{i=1}^n \Xi_{i, \eta_k} \right) > \delta.$$

From Lemma 2.4 with $\mathbf{s} = \{0\}$, there is a subsequence (which we again denote by $\{\xi_k\}$) and a global solution $\xi_0: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 such that $\xi_k(t) \rightarrow \xi_0(t)$ uniformly in compact subsets of \mathbb{R} . From (G1), $\xi_0(t) \rightarrow \Xi_{i,0}$, for some $1 \leq i \leq n$. It follows that, for each $\ell \in \mathbb{N}^*$, there are $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $d(\xi_k(t_\ell), \Xi_{i,0}) < \frac{1}{\ell}$, for each $k \geq k_\ell$. From (3.15), there exists $t'_\ell > t_\ell$ such that $d(\xi_{k_\ell}(t), \Xi_{i,0}) < \delta$ for all $t \in [t_\ell, t'_\ell)$ and $d(\xi_{k_\ell}(t'_\ell), \Xi_{i,0}) = \delta$. From the continuity of \mathcal{T}_0 and the \mathcal{T}_0 -invariance of $\Xi_{i,0}$ we have $t_{\ell'} - t_\ell \xrightarrow{\ell \rightarrow \infty} \infty$. From Lemma 2.4 with $\mathbf{s} = \{t_\ell\}_{\ell=1}^\infty$, taking subsequences if necessary, there is a global solution $\xi_1: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 such that $\xi_1(t) = \lim_{\ell \rightarrow \infty} \xi_{k_\ell}(t + t_\ell)$ uniformly in compact subsets of \mathbb{R} . Then $\text{dist}_H(\xi_1(t), \Xi_{i,0}) \leq \delta$ for all $t \leq 0$ and, consequently, $\xi_1(t) \rightarrow \Xi_{i,0}$ as $t \rightarrow -\infty$. From (G1) and (G2), $\xi_1(t) \rightarrow \Xi_{j,0} \in \Xi_0$ as $t \rightarrow \infty$ with $i \neq j$. From the fact that $\xi_{k_\ell} \rightarrow \xi_1(t)$ uniformly in compact subsets of \mathbb{R} we have, for each $m \in \mathbb{N}$, a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $\text{dist}_H(\xi_k(t_m), \Xi_j) < \frac{1}{m}$ for all $k \geq k_m$. Again, from (3.15), there exists $t'_m > t_m$ such that $\text{dist}_H(\xi_{k_m}(t), \Xi_j) < \delta$ for all $t \in [t_m, t'_m)$ and $\text{dist}_H(\xi_{k_m}(t'_m), \Xi_j) = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 such that $\xi_2(t) \rightarrow \Xi_j$ as $t \rightarrow -\infty$ and $\xi_2(t) \rightarrow \Xi_{r,0} \in \Xi_0$ as $t \rightarrow \infty$ with $r \notin \{i, j\}$. In a finite number of steps we construct a homoclinic structure and arrive at a contradiction with (G2). This proves that there is an $\eta_1 > 0$ such that, for all global solution ξ_η in \mathcal{A}_η with $\eta \leq \eta_1$, we have

$$\lim_{t \rightarrow \infty} \text{dist}_H(\xi_\eta(t), \Xi_{i,0}) = 0.$$

The proof that there is an $\eta_2 > 0$ such that for all global solutions ξ_η of \mathcal{T}_η in \mathcal{A}_η with $\eta \leq \eta_2$ we have

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_\eta(t), \Xi_{j,0}) = 0$$

is similar. This completes the proof that, for all suitably small η , \mathcal{T}_η satisfies (G1).

Let us prove that, for all suitably small η , \mathcal{T}_η satisfies (G2). Again we argue by contradiction. Assume that there is a collection $\Xi_{1,0}, \dots, \Xi_{p,0}$ in Ξ_0 , a sequence $\eta_k \rightarrow 0$, global solutions ξ_k of \mathcal{T}_{η_k} in \mathcal{A}_{η_k} , and times t_1^k, \dots, t_p^k such that

$$\text{dist}_H(\xi_k(0), \Xi_{i,0}) < \frac{1}{k}, \quad \text{dist}_H(\xi_k(t_i^k), \Xi_{i+1,0}) < \frac{1}{k}, \quad 1 \leq i \leq p,$$

where $\Xi_{1,0} = \Xi_{p+1,0}$. One can choose a fixed sequence $\Xi_{1,0}, \dots, \Xi_{p,0}$ since there are only a finite number of possible sequences from Ξ_0 . Proceeding as in the proof of (G1) we construct a homoclinic structure for \mathcal{T}_0 and arrive at a contradiction. \square

In the pictorial example of Figure 3.6 we may consider the stable node, the unstable focus and the closure of the homoclinic orbit (adding the saddle to it) as the three isolated invariant sets and we will have an attractor for a dynamically gradient semigroup relative to these isolated invariant sets.

A possible perturbation is described in Figure 3.7, with the isolated invariant sets being the stable node, the unstable focus and the closure of the intersection

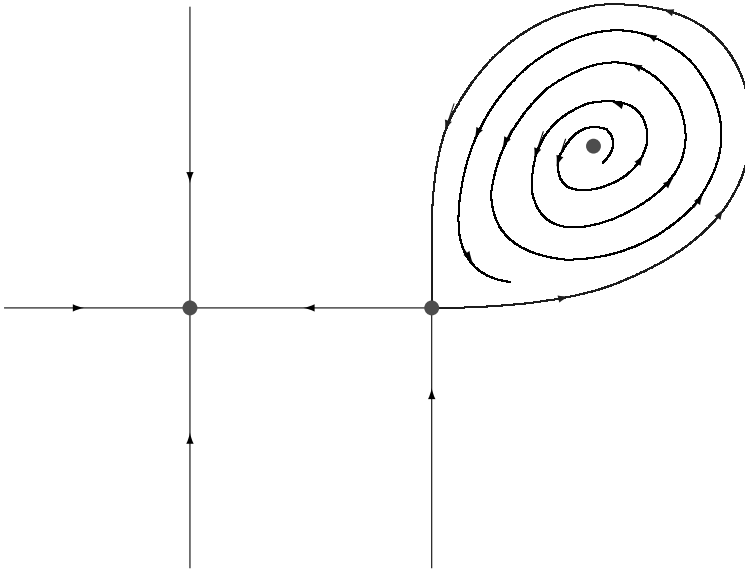


FIGURE 3.6. Unperturbed pictorial example.

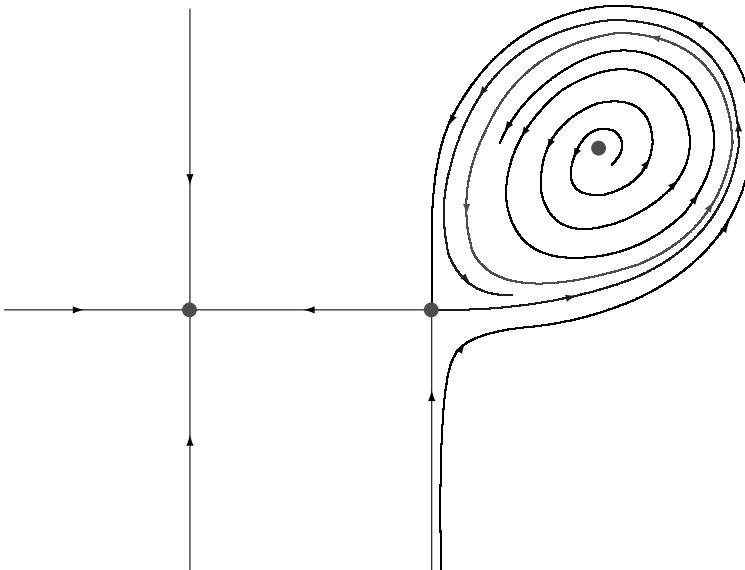


FIGURE 3.7. Perturbed pictorial example.

between the stable manifold of the periodic orbit and the unstable manifold of the saddle.

REMARK 3.43. As examples we may consider each one of the examples considered in Section 2, as well as several of those considered in Section 3, and consider a small C^1 (in the phase space) perturbation of the nonlinearity. The resulting problem (under some mild assumptions) will give rise to associated gradient semigroups.

It is an interesting exercise to consider the several examples and the appropriate conditions which ensure the applicability of Theorem 3.42, and we leave this task to the reader.

Our next result establishes that, for suitably small values of the parameter, if there are connections between two isolated invariant sets of the perturbed problem, then there will be an *indirect* connection between these isolated invariant sets of the unperturbed problem. This feature is essential for the structural stability of Morse-Smale Semigroups in Section 2.

LEMMA 3.44. *With the hypothesis of Theorem 3.42, if $\eta_k \rightarrow 0$ and $\xi_k: \mathbb{R} \rightarrow X$ is a global solution of \mathcal{T}_{η_k} with*

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_k(t), \Xi_{i, \eta_k}) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\xi_k(t), \Xi_{j, \eta_k}) = 0,$$

then there exist global solutions $\xi^\ell: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 , for $1 \leq \ell \leq m-1$, $m \leq n$, and isolated invariant sets $\{\Xi_{n_\ell}\}_{1 \leq \ell \leq m}$ in Ξ_0 such that $\Xi_{n_1} = \Xi_{i,0}$, $\Xi_{n_{m-1}} = \Xi_{j,0}$ and

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi^\ell(t), \Xi_{n_\ell}) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\xi^\ell(t), \Xi_{n_{\ell+1}}) = 0,$$

for $1 \leq \ell \leq m-1$.

PROOF. Fix $0 < \delta < \epsilon_0(\Xi_0)$. Since

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_k(t), \Xi_{i, \eta_k}) = 0 \quad \text{and} \quad \lim_{\eta \rightarrow 0} \text{dist}_H(\Xi_{i, \eta}, \Xi_{i,0}) = 0,$$

taking subsequences, if necessary, we may assume that $\xi_k(t) \rightarrow \xi^1(t)$ uniformly in compact subsets of \mathbb{R} with $\xi^1(t) \xrightarrow{t \rightarrow -\infty} \Xi_{n_1} := \Xi_{i,0}$. Also, (G1) implies that there exists an isolated invariant set $\Xi_{n_2} \in \Xi_0$ (different from Ξ_{n_1}) such that $\xi^1(t) \xrightarrow{t \rightarrow \infty} \Xi_{n_2}$. If $\Xi_{n_2} = \Xi_{j,0}$ the proof is done, otherwise there are $t'_\ell > t_\ell$ such that $\text{dist}_H(\xi_{k_\ell}(t_\ell), \Xi_{n_2}) < \frac{\delta}{\ell}$, $\text{dist}_H(\xi_{k_\ell}(t), \Xi_{n_2}) < \delta$ for all $t \in [t_\ell, t'_\ell]$ and $\text{dist}_H(\xi_{k_\ell}(t'_\ell), \Xi_{n_2}) = \delta$. Taking subsequences, if necessary, let $\xi^2(t) = \lim_{\ell \rightarrow \infty} \xi_{k_\ell}(t + t'_\ell)$. Then, since $t'_\ell - t_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$, $\text{dist}_H(\xi^2(t), \Xi_{n_2}) \leq \delta$ for all $t \leq 0$ and consequently $\text{dist}_H(\xi^2(t), \Xi_{n_2}) \xrightarrow{t \rightarrow -\infty} 0$. Furthermore, $\text{dist}_H(\xi^2(t), \Xi_{n_3}) \xrightarrow{t \rightarrow \infty} 0$ for some $\Xi_{n_3} \in \Xi_0$ (different from Ξ_{n_1} and Ξ_{n_2}). This reasoning can be done as long as Ξ_{n_ℓ} is different from $\Xi_{j,0}$, and since there are only finitely many isolated invariant sets, the proof is completed in finitely many steps. \square

5. Examples

EXAMPLE 3.45. If $f \in C^2(\mathbb{R})$, $\limsup_{|u| \rightarrow \infty} \frac{f(u)}{u} < 0$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\epsilon \geq 0$ and $a \in C(\bar{\Omega}, \mathbb{R}^n)$, consider the initial value problem

$$\begin{cases} u_{tt} + \beta u_t = \Delta u + \epsilon a(x) \cdot \nabla u + f(u), & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \partial\Omega, t > 0, \\ u(\cdot, 0) = u_0 \in H_0^1(\Omega), \\ u_t(\cdot, 0) = v_0 \in L^2(\Omega), \end{cases}$$

with Dirichlet boundary condition. This equation generates a semigroup \mathcal{T}_ϵ in $H_0^1(\Omega) \times L^2(\Omega)$, which is dynamically gradient if there exists a finite number of equilibria of $\Delta u = -f(u)$, with all of them hyperbolic, and $\epsilon \geq 0$ is suitably small (see [74]). Hence, for ϵ sufficiently small, the semigroups \mathcal{T}_ϵ are gradient as an application of Theorem 3.42.

EXAMPLE 3.46. For each $1 \leq m \in \mathbb{N}$, the *time one map* for the problem

$$(3.16) \quad \begin{cases} \dot{r} = \begin{cases} \pi^{-1} \left(1 - \frac{1}{2m+1} - r\right)^3 \sin \frac{\pi}{1-r}, & r < 1 - \frac{1}{2m+1}, \\ -\left(1 - \frac{1}{2m+1} - r\right)^2, & r \geq 1 - \frac{1}{2m+1}, \end{cases} \\ \dot{\theta} = 2\pi \end{cases}$$

has the attractor $\mathcal{A}_m = \{|r| \leq 1 - \frac{1}{2m+1}\}$, which is the union of unstable manifolds of Ξ_j , $1 \leq j \leq 2m+1$, where Ξ_j is the 1-periodic solution corresponding to $r = 1 - \frac{1}{j}$, $1 \leq j \leq 2m+1$. These periodic solutions are normally hyperbolic (if k is even, the orbit is unstable, if k is odd, the orbit is stable). In this case, it is easy to see that the attractor \mathcal{A}_m is the union of the unstable manifolds of the periodic solutions $\{\Xi_j\}_{1 \leq j \leq 2m+1}$. Theorem 3.42 implies that any small perturbation of the vector field in (3.16) will lead us to a dynamically gradient semigroup relative to a disjoint collection of isolated invariant sets and the attractor is characterized.

Notes. The *Fundamental Theorem of Dynamical Systems* (as named in [174]) states that every dynamical system on a compact metric space (the one defined on a global attractor, for instance) has a geometrical structure described by a (finite or countable) number of sets with an intrinsic recurrent dynamics and a gradient-like dynamics outside them, and has been considered in different frameworks, as in the case of flows ([105]) and semiflows on compact spaces ([192]), or even compact and noncompact topological spaces ([146, 181, 182]). In other words, the attractor can be always described by a (finite or countable) number of invariants and connections between them. This intrinsic dynamics is what is described by dynamically gradient semigroups. Note that, given a system of differential or partial differential equations, the determination of a Lyapunov function is usually a difficult task. Thus, the characterization of gradient systems as dynamically gradient systems for infinite-dimensional spaces enlarges the class of systems where a detailed structure of the attractors is available (see, for instance, [112] for this equivalence in the case of multivalued semigroups, or [48] for the case of stochastic differential equations). The determination of a Lyapunov function describing all the isolated invariant sets within a global attractor allows a formal description of this gradient attractor from its energy levels ([4]), which joins invariants, which are not connected among themselves, with a same value given by the Lyapunov functional. Sometimes, a system can be proved to be gradient by a direct description of the Morse sets and their gradient-like dynamics ([129]). The robustness under autonomous ([5]) and non-autonomous ([6]) perturbation of dynamically gradient semigroups allows us to conclude the robustness of Lyapunov functionals under perturbation.

Morse-Smale Semigroups

In this chapter we prove that a suitable small perturbation of a Morse-Smale semigroup has an attractor with the *same phase diagram structure* (isolated invariant sets and connections) as the attractor for the unperturbed semigroup. We show that Morse-Smale semigroups are a special case of dynamically gradient semigroups. Then, we show that the structure of the global attractor is robust under perturbation, that is, not only do the isolated invariant sets behave continuously under perturbations, but they also maintain the same network of connections. This leads to an isomorphism between the phase diagrams of the limit and the perturbed attractors (they give rise to the same oriented graph with the vertices being the isolated invariant sets and there is an oriented edge between two vertices if there is a global solution connecting the corresponding isolated invariant sets).

1. Basic notions

We begin our discussion with the definition of the nonwandering set for a semigroup \mathcal{T} .

DEFINITION 5.1. Let \mathcal{T} be a semigroup with a global attractor \mathcal{A} . We define the **nonwandering set** $\Omega_{\mathcal{T}}$ of \mathcal{T} as the set of points $x \in X$ such that given $t_0 \geq 0$ and a neighborhood V_x of x in X , there exists $t \geq t_0$ such that $T(t)V_x \cap V_x \neq \emptyset$. Clearly $\Omega_{\mathcal{T}} \subset \mathcal{A}$.

With this at hand, we can define the concept of a Morse-Smale semigroup (introduced in 1960 by S. Smale in [195]).

DEFINITION 5.2. Let \mathcal{T} be a semigroup with a global attractor \mathcal{A} . We say that \mathcal{T} is a **Morse-Smale semigroup** if it satisfies the following conditions:

- (a) \mathcal{T} is **reversible**, that is, $T(t)|_{\mathcal{A}}$ is injective, $T(t)$ is differentiable in \mathcal{A} and $D_x T(t)(z): X \rightarrow X$ is an injective bounded linear operator, for all $t \geq 0$ and $z \in \mathcal{A}$;
- (b) there exists a finite number of hyperbolic equilibrium points z_1, \dots, z_n and a finite number of normally hyperbolic periodic orbits $\xi_1, \dots, \xi_m: \mathbb{R} \rightarrow X$ of \mathcal{T} such that

$$\Omega_{\mathcal{T}} = \bigcup_{i=1}^n \{z_i\} \cup \bigcup_{j=1}^m \xi_j(\mathbb{R}).$$

We will often represent and treat $\Omega_{\mathcal{T}}$ as a list of sets

$$\{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\},$$

and write $\Xi \in \Omega_{\mathcal{T}}$ to mean that either $\Xi = \{z_i\}$, $1 \leq i \leq n$, or $\Xi = \xi_j(\mathbb{R})$, $1 \leq j \leq m$;

- (c) if $\Xi \in \Omega_{\mathcal{T}}$, then $\dim W_{loc}^u(\Xi) < \infty$;
- (d) if $\Xi, \hat{\Xi} \in \Omega_{\mathcal{T}}$, then $W^u(\Xi)$ and $W_{loc}^s(\hat{\Xi})$ intersect transversally, that is, if $z \in W^u(\Xi) \cap W_{loc}^s(\hat{\Xi})$, then $T_z W^u(\Xi) + T_z W_{loc}^s(\hat{\Xi}) = X$, where $T_z W^u(\Xi)$ and $T_z W_{loc}^s(\hat{\Xi})$ are the tangent spaces of $W^u(\Xi), W_{loc}^s(\hat{\Xi})$, respectively, in X at the point z .

REMARK 5.3. Note that in item (d) of Definition 5.2, the manifolds $W^u(\Xi)$ and $W_{loc}^s(\hat{\Xi})$ may not have intersection at all, and (d) is automatically satisfied.

Our first goal will be to relate the Morse-Smale semigroups and the dynamically gradient semigroups, that is, to express both concepts in the same terminology. We begin by showing that the ω -limit and the α_ϕ -limit of points are contained in the nonwandering set.

LEMMA 5.4. *Let \mathcal{T} be a semigroup with a global attractor \mathcal{A} and a nonwandering set $\Omega_{\mathcal{T}}$. Then $\Omega_{\mathcal{T}}$ is compact, $T(t)\Omega_{\mathcal{T}} \subset \Omega_{\mathcal{T}}$ for all $t \geq 0$, $\omega(x) \subset \Omega_{\mathcal{T}}$ for all $x \in X$ and $\alpha_\phi(x) \subset \Omega_{\mathcal{T}}$ for all $x \in \mathcal{A}$.*

PROOF. To show the compactness of $\Omega = \Omega_{\mathcal{T}}$ it suffices to show that it is closed, since $\Omega \subset \mathcal{A}$. Let $\{z_n\} \subset \Omega$ such that $z_n \rightarrow z \in \mathcal{A}$, $t_0 \geq 0$ and V_z a neighborhood of z in X . Since $z_n \rightarrow z$, there exists $n_0 \in \mathbb{N}$ such that V_z is a neighborhood of $z_{n_0} \in \Omega$, thus there exists $t \geq t_0$ such that $T(t)V_z \cap V_z \neq \emptyset$, and shows that $z \in \Omega$.

Let $z \in \Omega$, $t, t_0 \geq 0$ and V a neighborhood of $T(t)z$. From the continuity of $T(t)$, $T(t)^{-1}(V)$ is a neighborhood of z and hence, there exists $\tau \geq t_0$ such that $T(\tau)(T(t)^{-1}(V)) \cap T(t)^{-1}(V) \neq \emptyset$, which implies that $T(\tau)V \cap V \neq \emptyset$, $\tau \geq t_0$. From this we conclude that $T(t)z \in \Omega$ and that $T(t)\Omega \subset \Omega$, for all $t \geq 0$.

For the last statement, let $z \in \omega(x) \subset \mathcal{A}$, for some $x \in X$, $t_0 \geq 0$ and V a neighborhood of z in X . Let $\{t_n\} \subset [0, \infty)$ such that $t_n \rightarrow \infty$ and $z = \lim_{n \rightarrow \infty} T(t_n)x$, thus there exists $n_0 \in \mathbb{N}$ such that $T(t_n)x \in V$ for all $n \geq n_0$ and we choose $n_1 \geq n_0$ such that $t := t_{n_1} - t_{n_0} \geq t_0$. Thus $T(t_{n_1})x \in V$ and $T(t_{n_1})x = T(t_{n_1} - t_{n_0})T(t_{n_0})x = T(t)T(t_{n_0})x \in T(t)V$, which shows us that $T(t)V \cap V \neq \emptyset$ and consequently $z \in \Omega$. The proof for the α_ϕ -limit is analogous. \square

COROLLARY 5.5. *With the hypothesis of Lemma 5.4, if $X = \mathcal{A}$ and \mathcal{T} is injective, then $\Omega_{\mathcal{T}}$ is invariant.*

PROOF. To see this let $z \in \Omega_{\mathcal{T}}$. Since $T(s)$ is injective and \mathcal{A} is compact, it follows that $T(s)$ is a homeomorphism and thus there exists a unique $x \in \mathcal{A}$ such that $T(s)x = z$. Let $t_0 \geq 0$ and V a neighborhood of x in X . Again, since $T(s)$ is a homeomorphism, $T(s)V$ is a neighborhood of z and thus there exists $t \geq t_0$ such that $T(t)(T(s)V) \cap T(s)V \neq \emptyset$. Therefore $T(t)V \cap V \neq \emptyset$, which concludes the invariance of $\Omega_{\mathcal{T}}$. \square

Now we can show that every Morse-Smale semigroup is a generalized dynamically gradient semigroup. We start with some preparatory results which will be useful throughout the chapter.

PROPOSITION 5.6. *Let \mathcal{T} be a Morse-Smale semigroup with nonwandering set*

$$\Omega_{\mathcal{T}} = \{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\},$$

where the z_i is a hyperbolic equilibria and $\xi_j: \mathbb{R} \rightarrow X$ is a normally hyperbolic periodic solution, $1 \leq i \leq n$, $1 \leq j \leq m$. Let $\Xi, \hat{\Xi} \in \Omega$. If $W^u(\Xi) \cap W_{loc}^s(\hat{\Xi}) \neq \emptyset$, then $\dim W^u(\Xi) \geq \dim W^u(\hat{\Xi})$.

PROOF. Since \mathcal{T} is a Morse-Smale semigroup, if $z \in W^u(\Xi) \cap W_{loc}^s(\hat{\Xi})$ is a point of transversal intersection, then $T_z W^u(\Xi) + T_z W_{loc}^s(\hat{\Xi}) = X$. For $\hat{z} \in \hat{\Xi}$ we have $T_{\hat{z}} W_{loc}^u(\hat{\Xi}) \oplus T_{\hat{z}} W_{loc}^c(\hat{\Xi}) \oplus T_{\hat{z}} W_{loc}^s(\hat{\Xi}) = X$ and

$$\begin{aligned} \dim W^u(\Xi) &= \dim T_z W^u(\Xi) \geq \dim T_{\hat{z}} W^u(\hat{\Xi}) + \dim T_{\hat{z}} W_{loc}^c(\hat{\Xi}) \\ &= \dim W^u(\hat{\Xi}) + \dim T_{\hat{z}} W_{loc}^c(\hat{\Xi}), \end{aligned}$$

where $\dim T_{\hat{z}} W_{loc}^c(\hat{\Xi}) = 0$ if $\hat{\Xi}$ is a hyperbolic equilibrium and $\dim T_{\hat{z}} W_{loc}^c(\hat{\Xi}) = 1$ if $\hat{\Xi}$ is a normally hyperbolic periodic orbit. \square

LEMMA 5.7. *Let \mathcal{T} be a Morse-Smale semigroup with nonwandering set*

$$\Omega_{\mathcal{T}} = \{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\},$$

where the z_i is a hyperbolic equilibria and $\xi_j: \mathbb{R} \rightarrow X$ is a normally hyperbolic periodic solution, $1 \leq i \leq n$, $1 \leq j \leq m$. If $W^u(\Xi) \cap W_{loc}^s(\hat{\Xi}) \neq \emptyset$, then there exists a submanifold $D^0 \subset W^u(\Xi)$ such that $D^0 \cap W_{loc}^s(\hat{\Xi}) = \{x^0\}$ and $T_{x^0} D^0 \oplus T_{x^0} W_{loc}^s(\hat{\Xi}) = X$.

PROOF. Let $x^0 \in W^u(\Xi) \cap W_{loc}^s(\hat{\Xi})$ be a point of transversal intersection. There exist a neighborhood V of x^0 in $W^u(\Xi)$, a neighborhood W of 0 in $T_{x^0} W^u(\Xi)$ and a C^1 open map $\psi: W \rightarrow V$. By Proposition 5.6, $\dim W = \dim W^u(\Xi) \geq \dim W^u(\hat{\Xi})$ and hence there exists a vector subspace F of $T_{x^0} W^u(\Xi)$ such that $F \oplus T_{x^0} W_{loc}^s(\hat{\Xi}) = X$.

Then $D^0 = \psi(F)$ has the desired property. \square

We now discuss the *openness of transversality*. Such results can be found in [144] - where they are referred to as *Thom's Transversality Theorems* - in a very abstract and general framework. We only use the rather particular property of these results which are stated and proved next.

LEMMA 5.8. *Let X be a Banach space, X_1, X_2 vector subspaces of X such that X_1 is closed and $X_1 \oplus X_2 = X$. Let also $\theta, \tilde{\theta}: X_1 \rightarrow X_2$, $\sigma, \tilde{\sigma}: X_2 \rightarrow X_1$ be continuously differentiable maps such that $\theta(0) = \sigma(0) = 0$ and $\text{Lip } \theta, \text{Lip } \tilde{\theta}, \text{Lip } \sigma, \text{Lip } \tilde{\sigma} < 1$. Consider the following immersed C^1 submanifolds*

$$M = \{y + \theta(y): y \in X_1, \|y\| \leq r\}, \quad N = \{\sigma(x) + x: x \in X_2, \|x\| \leq r\},$$

and

$$\tilde{M} = \{y + \tilde{\theta}(y): y \in X_1, \|y\| \leq r\}, \quad \tilde{N} = \{\tilde{\sigma}(x) + x: x \in X_2, \|x\| \leq r\}.$$

If $\tilde{\theta}$ is C^1 -close to θ and $\tilde{\sigma}$ is C^1 -close to σ , then \tilde{M} and \tilde{N} have a nonempty intersection.

PROOF. From the hypotheses, there exists a constant $c \in (0, 1)$ such that

$$\text{Lip } \theta, \text{Lip } \tilde{\theta}, \text{Lip } \sigma, \text{Lip } \tilde{\sigma} \leq c,$$

and also there exists $r > 0$ such that

$$(5.1) \quad \|\theta - \tilde{\theta}\|_{C^1(B_r^{X_1}(0), X_2)} \leq (1-c)r \text{ and } \|\sigma - \tilde{\sigma}\|_{C^1(B_r^{X_2}(0), X_1)} \leq (1-c)r.$$

Note that $y + \tilde{\theta}(y) = \tilde{\sigma}(x) + x$ is equivalent to $\tilde{\sigma}(\tilde{\theta}(y)) = y$. Define $g: \bar{B}_r^{X_1}(0) \rightarrow \bar{B}_r^{X_1}(0)$ by $g(x) = \tilde{\sigma}(\tilde{\theta}(x))$. We claim that g is well defined. Indeed, for all $x \in \bar{B}_r^{X_1}(0)$,

$$\|\tilde{\theta}(x)\| \leq \|\tilde{\theta}(x) - \theta(x)\| + \|\theta(x) - \theta(0)\| \leq \|\tilde{\theta}(x) - \theta(x)\| + cr \leq r.$$

Analogously,

$$\|\tilde{\sigma}(\tilde{\theta}(x))\| \leq \|\tilde{\sigma}(\tilde{\theta}(x)) - \sigma(\tilde{\theta}(x))\| + \|\sigma(\tilde{\theta}(x)) - \sigma(0)\| \leq \|\tilde{\sigma}(\tilde{\theta}(x)) - \sigma(\tilde{\theta}(x))\| + cr \leq r$$

for all $x \in \bar{B}_r^{X_1}(0)$. Note that g is a contraction, since

$$\|g(x) - g(y)\| \leq \text{Lip}(\tilde{\sigma})\text{Lip}(\tilde{\theta})\|x - y\| \leq c^2\|x - y\|.$$

Hence g has a unique fixed point, which shows that \tilde{M} and \tilde{N} intersect. \square

Using this lemma we can prove the *transitive property of connections* (see Figure 5.1).

PROPOSITION 5.9. *Let \mathcal{T} be a Morse-Smale semigroup with nonwandering set $\Omega_{\mathcal{T}} = \{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\}$ and $\Xi_1, \Xi_2, \Xi_3 \in \Omega_{\mathcal{T}}$. If $W^u(\Xi_1) \cap W_{loc}^s(\Xi_2) \neq \emptyset$ and $W^u(\Xi_2) \cap W_{loc}^s(\Xi_3) \neq \emptyset$, then $W^u(\Xi_1) \cap W_{loc}^s(\Xi_3) \neq \emptyset$.*

PROOF. From Lemma 5.7, if x_{12} is the point of transversal intersection between $W^u(\Xi_1)$ and $W_{loc}^s(\Xi_2)$, we can choose a submanifold D^0 of X such that $D^0 \cap W_{loc}^s(\Xi_2) = \{x_{12}\}$, $T_{x_{12}}D^0 \oplus T_{x_{12}}W_{loc}^s(\Xi_2) = X$ and $D^0 \subset W^u(\Xi_1)$. Choosing n_0 such that, if x_{23} is the point of transversal intersection between $W^u(\Xi_2)$ and $W_{loc}^s(\Xi_3)$, $x_{23} \in T^{n_0}(W_{loc}^u(\Xi_2))$, the global λ -lemma implies that there exist a neighborhood D_ρ^0 of x_{12} in D^0 and sets $R^n \subset T^n(D_\rho^0) \subset W^u(\Xi_1)$ such that

$$R^n \rightarrow T^{n_0}(W_{loc}^u(\Xi_2)),$$

in the C^1 norm. Hence, by the *openness of the transversality* (Lemma 5.8), there exists a point $x_{13} \in W^u(\Xi_1) \cap W_{loc}^s(\Xi_3)$ (in which the intersection is transversal). \square

Now we prove that a Morse-Smale semigroup is a dynamically gradient semigroup.

THEOREM 5.10. *Let \mathcal{T} be a Morse-Smale semigroup with global attractor \mathcal{A} and nonwandering set $\Omega_{\mathcal{T}} = \{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\}$. Then \mathcal{T} is a dynamically gradient semigroup with $\Omega_{\mathcal{T}}$ as its disjoint set of isolated invariant families.*

PROOF. Let us first prove that \mathcal{T} satisfies (G2). Assume that there exists a homoclinic structure in $\Omega_{\mathcal{T}}$. From Proposition 5.9 we obtain that there is a homoclinic solution, that is, there exist $\Xi \in \Omega_{\mathcal{T}}$ and a global solution $\xi: \mathbb{R} \rightarrow X$ such that $\xi(t_0) \notin \Omega$, for some $t_0 \in \mathbb{R}$, and $\xi(t) \xrightarrow{t \rightarrow \pm\infty} \Xi$. Let $x^0 = \xi(t_0)$ (which we can take in $W_{loc}^s(\Xi)$), a neighborhood V of x^0 in X and $D^0 \subset V$ a submanifold transversal to $W_{loc}^s(\Xi)$ in x^0 . Then, the global λ -lemma implies that there exist a neighborhood D_ρ^0 of x^0 in D^0 and subsets S_n of $T^n(D_\rho^0) \subset T^n(V)$ such that $S_n \rightarrow T^{n_0}(W_{loc}^u(\Xi))$, where $n_0 \in \mathbb{N}$ is such that $x^0 \in T^{n_0}(W_{loc}^u(\Xi))$, which proves that $x^0 \in \Omega_{\mathcal{T}}$ and gives us a contradiction. Therefore \mathcal{T} satisfies (G2).

Now we prove that \mathcal{T} satisfies (G1). Let $\xi: \mathbb{R} \rightarrow \mathcal{A}$ be a global solution with $\xi(\mathbb{R}) \notin \Omega$ and let $x = \xi(0)$. From Lemma 5.4 we know that $\omega(x), \alpha_\xi(x) \subset \Omega_{\mathcal{T}}$. Since $\omega(x)$ and $\alpha_\xi(x)$ are both connected it follows immediately that (G1) is satisfied. \square

Now we prove the converse result.

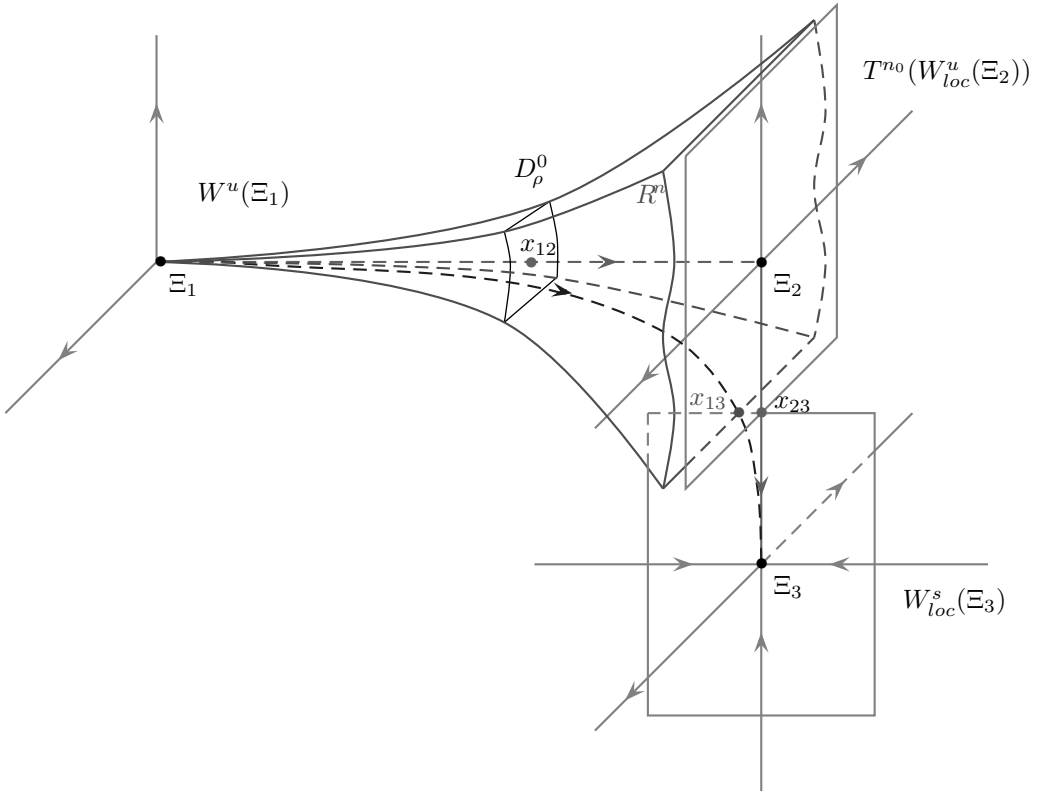


FIGURE 5.1. Indirect connections between Ξ_1 and Ξ_3 generate a direct connection between Ξ_1 and Ξ_3 .

THEOREM 5.11. *Let \mathcal{T} be a reversible dynamically gradient semigroup associated with the disjoint set of isolated invariants given by $\Xi = \{z_1, \dots, z_n, \xi_1(\mathbb{R}), \dots, \xi_m(\mathbb{R})\}$, where each z_i is a hyperbolic equilibrium point and $\xi_j: \mathbb{R} \rightarrow X$ is a normally hyperbolic periodic orbit, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Assume that if $\Xi \in \Xi$, then $\dim W_{loc}^u(\Xi) < \infty$ and that, if $\Xi, \hat{\Xi} \in \Xi$, then $W^u(\Xi)$ and $W_{loc}^s(\hat{\Xi})$ intersect transversally. Then \mathcal{T} is a Morse-Smale semigroup and its nonwandering set $\Omega_{\mathcal{T}}$ coincides with Ξ .*

PROOF. It just remains to show the last assertion, that is, that $\Omega_{\mathcal{T}} = \Xi$. It is clear that $\Xi \subset \Omega_{\mathcal{T}}$. Now assume that there exists $x \in \Omega_{\mathcal{T}} \setminus \Xi$ and consider a global solution $\xi_1: \mathbb{R} \rightarrow X$ through x . By (G1), there exist Ξ_1 and Ξ_2 in Ξ such that $\Xi_1 \neq \Xi_2$ and

$$\Xi_1 \xrightarrow[t \rightarrow -\infty]{} \xi_1(t) \xrightarrow[t \rightarrow \infty]{} \Xi_2.$$

Since $x \in \Omega_{\mathcal{T}}$ and using the already constructed ξ_1 , it is possible to construct a global solution $\xi_2: \mathbb{R} \rightarrow X$ such that

$$\Xi_2 \xrightarrow[t \rightarrow -\infty]{} \xi_2(t) \xrightarrow[t \rightarrow \infty]{} \Xi_3,$$

for some $\Xi_3 \in \Xi$, and by (G2) we must have $\Xi_3 \notin \{\Xi_1, \Xi_2\}$. Using this process inductively, we reach a contradiction since Ξ only has a finite number of isolated invariant families and therefore, $\Omega_{\mathcal{T}} \subset \Xi$ and this completes the proof. \square

We can now join our results to obtain the following.

COROLLARY 5.12. *Let \mathcal{T} be a reversible semigroup with finitely many equilibria and finitely many periodic orbits $\Xi = \{z_1, \dots, z_n, \xi_i(\mathbb{R}), \dots, \xi_m(\mathbb{R})\}$, where each z_i is a hyperbolic equilibrium point and $\xi_j: \mathbb{R} \rightarrow X$ is a normally hyperbolic periodic orbit, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Assume also that, if $\Xi \in \Xi$, then $\dim W_{loc}^u(\Xi) < \infty$ and, if $\Xi_1, \Xi_2 \in \Xi$, then $W^u(\Xi_1)$ and $W_{loc}^s(\Xi_2)$ intersect transversally. Then \mathcal{T} is a Morse-Smale semigroup if and only if it is a dynamically gradient semigroup with respect to Ξ .*

2. Structural stability of Morse-Smale semigroups

In this subsection we explore the behavior of the structure of the global attractor of a Morse-Smale semigroup under small perturbations. More precisely, we establish that the phase diagrams, for the global attractor of a Morse-Smale semigroup and of a suitably small perturbation of it, are isomorphic, that is, the connections between the isolated invariant sets (equilibria and periodic orbits) are the same for the limiting semigroup and the perturbed semigroup. We start with the definition of phase diagram commutativity.

DEFINITION 5.13. Let \mathcal{T} and \mathcal{S} be two dynamically gradient semigroups. We say that there exists a **phase diagram isomorphism** between the attractors of \mathcal{T} and \mathcal{S} if they satisfy the following condition: if $\Xi_{\mathcal{T}} = \{\Xi_1, \dots, \Xi_p\}$ and $\Xi_{\mathcal{S}} = \{\hat{\Xi}_1, \dots, \hat{\Xi}_p\}$ are the disjoint collection of isolated invariant families for \mathcal{T} and \mathcal{S} , respectively, then there exists a bijection $\mathcal{D}: \Xi_{\mathcal{T}} \rightarrow \Xi_{\mathcal{S}}$ such that, for some $1 \leq i, j \leq p$, there exists a global solution $\xi: \mathbb{R} \rightarrow X$ of \mathcal{T} satisfying

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi(t), \Xi_i) = 0 \text{ and } \lim_{t \rightarrow \infty} \text{dist}_H(\xi(t), \Xi_j) = 0,$$

if and only if there is a global solution $\psi: \mathbb{R} \rightarrow X$ of \mathcal{S} such that

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\psi(t), \mathcal{D}(\Xi_i)) = 0 \text{ and } \lim_{t \rightarrow \infty} \text{dist}_H(\psi(t), \mathcal{D}(\Xi_j)) = 0.$$

We can now state the main result of this chapter.

THEOREM 5.14. *Let $\{\mathcal{T}_\eta\}_{\eta \in [0,1]}$ be a family of semigroups which is continuous and collectively asymptotically compact at $\eta = 0$. Assume that*

- (a) \mathcal{T}_η has global attractor \mathcal{A}_η for each $\eta \in [0, 1]$ and $\bigcup_{\eta \in (0,1]} \mathcal{A}_\eta$ is bounded;
- (b) $\{\mathcal{T}_\eta\}_{\eta \in [0,1]} \subset C^1(X)$ and $\mathbb{R}^+ \times [0, 1] \ni (t, \eta) \mapsto T_\eta(t)'(x) \in \mathcal{L}(X)$ is continuous for each $x \in X$;
- (c) \mathcal{T}_0 is a dynamically gradient semigroup relative to the disjoint collection of isolated invariant sets $\Xi_0 = \{z_{1,0}, \dots, z_{n,0}, \xi_{1,0}(\mathbb{R}), \dots, \xi_{m,0}(\mathbb{R})\}$ where $z_{i,0}$ is a hyperbolic equilibria and $\xi_{j,0}: \mathbb{R} \rightarrow X$ is a normally hyperbolic periodic solution, $1 \leq i \leq n$, $1 \leq j \leq m$.

Then

- (i) *there is an $\eta_0 \in (0, 1]$ such that, for each $\eta \in (0, \eta_0]$, there are hyperbolic equilibria $\{z_{1,\eta}, \dots, z_{n,\eta}\}$ and normally hyperbolic periodic solutions $\{\xi_{1,\eta}(\mathbb{R}), \dots, \xi_{m,\eta}(\mathbb{R})\}$ such that*

$$\Xi_\eta = \{z_{1,\eta}, \dots, z_{n,\eta}, \xi_{1,\eta}(\mathbb{R}), \dots, \xi_{m,\eta}(\mathbb{R})\}$$

is a disjoint collection of isolated invariant sets for \mathcal{T}_η , for all suitably small $\eta > 0$, and

$$(5.2) \quad \lim_{\eta \rightarrow 0} \text{dist}_H(z_{j,\eta}, z_{j,0}) = 0 \quad \text{and} \quad \lim_{\eta \rightarrow 0} \text{dist}_H(\xi_{i,\eta}(\mathbb{R}), \xi_{i,0}(\mathbb{R})) = 0;$$

- (ii) *there exist $\delta > 0$ and $\eta_1 \in (0, \eta_0]$ such that, if $\eta \in (0, \eta_1]$ and $\Xi_\eta \in \Xi_\eta$, then Ξ_η is the maximal invariant set in $\mathcal{O}_\delta(\Xi_\eta)$;*
- (iii) *there exists an $\eta_2 \in (0, \eta_1]$ such that \mathcal{T}_η is a dynamically gradient semigroup relative to Ξ_η , for all $\eta \in (0, \eta_2]$.*

In addition, if \mathcal{T}_0 is a Morse-Smale semigroup, then

- (iv) *there exists an $\eta_3 \in (0, \eta_2]$ such that \mathcal{T}_η is a Morse-Smale semigroup and there exists a phase diagram isomorphism between the attractors of \mathcal{T}_η and \mathcal{T}_0 , for all $\eta \in (0, \eta_3]$.*

PROOF. Items (i) and (ii) follow from Theorems 4.24 and 4.38. From Theorem 3.42, item (iii) follows. It remains to prove (iv). We can chose η_2 in such a way that each equilibrium $z^* \in \{z_{1,\eta}, \dots, z_{n,\eta}\}$ is hyperbolic and each periodic solution $\xi^* \in \{\xi_{1,\eta}, \dots, \xi_{m,\eta}\}$ is normally hyperbolic, $0 < \eta < \eta_2$.

Define $\mathcal{D}: \Xi_0 \rightarrow \Xi_\eta$ by

$$\mathcal{D}(z_i) = z_{i,\eta}, \text{ for } i = 1, \dots, n, \quad \text{and} \quad \mathcal{D}(\xi_j) = \xi_{j,\eta}, \text{ for } j = 1, \dots, m,$$

which is clearly a bijection.

Now if $W^u(\Xi_{i,0}) \cap W^s_{loc}(\Xi_{j,0}) \neq \emptyset$, then using the results of Section 2, which ensure that the unstable and stable manifolds behave continuously in the C^1 norm, and Lemma 5.8, there exists $\eta_3 \in (0, \eta_2]$ such that $W^u(\Xi^*_{i,\eta}) \cap W^s_{loc}(\Xi_{j,\eta}) \neq \emptyset$, for all $\eta \in (0, \eta_3]$.

This shows that if there exists a connection between $\Xi_{i,0}$ and $\Xi_{j,0}$, then there exists a connection between $\Xi_{i,\eta}$ and $\Xi_{j,\eta}$. It remains to show the converse.

Assume that we have a sequence $\eta_k \rightarrow 0$ and a sequence of global solutions $\xi_k: \mathbb{R} \rightarrow X$ of \mathcal{T}_{η_k} such that

$$\lim_{t \rightarrow -\infty} \text{dist}_H(\xi_k(t), \Xi_{i,\eta_k}) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \text{dist}_H(\xi_k(t), \Xi_{j,\eta_k}) = 0.$$

Then we know by Lemma 3.44 that there is a finite collection of global solutions $\varphi_\ell: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 , $\ell = 1, \dots, m - 1$ ($m \leq n$), together with isolated invariant sets Ξ_{n_ℓ} in Ξ_0 , such that $\Xi_{n_1} = \Xi_{i,0}$, $\Xi_{n_m} = \Xi_{j,0}$ and

$$\Xi_{n_\ell} \xrightarrow[t \rightarrow -\infty]{} \varphi_\ell(t) \xrightarrow[t \rightarrow \infty]{} \Xi_{n_{\ell+1}}, \text{ for all } \ell = 1, \dots, m - 1.$$

But Proposition 5.9 ensures the existence of a global solution $\varphi: \mathbb{R} \rightarrow X$ of \mathcal{T}_0 such that

$$\Xi_i \xrightarrow[t \rightarrow -\infty]{} \varphi(t) \xrightarrow[t \rightarrow \infty]{} \Xi_j,$$

and we conclude the proof. □

3. Examples

In this section we will describe a few applications for Theorem 5.14, and also an example where we lose connections when making a perturbation on a semigroup due to the lack of transversality.

3.1. Automatic transversality for a class of ODE's. Following [125], let us consider an ordinary differential equation

$$(5.3) \quad \begin{cases} \dot{x} = f(x), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $f: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 map, with Ω convex and open. We consider \mathcal{L} the set of $n \times n$ matrices of the form

$$(5.4) \quad L = \begin{bmatrix} a_1 & b_1 & 0 & c_1 \\ c_2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & b_{n-1} \\ b_n & 0 & c_n & a_n \end{bmatrix},$$

where a_i, b_i and c_i satisfy $b_i, c_i \geq 0$, for $i = 1, \dots, n$, and

$$\prod_{i=1}^n b_i + \prod_{i=1}^n c_i > 0.$$

If $\{e_j\}_{j=1}^n$ is a basis in E and $\mathcal{L}(E)$ is the set of linear operators on E , we define $\mathcal{L}\{e_j\} \subset \mathcal{L}(E)$ the subset of operators for which the matrix representation with respect to the basis $\{e_j\}_{j=1}^n$ belongs to \mathcal{L} . Thus, from [125, Theorem 5], if $f: \Omega \subset E \rightarrow E$ is a C^1 function, with Ω convex and open, and is such that there exists a fixed basis $\{e_j\}_{j=1}^n$ such that for each $x \in \Omega$, the derivative $f'(x) \in \mathcal{L}\{e_j\}$, then given Ξ_1, Ξ_2 two critical elements of (5.3) (either a hyperbolic point or a normally hyperbolic orbit) we have $W^u(\Xi_1)$ and $W^s(\Xi_2)$ intersect transversally, provided that:

- (i) Ξ_1 or Ξ_2 is a normally hyperbolic orbit;
- (ii) Ξ_1 and Ξ_2 are both hyperbolic points and the spectra $\sigma(S_k)$ of the operators $S_k = e^{f(\Xi_k)}$, $k = 1, 2$, have the property that there is $h \in \{1, \dots, (\tilde{n} - 1)/2 + 1\}$ such that

$$\nu_h^- > 1 \quad \text{and} \quad \mu_{h+1}^+ < 1,$$

where $\tilde{n} = n$ if n is odd and $\tilde{n} = n + 1$ if n is even, and ν_h^- and μ_h^+ are given in [125, Theorem 2].

Thus, if (5.3) generates a reversible semigroup \mathcal{T} with a global attractor \mathcal{A} , with a finite number of hyperbolic equilibria and of normally hyperbolic orbits, then \mathcal{T} is a Morse-Smale semigroup.

3.2. Reaction-diffusion equations. Let Ω be a bounded smooth domain in \mathbb{R}^n , $n \geq 1$, and let f be a C^k -function, $k \geq 2$, from $\bar{\Omega} \times \mathbb{R}$ into \mathbb{R} . We consider the equation

$$(5.5) \quad \begin{cases} u_t = \Delta u + f(x, u), & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u = u_0, & t = 0, x \in \Omega, \end{cases}$$

where $u_0 \in W_0^{1,p}(\Omega)$, $p > n$. Since $W_0^{1,p}(\Omega)$, $p > n$, is continuously embedded in $C^0(\bar{\Omega})$, the equation (5.5) defines a *local semiflow* \mathcal{T}_f on $X = W_0^{1,p}(\Omega)$, $p > n$. This semiflow is dynamically gradient, with associated Lyapunov function

$$(5.6) \quad \Phi(\varphi) = \int_{\Omega} \left(\frac{1}{2} |\nabla \varphi(x)|^2 - F(x, \varphi(x)) \right) dx,$$

which decreases along nonconstant trajectories of \mathcal{T}_f . Here $F(x, y)$ is the primitive $F(x, y) = \int_0^y f(x, s) ds$ of $f(x, \cdot)$. To obtain a semigroup (that is, a global semiflow), we will add the following dissipative condition: we assume that there exist a positive constant C_0 and a constant $\mu < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary conditions, such that, for any $y \in \mathbb{R}$ and $x \in \bar{\Omega}$, we have

$$(5.7) \quad f(x, y)y < C_0 + \mu y^2, \quad F(x, y) < C_0 + \frac{1}{2} \mu y^2.$$

Under this additional assumption (5.7), we obtain the following proposition.

PROPOSITION 5.15. *If condition (5.7) holds, then all the solutions of (5.5) are global, that is, they exist for all times $t \geq 0$. In addition, the semigroup \mathcal{T}_f admits a global attractor \mathcal{A}_f in X and there exists a positive constant K_0 , depending only on C_0 , μ and Ω , such that*

$$(5.8) \quad \sup_{\varphi \in \mathcal{A}_f} (\|\varphi\|_{W^{2,p}} + \|\varphi\|_{L^\infty}) \leq K_0.$$

Let \mathcal{G}^k , $k \geq 1$, be the subset of functions of $C^k(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfying (5.7), endowed with the Whitney C^k topology. It follows from [34, Theorem 1.1] that there exists a generic set \mathcal{G}_{MS}^k in \mathcal{G}^k , $k \geq 2$, such that, for any $f \in \mathcal{G}_{MS}^k$, every equilibrium point φ of (5.5) is hyperbolic and, if φ_- and φ_+ are any two such equilibria, their stable and unstable manifolds intersect transversally, that is,

$$W^u(\varphi_-) \pitchfork W^s(\varphi_+).$$

Thus, from [34, Theorem 1.1], we at once deduce the following theorem.

THEOREM 5.16. *For any f belonging to the generic set \mathcal{G}_{MS}^k , $k \geq 2$, the semigroup \mathcal{T}_f generated by (5.5) is a C^2 semigroup, which is a Morse-Smale semigroup on X with global attractor \mathcal{A}_f satisfying the estimates (5.8). Moreover, since \mathcal{T}_f is dynamically gradient, its nonwandering set $\Omega_{\mathcal{T}_f}$ reduces to a finite set of hyperbolic equilibrium points $\mathcal{Z}_f = \{\varphi_1, \dots, \varphi_m\}$.*

We now fix a function $f_0 \in \mathcal{G}_{MS}^k$, $k \geq 2$, and consider perturbations of equation (5.5), that is, we consider the equation

$$(5.9) \quad \begin{cases} u_t = \Delta u + \eta a(x) \cdot \nabla u + f_0(x, u), & t > 0, \ x \in \Omega, \\ u = 0, & t > 0, \ x \in \partial\Omega, \\ u = u_0, & t = 0, \ x \in \Omega, \end{cases}$$

where $u_0 \in X$, $a: \bar{\Omega} \rightarrow \mathbb{R}^n$. Then, from Theorem 5.14, for all η suitably small the semigroup associated to (5.9) is Morse-Smale and there is a phase diagram isomorphism between the attractors of the perturbed and unperturbed semigroups.

3.3. Application to the damped wave equation. Let Ω be a bounded smooth domain in \mathbb{R}^n , $n = 1, 2, 3$. Let f be a C^k -function, $k \geq 2$, from $\overline{\Omega} \times \mathbb{R}$ into \mathbb{R} . We choose a positive constant $\gamma > 0$ and consider the damped wave equation

$$(5.10) \quad \begin{cases} u_{tt} + \gamma u_t - \Delta u = f(x, u), & t > 0, x \in \Omega, \\ u(x, t) = 0, & t > 0, x \in \partial\Omega, \\ (u(x, 0), u_t(x, 0)) = (u_0, v_0), \end{cases}$$

where $(u_0, v_0) \in Y$ with $Y = H_0^1(\Omega) \times L^2(\Omega)$ if $n = 1$ and $Y = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ if $n = 2$ or 3 . Then (5.10) generates a local semiflow $S_f(t)$ on Y . This semiflow is dynamically gradient with associated Lyapunov function

$$(5.11) \quad \Phi(\varphi, \psi) = \int_{\Omega} \left(\frac{1}{2} |\nabla \varphi(x)|^2 + \frac{1}{2} |\psi(x)|^2 - F(x, \varphi(x)) \right) dx,$$

which decreases along nonconstant trajectories of $S_f(\cdot)$. To obtain a global semiflow, as in the parabolic case, we will assume the dissipative condition (5.7). Under this additional assumption (5.7), again as in the parabolic case, we obtain the following proposition.

PROPOSITION 5.17. *If condition (5.7) holds, then all the solutions of (5.10) are global. In addition, the semiflow $S_f(t)$ admits a compact global attractor \mathcal{A}_f^D in Y and there exists a positive constant K_0^D , depending only on C_0 , μ , γ and Ω , such that*

$$(5.12) \quad \sup_{(\varphi, \psi) \in \mathcal{A}_f^D} (\|(\varphi, \psi)\|_{H^2 \times H^1} + \|\varphi\|_{L^\infty}) \leq K_0^D.$$

For $k \geq 3$ (and $k \geq 2$ if $n = 1$), we introduce the space $\mathcal{G}_D^k = \{f \in C^k(\overline{\Omega} \times \mathbb{R}, \mathbb{R})\}$ (respectively $\mathcal{G}_D^k = \{f \in C^k(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) : f(x, 0) = 0, \text{ for all } x \in \partial\Omega\}$ in the case of Dirichlet boundary conditions if $n \geq 2$) which, in addition, satisfy the dissipative condition (5.7). We endow this space with the Whitney C^k topology. It follows from [36, Theorem 1.1] that there exists a generic set $\mathcal{G}_{D,MS}^k$ in \mathcal{G}_D^k , $k \geq 2$, such that, for any $f \in \mathcal{G}_{D,MS}^k$, every equilibrium point $(\varphi, 0)$ of (5.10) is hyperbolic and, if $(\varphi_-, 0)$ and $(\varphi_+, 0)$ are any two such equilibria, their stable and unstable manifolds intersect transversally, that is,

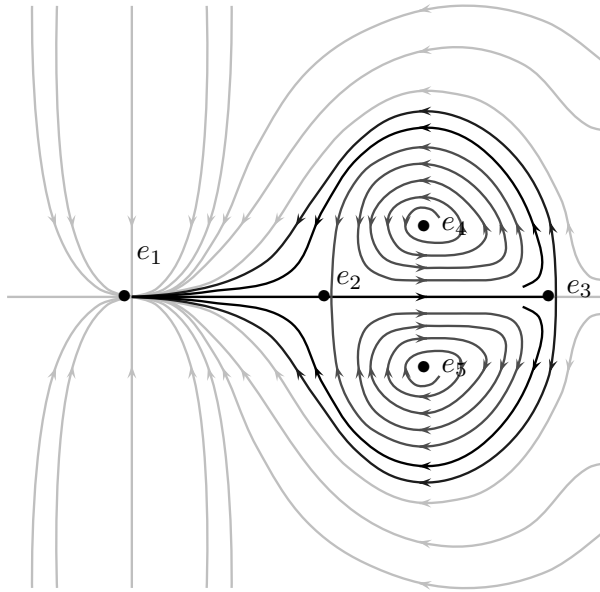
$$W^u((\varphi_-, 0)) \pitchfork W^s((\varphi_+, 0)).$$

Thus, from [36, Theorem 1.1], we at once deduce the following theorem.

THEOREM 5.18. *For any f belonging to the generic set $\mathcal{G}_{D,MS}^k$, $k \geq 2$, the semigroup S_f , generated by (5.10), is C^2 , and is a C^1 -reversible Morse-Smale semigroup on Y , with global attractor \mathcal{A}_f^D satisfying the estimates (5.12). Moreover, since S_f is dynamically gradient, its nonwandering set Ω_{S_f} reduces to a finite set of hyperbolic equilibrium points $\mathcal{Z}_f^D = \{(\varphi_1, 0), \dots, (\varphi_m, 0)\}$.*

From Theorem 5.14, suitably small perturbations (as in the previous example) of the semigroup associated to (5.10) are Morse-Smale and there is a phase diagram isomorphism between the attractors of the perturbed and unperturbed semigroups.

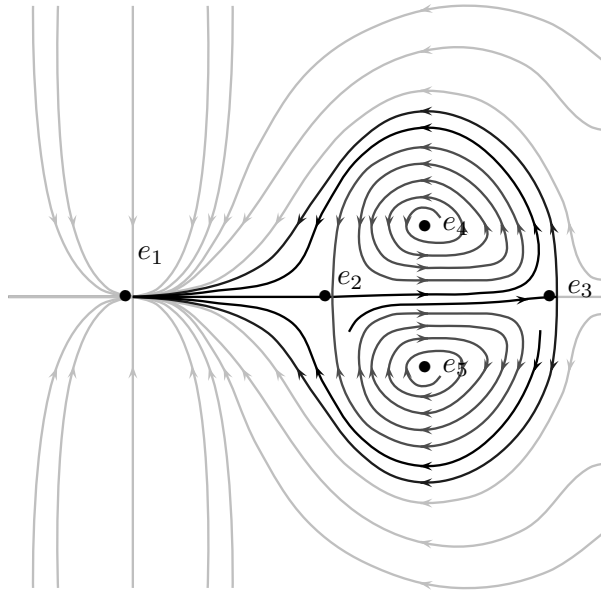
3.4. An example of nontransversality. In this subsection we recall the brief example done in the introduction in which we *do not* have the structural stability, due to a lack of transversality in the intersection of the unstable manifold and the local stable manifold of two equilibria.

FIGURE 5.2. The *alien head*.

Consider the attractor given in Figure 5.2: note that the unstable manifold $W^u(e_2)$ of e_2 and the local stable manifold $W_{loc}^s(e_3)$ of e_3 do not intersect transversally.

Now, we can make a small autonomous perturbation to this attractor to obtain the attractor given in Figure 5.3, and we can see that there is no phase diagram isomorphism between these objects, since the connection between e_2 and e_3 is lost.

Notes. It is always a very difficult task to show the internal characterization of global attractors. Morse-Smale semigroups have been considered in many works and are the class of examples which shows the most detailed structure of attractors for infinite-dimensional dynamical systems ([35, 36, 133, 147, 147, 175, 176, 186]). The concept was firstly developed in a finite-dimensional setting (see, for instance, [121, 178–180]). First results on Morse-Smale semigroups can be found in [141–143].

FIGURE 5.3. The perturbed *alien head*.

Non-autonomous Dynamical Systems and Their Attractors

Recently, the analysis of qualitative properties of *non-autonomous problems* in general phase spaces (infinite-dimensional Banach spaces or general metric spaces) has received a lot of attention (see for instance [153], [140], [201], [202], [78], [89], [43], [98], [190], [181], [116], [79], [99], [149] and [150]). In particular, the study of *pullback*, *skew-product* and *uniform attractors* has developed a wide and deep research area, providing qualitative information for the asymptotic dynamics of an increasing number of non-autonomous models of phenomena from different areas of knowledge such as, among others, Physics, Biology, Economics and Engineering.

To motivate our approach, we consider a simple non-autonomous ordinary differential equation in \mathbb{R}^n . Let \mathbb{K} be either \mathbb{R}^+ or \mathbb{R} and let $f: \mathbb{K} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a family of vector fields (one for each $t \in \mathbb{K}$). Assume that f satisfies *good* conditions, in the sense that we have existence and uniqueness of solutions for the initial value problem

$$(6.1) \quad \begin{cases} \dot{u} = f(t, u), & t > s, \\ u(s) = u_0 \in \mathbb{R}^n, \end{cases}$$

for each $s \in \mathbb{K}$ and $u_0 \in \mathbb{R}^n$. Suppose also that the solution $u(\cdot, s, u_0)$ of (6.1) is defined in $[s, \infty)$ for each $s \in \mathbb{K}$ and $u_0 \in \mathbb{R}^n$ and that the map $\mathcal{P} \times \mathbb{R}^n \ni (t, s, u_0) \mapsto u(t, s, u_0) \in \mathbb{R}^n$ is continuous, where $\mathcal{P} = \{(t, s) \in \mathbb{K}^2: t \geq s\}$.

For each t , $f(t, \cdot)$ is the vector field that drives the solution at time t . Hence, the path described by the solution in \mathbb{R}^n between s and $s + \tau$ will depend on the vector fields $f(t, \cdot)$ for $t \in [s, s + \tau]$, and therefore it will depend on both the initial time s and elapsed time τ . When $f(t, \cdot) = f(\cdot)$ is independent of t , that is, (6.1) is autonomous, the path described by the solution will no longer depend on s but only on the elapsed time τ .

The change from a time independent to a time dependent f may easily be underestimated in the study of the asymptotics of (6.1). Hence, we insist on calling the readers' attention to the fact that in the *autonomous case* there is only one vector field driving the solution at all times, whereas in the *non-autonomous case* there are infinitely many vector fields driving the solution (one for each instant of time), and these vector fields evolve with time, in a prescribed way, just as the solution does. Once we are aware of the inherent difficulty of non-autonomous problems we can start to come up with ways to look at its asymptotic dynamics.

When $\mathbb{K} = \mathbb{R}$ we can define the solution operator $S(t, s)u_0 = u(t, s, u_0)$ for each $(t, s) \in \mathcal{P}$ and the family $\mathcal{S} = \{S(t, s): (t, s) \in \mathcal{P}\} \subset C(\mathbb{R}^n)$ is an *evolution process* in \mathbb{R}^n , that is, a family $\mathcal{S} = \{S(t, s): (t, s) \in \mathcal{P}\} \subset C(\mathbb{R}^n)$ such that $S(t, t)x = x$,

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $S(t, s) = S(t, r)S(r, s)$, for all $t \geq r \geq s$, and such that the map $\mathcal{P} \times \mathbb{R}^n \ni (t, s, x) \mapsto S(t, s)x \in \mathbb{R}^n$ is continuous.

When f is independent of time, and hence $u(t, s, f, u_0) = u(t - s, 0, f, u_0)$, the asymptotic dynamics can be achieved by either making $t \rightarrow \infty$, that is, seeing what happens to the state at the final time t when t is driven further and further to the future, or making $s \rightarrow -\infty$, and thus seeing what happens to the state at time t when the initial time s is driven further and further to the past.

Now if f is time dependent these two asymptotic dynamics give rise to completely different scenarios. We may study the asymptotics with respect to the elapsed time $t - s$ (when $t - s \rightarrow \infty$) or with respect to s (when $s \rightarrow -\infty$ and t is arbitrary but fixed). These are called, respectively, *forward* and *pullback dynamics*, and are in general unrelated. It is only natural that they are unrelated, since the set of vector fields driving the solution depends on time.

For an evolution process \mathcal{S} , a *pullback attractor* is a family $\{A(t)\}_{t \in \mathbb{R}}$ such that $\bigcup_{t \in \mathbb{R}} A(t)$ is bounded, $A(t)$ is compact for each $t \in \mathbb{R}$, $S(t, s)A(s) = A(t)$ for all $t \geq s$ and $\text{dist}_H(S(t, s)B, A(t)) \rightarrow 0$ as $s \rightarrow -\infty$ for each bounded subset B of \mathbb{R}^n and for each $t \in \mathbb{R}$.

A *bounded global solution* of \mathcal{S} is a function $\xi: \mathbb{R} \rightarrow \mathbb{R}^n$ such that $S(t, s)\xi(s) = \xi(t)$ for each $t \geq s$, and in this case, assuming that it has a bounded pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$, we can characterize it by

$$A(t) = \{\xi(t) : \xi \text{ is a bounded global solution of } \mathcal{S}\}.$$

Therefore this object has much to say about the dynamics of (6.1), since it consists of all bounded global solutions of (6.1). It is also important to know that $\{A(t)\}_{t \in \mathbb{R}}$ may not have any forward attraction property (see, for instance, [78, Section 1.3] or [162]). In particular, when f is independent of time, it coincides with the global attractor of the associated semigroup.

One must note that the theory of pullback attractors requires that the vector fields are defined for all times in the real line \mathbb{R} (that is, $\mathbb{K} = \mathbb{R}$). In many models the backwards history or evolution is simply not available, and the vector fields are only known after a given fixed initial time, say, for each time in $\mathbb{K} = \mathbb{R}^+$. If we artificially define the vector fields for negative times, the pullback attractors will strongly depend on such extension, since most of the pullback asymptotics will be driven by these artificially introduced vector fields.

When f is time independent and $\mathbb{K} = \mathbb{R}^+$, the following important remark is due: in many situations, there is absolutely no reason to extend f for negative values of time by the same time independent vector field f . Yet, we speak about global solutions for such autonomous problems as we did in Chapter 1. The explanation of that will come together with the natural framework that we will establish for the non-autonomous case. Also, when $\mathbb{K} = \mathbb{R}^+$, the pullback attractor does not play a direct role in the asymptotic dynamics of (6.1). Nonetheless, as we will see later, it appears again in a natural and essential way, in the understanding of the asymptotic behavior of solutions of (6.1) within the appropriate framework. Let us pursue this a little further.

The crucial point here is to understand that, for a non-autonomous differential equation, there are infinitely many vector fields (one for each time) driving the evolution, whereas in the autonomous case there is only one. Therefore the evolution of the vector fields (called *driving semigroup*) must be taken into account as well.

Once one realizes this feature it becomes clear how rich and difficult the subject *dynamics of non-autonomous dynamical systems* really is.

This approach has been followed by Chepyzhov and Vishik in their seminal work on the forward dynamics of non-autonomous problems; see [98, Chapter VII]. Assuming the backwards uniqueness property for the driving semigroup, they consider the *uniform attractor* for a family of semiprocesses, indexed in the symbol space. They prove that the uniform attractor can be characterized as the union of the kernel sections at zero corresponding to all the elements in the global attractor of the driving semigroup, for which the driving is naturally defined, in a unique way, for negative times. It occurs, very often and in a natural way, that backwards uniqueness does not hold in the attractor of the driving semigroup. Having that in mind, we present the theory *avoiding the requirement of backwards uniqueness* for the driving semigroup. In fact, we show that nonuniqueness is what naturally occurs.

One can already see the difficulties that arise when dealing with non-autonomous problems by looking at simple concepts, such as hyperbolicity for instance. Indeed, in the autonomous case we choose an equilibrium (fixed point of the solution operator), linearize the solution operator around it and compute the spectrum of the linearized operator to decide if it is hyperbolic or not. In the non-autonomous context we have no way to identify which solutions play the special role of equilibria and, even if we were able to single out these solutions, we would not know how to verify if they are hyperbolic or not. Even the simple matter of obtaining a hyperbolic global solution as a small non-autonomous perturbation of a hyperbolic equilibrium involves highly nontrivial results on the robustness of exponential dichotomies (see [73] and [75]).

In what follows, we introduce a general method that provides a way to study a non-autonomous differential equation, and explain some important features from autonomous equations which are often overlooked. The method consists in considering the family of nonlinearities as a base flow driven by the time shift operator acting on the closure of $\{f(t + \cdot, \cdot) : t \in \mathbb{R}^+\}$ where nonlinearity $f(\cdot, \cdot)$ is the nonlinearity occurring in the original equation (see [193] and [194]).

To ensure the proper understanding of the framework, we will continue speaking about the problem (6.1), and leave the more general abstract framework to the next section. We consider \mathcal{C} the set of continuous functions $f: \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for each bounded subset $B \subset \mathbb{R}^n$ there exist a constant $L_f(B) > 0$ and an increasing continuous function $w_{f,B}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $w_{f,B}(0) = 0$ such that

$$(6.2) \quad \|f(t_1, x_1) - f(t_2, x_2)\| \leq w_{f,B}(|t_1 - t_2|) + L_f(B)\|x_1 - x_2\|,$$

for $t_1, t_2 \in \mathbb{R}^+$ and $x_1, x_2 \in B$.

In \mathcal{C} we consider the metric ρ of the uniform convergence in compact subsets of $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose, in addition, that f is *dissipative*, that is, there exist $M > 0$ and $\delta > 0$ such that

$$(6.3) \quad \langle f(t, x), x \rangle \leq -\delta \quad \text{for } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n \text{ with } \|x\| > M,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n .

Denote by Σ the closure, with respect to ρ , of the set of all forward translates of f , that is,

$$\Sigma(f) = \overline{\{f(s + \cdot, \cdot) : s \in \mathbb{R}^+\}}^\rho,$$

known as the **positive hull** of the function f in the metric space (\mathcal{C}, ρ) (see [98], [193] and [194] for more details), and define the *shift operator* $\Theta(t): \Sigma(f) \rightarrow \Sigma(f)$ by

$$\Theta(t)f(\cdot, \cdot) = f(t + \cdot, \cdot).$$

It is simple to show that $\Theta = \{\Theta(t): t \in \mathbb{R}^+\}$ defines a semigroup in $\Sigma(f)$, which we usually denote simply by Θ , called the **driving semigroup**. Also, it follows from the Arzelà-Ascoli Theorem that $\Sigma(f)$ is compact and Θ has a global attractor \mathcal{S} in $\Sigma(f)$.

Thus, to treat the asymptotics of a non-autonomous differential equation, we consider a combination of a base flow Θ on $\Sigma(f)$ and, for each $\sigma \in \Sigma(f)$, the semiflow $\mathbb{R}^+ \times \mathbb{R}^n \ni (t, u_0) \mapsto \mathcal{K}(t, \sigma)u_0 \in \mathbb{R}^n$ where, for each $u_0 \in \mathbb{R}^n$, the map $\mathbb{R}^+ \ni t \mapsto \mathcal{K}(t, \sigma)u_0 \in \mathbb{R}^n$ is the solution of the initial value problem

$$(6.4) \quad \begin{cases} \dot{u} = \sigma(t, u), & t > 0, \\ u(0) = u_0 \in \mathbb{R}^n. \end{cases}$$

Note that the conditions imposed on f ensure that the solutions of (6.4) are defined for all $t \geq 0$. Also the map $\mathbb{R}^+ \times \Sigma(f) \ni (t, \sigma) \mapsto \mathcal{K}(t, \sigma) \in \mathcal{C}(\mathbb{R}^n)$ satisfies

- (i) $\mathcal{K}(0, \sigma)x = x$ for all $x \in \mathbb{R}^n$ and $\sigma \in \Sigma(f)$;
- (ii) $\mathcal{K}(t + s, \sigma) = \mathcal{K}(t, \Theta(s)\sigma)\mathcal{K}(s, \sigma)$ for all $t \geq s$ and $\sigma \in \Sigma(f)$;
- (iii) the map $\mathbb{R}^+ \times \Sigma(f) \times \mathbb{R}^n \ni (t, \sigma, x) \mapsto \mathcal{K}(t, \sigma)x \in \mathbb{R}^n$ is continuous.

The second property is commonly known as the *cocycle property* and, due to this name, the map \mathcal{K} is called *cocycle semiflow*, or simply *cocycle*. These properties define a cocycle and the association with the differential equation (6.4) is no longer needed. In any case, one interprets $\mathcal{K}(t, \sigma)u_0$ as the solution at time t that has started in the state u_0 at time zero subject to the non-autonomous driving term $\sigma \in \Sigma(f)$.

Now we can talk about *evolution processes* and *pullback attractors*. If $\eta: \mathbb{R} \rightarrow \Sigma(f)$ is a global solution for the semigroup Θ , then the family $\mathcal{S}_\eta = \{S_\eta(t, s): t \geq s\} \subset \mathcal{C}(\mathbb{R}^n)$ defined by

$$S_\eta(t, s) = \mathcal{K}(t - s, \eta(s)) \quad \text{for } t \geq s,$$

defines an evolution process. In fact, for $t \geq s \geq r$, using the cocycle property, we have

$$S_\eta(t, s)S_\eta(s, r) = \mathcal{K}(t - s, \Theta(s - r)\eta(r))\mathcal{K}(s - r, \eta(r)) = \mathcal{K}(t - r, \eta(r)) = S_\eta(t, r),$$

and all other properties of an evolution process are satisfied, using the properties of the driving semigroup Θ and the cocycle \mathcal{K} . It is not difficult to show (we will see this later in the chapter) that, under the conditions imposed on f , \mathcal{S}_η has a pullback attractor $\{A_\eta(t)\}_{t \in \mathbb{R}}$.

REMARK 6.1. Assume that $\mathbb{K} = \mathbb{R}^+$ and also that f is independent of time. Then $\Sigma(f) = \{f\}$ and $\mathcal{S} = \{f\}$. From the Θ -invariance of \mathcal{S} , there is a global solution $\eta: \mathbb{R} \rightarrow \{f\}$ and we can define

$$S_\eta(t + s, s) = \mathcal{K}(t, \eta(s)) = \mathcal{K}(t, f),$$

for all $t \geq 0$ and $s \in \mathbb{R}$. Therefore at ∞ the vector field f can indeed be defined for all $t \in \mathbb{R}$. Readers should know that this remark really deserves deep reflection.

Together, \mathcal{K} and Θ determine the non-autonomous problem (6.1), and the pair $(\mathcal{K}, \Theta)_{(\mathbb{R}^n, \Sigma(f))}$ will be called a *non-autonomous dynamical system* on $(\mathbb{R}^n, \Sigma(f))$, and when there is no confusion, it will be simply denoted by the cocycle \mathcal{K} . With this object one can also define an associated semigroup (see [98], [194] and [193]) $\Pi = \{\Pi(t) : t \geq 0\}$ on $\mathbb{X} = \mathbb{R}^n \times \Sigma(f)$, with the metric $d_{\mathbb{X}}((x_1, \sigma_1), (x_2, \sigma_2)) = \|x_1 - x_2\| + \rho(\sigma_1, \sigma_2)$, by setting

$$\Pi(t)(x, \sigma) = (\mathcal{K}(t, \sigma)x, \Theta(t)\sigma), \text{ for } t \geq 0 \text{ and } (x, \sigma) \in \mathbb{X},$$

and the properties of Θ and \mathcal{K} ensure that Π satisfies all the properties of a semigroup. This semigroup Π is called the *skew-product semiflow* associated with \mathcal{K} . Also, the set $\{(x, \sigma) \in \mathbb{X} : \|x\| \leq M, \sigma \in \mathcal{S}\}$ is a compact subset of \mathbb{X} which attracts bounded subsets of \mathbb{X} . It follows that Π has a global attractor \mathbb{A} in \mathbb{X} .

We will see later that in fact we have

$$(6.5) \quad \mathbb{A} = \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_{\eta}(t) \times \{\eta(t)\}$$

where the union is taken over all global solutions η of Θ in \mathcal{S} .

The asymptotic set of states, for $t \rightarrow \infty$, of the solutions of (6.1) is \mathcal{A} = the projection on the first coordinate of \mathbb{A} , and it is called the *uniform attractor* for (6.1), in the terminology of Chepyzhov and Vishik [98]. Thus, if we only look at the uniform attractor we may lose perspective of what is indeed the asymptotic dynamics of (6.1). That is, if $\sigma \in \mathcal{S}$, a global solution $\xi : \mathbb{R} \rightarrow \mathcal{S}$ of Θ with $\xi(0) = \sigma$ corresponds to a natural extension (given by the evolution of (6.1)) of σ for negative times, that is, we can consider the extension $\sigma(s + t, x) = \xi(s)(t, x)$, for all $t \geq 0$ and $s \in \mathbb{R}$. Of course if $s \geq 0$ we have $\xi(s)(t, x) = \Theta(s)\xi(0)(t, x) = \sigma(t + s, x)$, for all $t \geq 0$.

Another important remark here is that in general Θ is not injective in \mathcal{S} , and we give a simple example to settle this once and for all.

EXAMPLE 6.2. Consider the function $f(t, x) = \alpha(t)g(x)$ for $t \geq 0$ and $x \in \mathbb{R}^n$, where $g \in \mathcal{C}$ satisfies (6.2) and (6.3), and $\alpha : \mathbb{R}^+ \rightarrow [1, 2]$ a globally Lipschitz function with $\alpha(t) = 1$ if $t \in ((2n)^2, (2n + 1)^2)$, $\alpha(t) = 2$ if $t \in ((2n + 1)^2 + 1, (2n + 2)^2 - 1)$, $n \in \mathbb{N}$, and linear elsewhere.

It is clear that $f \in \mathcal{C}$ and it also satisfies (6.2) and (6.3). For this such f , the function $\sigma(t, x) = 2g(x)$, for $t \geq 0$, is in \mathcal{S} and there are infinitely many global solutions $\eta : \mathbb{R} \rightarrow \mathcal{S}$ of Θ such that $\eta(0) = \sigma$. In fact, considering $t_n^1 = \frac{(2n+1)^2 + (2n+2)^2}{2}$ and $\eta_n^1(\cdot) = \Theta(\cdot + t_n^1)f : [-t_n^1, \infty) \rightarrow \Sigma(f)$, $\{\eta_n^1\}$ converges in the metric ρ , passing to a subsequence if needed, to $\eta_1(t) = 2g$ for all $t \in \mathbb{R}$, whereas if $t_n^2 = (2n + 1)^2 + 1$ and $\eta_n^2(\cdot) = \Theta(\cdot + t_n^2)f : [-t_n^2, \infty) \rightarrow \Sigma(f)$, $\{\eta_n^2\}$ converges in the metric ρ , passing to a subsequence if needed, to $\eta_2(t) = 2g$ for $t \geq 0$ and $\eta_2(t) = g$ if $t \leq -1$.

Clearly $\eta_1(0) = \eta_2(0) = 2g$, which already gives nonuniqueness, but one can repeat this reasoning and construct, for the same example, infinitely many global solutions of Θ through $\sigma = 2g \in \mathcal{S}$.

Having settled this, *we will not assume backwards uniqueness* for Θ in $\Sigma(f)$, and thus, given a non-autonomous differential equation such as (6.1), we need to deal with *four* different, but closely related, evolution operators:

- (a) the *semigroup* Θ on $\Sigma(f)$, which is associated to the dynamics of the time-dependent nonlinearities appearing in the equation;

- (b) the *non-autonomous dynamical system* $(\mathcal{K}, \Theta)_{(\mathbb{R}^n, \Sigma(f))}$ or the *cocycle* \mathcal{K} ;
- (c) the *skew-product semiflow* Π defined on the product space \mathbb{X} ;

and lastly, for each global solution $\eta: \mathbb{R} \rightarrow \Sigma(f)$ of Θ , the *evolution process* \mathcal{S}_η given by

$$(d) \quad \mathcal{S}_\eta(t, s)x = \mathcal{K}(t - s, \eta(s))x \text{ for all } t \geq s \text{ and } x \in \mathbb{R}^n.$$

To each one of the objects described above, we will present a corresponding notion of *attractor*, and each one of them will play its part in itemizing the asymptotic dynamics of (6.1). They are the following:

- (a) the *global attractor* \mathcal{S} for Θ in $\Sigma(f)$;
- (b) the *uniform attractor* \mathcal{A} for \mathcal{K} in \mathbb{R}^n ;
- (c) the *global attractor* \mathbb{A} for Π in \mathbb{X} ;
- (d) for each global solution $\eta: \mathbb{R} \rightarrow \Sigma(f)$ of Θ , the *pullback attractor* $\{A_\eta(t)\}_{t \in \mathbb{R}}$ of the evolution process \mathcal{S}_η .

REMARK 6.3. There is another notion of attractor, widely used in the literature and commonly associated to the non-autonomous dynamical systems $(\mathcal{K}, \Theta)_{X, \Sigma}$, called the **cocycle attractor** (see [153], for instance), and given by a family $\{A(\sigma)\}_{\sigma \in \mathcal{S}}$. We have chosen not to present it here, since its introduction would require the assumption that through each σ in the global attractor \mathcal{S} of $\Theta(t)$ passes a *unique global solution* $\eta: \mathbb{R} \rightarrow \Sigma(f)$.

In fact, if ξ and η are two distinct global solutions through σ , the cocycle attractor $A(\sigma)$ should coincide with the pullback attractor at time zero $A_\xi(0)$ of the evolution process

$$\mathcal{S}_\xi(t, s) = \mathcal{K}(t - s, \xi(s)),$$

which also coincides with the pullback attractor at time zero $A_\eta(0)$ of the evolution process

$$\mathcal{S}_\eta(t, s) = \mathcal{K}(t - s, \eta(s)),$$

and that will not happen, in general.

In the next section we present the precise definitions of all these objects, as well as conditions ensuring their existence in each case. Moreover, we present the relationship between pullback and uniform attractors, which leads to a detailed description of the uniform attractor and provides further understanding of its dynamical structures (associated to limiting evolution processes) and allows us to talk about upper and lower semicontinuity, and topological and geometrical structural stability, at least in the case of a suitable non-autonomous perturbation of a semigroup.

1. Pullback, uniform and skew-product attractors

Motivated by the discussion carried out in the previous section, we establish an abstract framework to work on. In order to study the asymptotics of a *non-autonomous problem* we need to consider the following families of operators: the driving semigroup and the associated cocycle, the skew-product semiflow, and the evolution processes obtained from the cocycle considering global solutions of the driving semigroup.

DEFINITION 6.4. Let (X, d_X) and (Σ, d_Σ) be metric spaces. Consider $\Theta = \{\Theta(t): t \geq 0\} \subset C(\Sigma)$ a semigroup (in the sense of Definition 1.1) in Σ , called

the **driving semigroup**. A **cocycle** relative to Θ is a family $\mathcal{K} = \{\mathcal{K}(t, \sigma) \in C(X) : t \geq 0, \sigma \in \Sigma\}$ which satisfies

- (i) $\mathcal{K}(0, \sigma)x = x$, for all $x \in X$ and $\sigma \in \Sigma$;
- (ii) $\mathcal{K}(t + s, \sigma) = \mathcal{K}(t, \Theta(s)\sigma)\mathcal{K}(s, \sigma)$, for all $t, s \geq 0$ and $\sigma \in \Sigma$;
- (iii) the map $\mathbb{R}^+ \times \Sigma \times X \ni (t, \sigma, x) \mapsto \mathcal{K}(t, \sigma)x \in X$ is continuous.

Property (ii) is called the **cocycle property**. Define in $\mathbb{X} = X \times \Sigma$, with the product metric, the **skew-product semiflow** or **skew-product semigroup** $\Pi = \{\Pi(t) : t \geq 0\} \subset C(\mathbb{X})$ defined by

$$\Pi(t)(x, \sigma) = (\mathcal{K}(t, \sigma)x, \Theta(t)\sigma) \quad \text{for } t \geq 0 \text{ and } (x, \sigma) \in \mathbb{X}.$$

Using the semigroup property of Θ and the cocycle property of \mathcal{K} we have

$$\begin{aligned} \Pi(t + s)(x, \sigma) &= (\mathcal{K}(t + s, \sigma)x, \Theta(t + s)\sigma) \\ &= (\mathcal{K}(t, \Theta(s)\sigma)\mathcal{K}(s, \sigma)x, \Theta(t)\Theta(s)\sigma) = \Pi(t)\Pi(s)(x, \sigma), \end{aligned}$$

and since the other properties of Definition 1.1 of a semigroup are clearly satisfied, Π is in fact a semigroup in \mathbb{X} .

When Σ is a unitary set, say $\Sigma = \{\sigma_0\}$, we have $\Theta(t)\sigma_0 = \sigma_0$ for all $t \geq 0$, and defining

$$(6.6) \quad T(t)x = \mathcal{K}(t, \sigma_0)x, \text{ for all } t \geq 0 \text{ and } x \in X,$$

the skew-product semigroup Π is reduced to $\Pi(t)(x, \sigma_0) = (T(t)x, \sigma_0)$ for $t \geq 0$ and $\mathcal{T} = \{T(t) : t \geq 0\}$ defines a semigroup in X . This is the situation that arises when we consider an autonomous differential equation. For a non-autonomous differential equation, Σ will not be a unitary set.

Let us now derive conditions for the existence of an attractor for the skew-product semigroup Π in \mathbb{X} . Recall that a semigroup has a global attractor if and only if it is asymptotically compact and bounded dissipative; see Theorem 1.23.

Define $\pi_X : \mathbb{X} \rightarrow X$ as the usual projection in the first coordinate, that is, $\pi_X(x, \sigma) = x$ for each $(x, \sigma) \in \mathbb{X}$. Thus we obtain

$$(6.7) \quad \pi_X(\Pi(t)(x, \sigma)) = \mathcal{K}(t, \sigma)x, \text{ for all } t \geq 0 \text{ and } (x, \sigma) \in \mathbb{X},$$

and defining $\pi_\Sigma : \mathbb{X} \rightarrow \Sigma$ as the usual projection in the second coordinate, that is, $\pi_\Sigma(x, \sigma) = \sigma$ for each $(x, \sigma) \in \mathbb{X}$, we obtain

$$(6.8) \quad \pi_\Sigma(\Pi(t)(x, \sigma)) = \Theta(t)\sigma, \text{ for all } t \geq 0 \text{ and } (x, \sigma) \in \mathbb{X}.$$

The bounded dissipativity property of Π is equivalent to saying that there are bounded subsets B_0 in X and S_0 in Σ such that, for bounded subsets $B \subset X$ and $S \subset \Sigma$ there exists $t_0 = t_0(B, S) \geq 0$ such that $\Pi(t)(B \times S) \subset B_0 \times S_0$ for all $t \geq t_0$. This implies that for any bounded subset $S \subset \Sigma$ there exists $t_0 = t_0(S) \geq 0$ such that $\Theta(t)S \subset S_0$ for all $t \geq t_0$, and also that for bounded subsets $B \subset X$ and $S \subset \Sigma$ there exists $t_0 = t_0(B, S) \geq 0$ such that $\mathcal{K}(t, \sigma)B \subset B_0$, for all $t \geq t_0$ and $\sigma \in S$.

The asymptotic compactness property of Π is equivalent to saying that given a sequence $\{t_n\}$ in \mathbb{R}^+ such that $t_n \rightarrow \infty$ and a bounded sequence $\{(x_n, \sigma_n)\}$ in \mathbb{X} , $\{\Pi(t_n)(x_n, \sigma_n)\}$ has a convergent subsequence in \mathbb{X} , which is equivalent to saying that given a sequence $\{t_n\}$ in \mathbb{R}^+ such that $t_n \rightarrow \infty$, and bounded sequences $\{x_n\}$ in X and $\{\sigma_n\}$ in Σ we have

- $\{\Theta(t_n)\sigma_n\}$ has a convergent subsequence in Σ and
- $\{\mathcal{K}(t_n, \sigma_n)x_n\}$ has a convergent subsequence in X .

This reasoning proves the following result on the characterization of the skew-product semigroups that possess a global attractor.

THEOREM 6.5. *Let (X, d_X) , (Σ, d_Σ) be metric spaces and $\mathbb{X} = X \times \Sigma$ with the product metric. Consider Θ a semigroup in Σ and \mathcal{K} a cocycle relative to Θ . If Π is the associated skew-product semigroup, then it has a global attractor \mathbb{A} in \mathbb{X} if and only if*

- (i) Θ has a global attractor \mathcal{S} in Σ ;
- (ii) \mathcal{K} is **bounded dissipative**, that is, there is a bounded subset B_0 of X such that, for bounded subsets $B \subset X$ and $S \subset \Sigma$, there exists $t_0 = t_0(B, S) \geq 0$ such that $\mathcal{K}(t, \sigma)B \subset B_0$, for all $t \geq t_0$ and $\sigma \in S$;
- (iii) \mathcal{K} is **asymptotically compact**, that is, given a sequence $\{t_n\}$ in \mathbb{R}^+ with $t_n \xrightarrow{n \rightarrow \infty} \infty$, and bounded sequences $\{x_n\}$ in X and $\{\sigma_n\}$ in Σ , the sequence $\{\mathcal{K}(t_n, \sigma_n)x_n\}$ has a convergent subsequence in X .

If we now return to our example (6.1), it is easy to see that any function in $\Sigma(f)$ must satisfy (6.2) and (6.3). Now if $\sigma \in \Sigma(f)$ and $u_0 \in \mathbb{R}^n$, the solution $\mathcal{K}(\cdot, \sigma)u_0$ of (6.4) must satisfy

$$\frac{d}{dt} \|\mathcal{K}(t, \sigma)u_0\|^2 = 2\langle \mathcal{K}(t, \sigma)u_0, \sigma(t, \mathcal{K}(t, \sigma)u_0) \rangle \leq -2\delta$$

and consequently, integrating both sides from 0 to t , we obtain $\|\mathcal{K}(t, \sigma)u_0\|^2 \leq \|u_0\|^2 - 2\delta t$, for all $t \geq 0$, such that $\|\mathcal{K}(t, \sigma)u_0\| > M$. Using this fact, we conclude that the solutions of (6.4) exist for all $t \geq 0$. Now if $B_M = \{u_0 \in \mathbb{R}^n : \|u_0\| \leq M\}$, then given a bounded subset B of \mathbb{R}^n we have

$$\sup_{\sigma \in \Sigma(f)} \text{dist}_H(\mathcal{K}(t, \sigma)B, B_M) \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

and since we have already seen that Θ has a global attractor in $\Sigma = \Sigma(f)$, all the conditions of Theorem 6.5 are satisfied.

Another important family of operators that naturally appears in the study of non-autonomous problems are the *evolution processes*, which we define next.

DEFINITION 6.6. A family $\mathcal{S} = \{S(t, s) : t \geq s\} \subset C(X)$ is called an **evolution process** in X if it satisfies:

- (i) $S(t, t)x = x$ for all $t \in \mathbb{R}$ and $x \in X$;
- (ii) $S(t, s) = S(t, r)S(r, s)$ for all $t \geq r \geq s$;
- (iii) the map $\mathcal{P} \times X \ni (t, s, x) \mapsto S(t, s)x \in X$ is continuous, where $\mathcal{P} = \{(t, s) \in \mathbb{R}^2 : t \geq s\}$.

In fact if $\eta: \mathbb{R} \rightarrow \Sigma$ is a global solution for Θ , the family $\mathcal{S}_\eta = \{S_\eta(t, s) : t \geq s\}$ defined by

$$S_\eta(t, s) = \mathcal{K}(t - s, \eta(s)) \quad \text{for } t \geq s,$$

is an evolution process in X .

We say that an evolution process $\mathcal{S} = \{S(t, s) : t \geq s\}$ is **autonomous** if the evolution depends only on the elapsed time $t - s$ rather than on t and s explicitly, that is, if $S(t, s) = S(t - s, 0)$ for all $t \geq s$. If an evolution process is not autonomous it is said to be **non-autonomous** or simply an **evolution process**. When $t, s \in \mathbb{Z}$ instead of \mathbb{R} , we say that $\mathcal{S} = \{S(t, s) : t \geq s \in \mathbb{Z}\}$ is a **discrete evolution process**.

Let us now define the notion of *invariance* for an evolution process \mathcal{S} . Such a notion is fundamentally connected to the possibility of constructing global solutions. In the autonomous case, a set is invariant if it is fixed by the evolution process and that notion is sufficient to study the asymptotic behavior. However, in the non-autonomous case, it may happen that no set is fixed by the evolution process. Note that simple objects which are very helpful in the description of the asymptotic behavior, like equilibria, may not exist. The notion of invariance for non-autonomous evolution processes which seems most suitable is for a *family of sets*, which is the following.

DEFINITION 6.7. Let X be a metric space and let \mathcal{S} be an evolution process in X . A family of sets $\{B(t)\}_{t \in \mathbb{R}}$ in X is said to be **\mathcal{S} -invariant** if $S(t, s)B(s) = B(t)$ for all $t \geq s$.

This notion does not reduce to the notion of invariance for an autonomous evolution process, but it has been shown to be suitable for the purpose of studying non-autonomous problems. A thorough discussion can be seen in [78].

For an autonomous evolution process, the asymptotic behavior can be studied making $t \rightarrow \infty$, called the *forward dynamics*, or fixing t and making $s \rightarrow -\infty$, called the *pullback dynamics*. In the non-autonomous case, these two dynamic regimes give rise to completely different scenarios. In what follows we will pursue the second scenario, defining *pullback attraction* for an evolution process.

DEFINITION 6.8. Let X be a metric space and let \mathcal{S} be an evolution process in X . We say that a set B_0 **\mathcal{S} -pullback attracts** a set B , at time t if

$$\lim_{s \rightarrow -\infty} \text{dist}_H(S(t, s)B, B_0) = 0.$$

With these notions of invariance and pullback attraction, we can define the notion of *pullback attractor* for an evolution process, but here we call the readers' attention to the fact that there are several distinct possible definitions of a pullback attractor in the literature. We choose to present the following.

DEFINITION 6.9. Let X be a metric space and let \mathcal{S} be an evolution process in X . A family $\{A(t)\}_{t \in \mathbb{R}}$ is called a **pullback attractor** for \mathcal{S} if $\bigcup_{t \in \mathbb{R}} A(t)$ is bounded, $A(t)$ is compact in X for each $t \in \mathbb{R}$, and $\{A(t)\}_{t \in \mathbb{R}}$ is \mathcal{S} -invariant and \mathcal{S} -pullback attracts bounded subsets of X at time t , for each $t \in \mathbb{R}$.

The pullback attractor, with this definition, is unique. In fact if $\{\tilde{A}(t)\}_{t \in \mathbb{R}}$ and $\{A(t)\}_{t \in \mathbb{R}}$ are pullback attractors for \mathcal{S} , since $B = \bigcup_{t \in \mathbb{R}} \tilde{A}(t)$ is bounded and $\{A(t)\}_{t \in \mathbb{R}}$ pullback attracts bounded subsets of X at time t , the \mathcal{S} -invariance property of pullback attractors implies that

$$\text{dist}_H(\tilde{A}(t), A(t)) = \text{dist}_H(S(t, s)\tilde{A}(s), A(t)) \leq \text{dist}_H(S(t, s)B, A(t)) \xrightarrow{s \rightarrow -\infty} 0.$$

This implies that $\tilde{A}(t) \subset A(t)$, for each $t \in \mathbb{R}$. The other inclusion follows in the same way, interchanging $\tilde{A}(t)$ and $A(t)$.

There is a vast literature on pullback attractors with results that characterize the evolution processes that possess pullback attractors. The interested reader can consult, for example, [78] and [153]. In the context of this book, the conditions for existence of pullback attractors will be drawn from the conditions for existence of a global attractor for the skew-product semigroup.

Now, as in the autonomous case, using *bounded global solutions*, one can characterize a pullback attractor, when it exists.

DEFINITION 6.10. Let X be a metric space and let \mathcal{S} be an evolution process in X . We say that a function $\xi: \mathbb{R} \rightarrow X$ is a **global solution for \mathcal{S}** if $S(t, s)\xi(s) = \xi(t)$ for all $t \geq s$, that is, if the family $\{\xi(t)\}_{t \in \mathbb{R}}$ is \mathcal{S} -invariant. If $\xi(\mathbb{R})$ is bounded in X , we say that ξ is a **bounded global solution**.

Within that context we have the following result on the characterization of pullback attractors.

THEOREM 6.11. *Let X be a metric space and let \mathcal{S} be an evolution process in X . If \mathcal{S} has a pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$, then for all $t \in \mathbb{R}$ we have*

$$A(t) = \{\xi(t) : \xi \text{ is a bounded global solution of } \mathcal{S}\}.$$

PROOF. From the \mathcal{S} -invariance of $\{A(t)\}_{t \in \mathbb{R}}$, proceeding as in Theorem 1.9, for each $x \in A(t_0)$, we can construct a global solution ξ such that $\xi(t_0) = x$ and $\xi(t) \in A(t)$ for all $t \in \mathbb{R}$.

On the other hand, if ξ is a bounded global solution of \mathcal{S} , then, since $\{A(t)\}_{t \in \mathbb{R}}$ \mathcal{S} -pullback attracts $B = \overline{\xi(\mathbb{R})}$ at time t , for each $n \in \mathbb{N}^*$, there exists $t_n \leq t$ such that $\text{dist}_H(S(t, t_n)B, A(t)) \leq \frac{1}{n}$ for all $s \leq t_n$. In particular

$$\text{dist}_H(S(t, t_n)\xi(t_n), A(t)) = \text{dist}_H(\xi(t), A(t)) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}^*.$$

It follows that $\xi(t) \in A(t)$ and completes the proof. \square

DEFINITION 6.12. Let X be a metric space and \mathcal{S} an evolution process in X . Given a subset $B \subset X$ and $t \in \mathbb{R}$, the **pullback ω -limit set** at time t of B is defined by

$$\omega(B, t) = \bigcap_{\sigma \leq t} \overline{\bigcup_{s \leq \sigma} S(t, s)B}.$$

Clearly $\omega(B, t)$ is closed for each $t \in \mathbb{R}$, and it is easy to see (cf. Proposition 1.13) that

$$(6.9) \quad \omega(B, t) = \{y \in X : \text{there are sequences } \{s_k\} \text{ in } (-\infty, t], s_k \rightarrow -\infty, \\ \text{and } \{x_k\} \text{ in } B, \text{ such that } y = \lim_{k \rightarrow \infty} S(t, s_k)x_k\}.$$

THEOREM 6.13. *Let X be a metric space and \mathcal{S} an evolution process in X . Assume that there exists a compact subset K of X such that K \mathcal{S} -pullback attracts bounded subsets of X at time t , for each $t \in \mathbb{R}$. Then, \mathcal{S} has a bounded pullback attractor $\{A(t) = \omega(K, t)\}_{t \in \mathbb{R}}$ and $\overline{\bigcup_{t \in \mathbb{R}} A(t)}$ is a compact subset of X .*

PROOF. Let us first show that $\omega(K, t)$ is nonempty. Given sequences $\{s_k\}$ in $(-\infty, t]$, with $s_k \rightarrow -\infty$, and $\{x_k\}$ in K , since K is compact and \mathcal{S} -pullback attracts itself at time t , $\{S(t, s_k)x_k\}$ must have a convergent subsequence, and consequently $\omega(K, t)$ is nonempty.

Clearly $\omega(K, t) \subset K$ for all $t \in \mathbb{R}$, since K \mathcal{S} -pullback attracts itself at time t , and $\overline{\bigcup_{t \in \mathbb{R}} \omega(K, t)} \subset K$. Since $\omega(K, t)$ is closed, we conclude that $\omega(K, t)$ is compact for all $t \in \mathbb{R}$ and that $\overline{\bigcup_{t \in \mathbb{R}} \omega(K, t)}$ is compact.

We now prove the \mathcal{S} -invariance of $\{\omega(K, t)\}_{t \in \mathbb{R}}$. If $t \geq s$ and $x \in \omega(K, t)$, there are sequences $\{s_k\}$ in $(-\infty, t]$, with $s_k \rightarrow -\infty$, and $\{x_k\}$ in K such that $S(t, s_k)x_k \rightarrow x$. Since $s_k \rightarrow -\infty$, we may assume that $s_k \leq s$ for all $k \in \mathbb{N}$. Now since K \mathcal{S} -pullback attracts itself at time s , $\{S(s, s_k)x_k\}$ has a convergent subsequence (which we denote the same) with limit, say, y . Clearly, $y \in \omega(K, s)$ and

from the continuity of $S(t, s)$ we have $S(t, s)y = x$. It follows that $S(t, s)\omega(K, s) \supset \omega(K, t)$. The proof of the other inclusion is simpler and it is left to the reader.

Next we prove that $\omega(K, t)$ \mathcal{S} -pullback attracts bounded subsets of X at time t . If there is a bounded subset B of X that is not \mathcal{S} -pullback attracted by $\omega(K, t)$ at time t , then there would exist $\epsilon > 0$ and sequences $\{s_k\}$ in $(-\infty, t]$, with $s_k \rightarrow -\infty$, and $\{x_k\}$ in B such that $\text{dist}_H(S(t, s_k)x_k, \omega(K, t)) \geq \epsilon$ for all k . Since K is compact and \mathcal{S} -pullback attracts B , $\{S(t, s_k)x_k\}$ must have a convergent subsequence with limit in $\omega(K, t)$. That contradiction proves that $\omega(K, t)$ \mathcal{S} -pullback attracts bounded subsets of X , and the proof is complete. \square

The next corollary follows directly from Theorems 6.5 and 6.13.

COROLLARY 6.14. *Let (X, d_X) , (Σ, d_Σ) be metric spaces and consider $\mathbb{X} = X \times \Sigma$ with the product metric. Consider also a semigroup Θ in Σ and a cocycle \mathcal{K} relative to Θ . If the associated skew-product semigroup Π has a global attractor \mathbb{A} in \mathbb{X} , then for any bounded global solution $\eta: \mathbb{R} \rightarrow \Sigma$ of Θ , the evolution process $\mathcal{S}_\eta = \{S_\eta(t, s) = \mathcal{K}(t - s, \eta(s)): t \geq s\}$ has a pullback attractor $\{A_\eta(t)\}_{t \in \mathbb{R}}$ and $\bigcup_{t \in \mathbb{R}} A_\eta(t) \subset \pi_X(\mathbb{A})$.*

We can now characterize the global attractor of the skew-product semigroup in terms of the pullback attractors of evolution processes associated to bounded global solutions of the driving semigroup.

THEOREM 6.15. *Let (X, d_X) , (Σ, d_Σ) be metric spaces and consider $\mathbb{X} = X \times \Sigma$ with the product metric. Consider also a semigroup Θ in Σ and a cocycle \mathcal{K} relative to Θ . If the associated skew-product semigroup Π has a global attractor \mathbb{A} in \mathbb{X} , then the driving semigroup Θ has a global attractor \mathcal{S} in Σ . If $\eta: \mathbb{R} \rightarrow \mathcal{S}$ is a bounded global solution for Θ , then the evolution process $\mathcal{S}_\eta = \{S_\eta(t, s): t \geq s\}$, given by $S_\eta(t, s) = \mathcal{K}(t - s, \eta(s))$, possesses a pullback attractor $\{A_\eta(t)\}_{t \in \mathbb{R}}$ with the property that $A_\eta(t) = \{x \in X: (x, \eta(t)) \in \mathbb{A}\}$. Moreover,*

$$\mathbb{A} = \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{\eta(t)\},$$

where the first union is taken over all bounded global solutions η of Θ .

PROOF. Note that a global bounded solution for Π is a continuous function $(x(\cdot), \eta(\cdot)): \mathbb{R} \rightarrow \mathbb{A}$ that satisfies

$$(x(t + s), \eta(t + s)) = \Pi(t)(x(s), \eta(s)) = (\mathcal{K}(t, \eta(s))x(s), \Theta(t)\eta(s)),$$

for all $t \geq 0$ and $s \in \mathbb{R}$. Thus $x(t + s) = \mathcal{K}(t, \eta(s))x(s) = S_\eta(t + s, s)x(s)$ and $\Theta(t)\eta(s) = \eta(t + s)$, for all $t \geq 0$ and $s \in \mathbb{R}$. It follows that $x: \mathbb{R} \rightarrow X$ is a bounded global solution for \mathcal{S}_η and η is a bounded global solution for Θ . Hence, any point of \mathbb{A} is of the form $(x(t), \eta(t))$, where $x(t) \in A_\eta(t)$ and $\eta: \mathbb{R} \rightarrow \Sigma$ is a bounded global solution for Θ , and therefore

$$\mathbb{A} \subset \bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_\eta(t) \times \{\eta(t)\},$$

where the first union is taken over all global bounded solutions η of Θ .

On the other hand, if $\eta: \mathbb{R} \rightarrow \mathcal{S}$ is a bounded global solution for Θ and $x_0 \in A_\eta(t_0)$, there exists a global solution $x: \mathbb{R} \rightarrow X$ of \mathcal{S}_η such that $x(t_0) = x_0$ and $x(t) \in A_\eta(t)$ for all $t \in \mathbb{R}$. Since $\bigcup_{t \in \mathbb{R}} A_\eta(t)$ is bounded, we have $x: \mathbb{R} \rightarrow X$ a bounded global solution of \mathcal{S}_η , and consequently $(x(\cdot), \eta(\cdot)): \mathbb{R} \rightarrow \mathbb{X}$ is a bounded

global solution of Π , showing that $(x(t), \eta(t)) \in \mathbb{A}$ for all $t \in \mathbb{R}$. Then, in particular, $(x_0, \eta(t_0)) \in \mathbb{A}$ and

$$\bigcup_{\eta} \bigcup_{t \in \mathbb{R}} A_{\eta}(t) \times \{\eta(t)\} \subset \mathbb{A},$$

where the first union is taken over all bounded global solutions η for Θ , which completes the proof. \square

An important object that appears in Theorem 6.15 is $\pi_X(\mathbb{A})$, which has the property of \mathcal{K} -attraction of bounded subsets of X , uniformly in bounded subsets of Σ . This object turns out to be, in our context, what is known in the literature as *uniform attractor* for the cocycle \mathcal{K} , which we explain next.

DEFINITION 6.16. Let (X, d_X) , (Σ, d_{Σ}) be metric spaces and $\mathbb{X} = X \times \Sigma$ with the product metric. Consider Θ a semigroup in Σ and \mathcal{K} a cocycle relative to Θ . The **uniform attractor** for the cocycle \mathcal{K} is the minimal closed subset \mathcal{A} of X such that for each bounded subset B of X and each bounded subset Γ of Σ we have

$$\lim_{t \rightarrow \infty} \sup_{\sigma \in \Gamma} \text{dist}_H(\mathcal{K}(t, \sigma)B, \mathcal{A}) = 0,$$

where the minimality condition means that if C is any closed subset of X such that

$$\lim_{t \rightarrow \infty} \sup_{\sigma \in \Gamma} \text{dist}_H(\mathcal{K}(t, \sigma)B, C) = 0,$$

for each bounded subset B of X and each bounded subset Γ of Σ , then we have $\mathcal{A} \subset C$.

From Theorem 6.5 the following result holds.

THEOREM 6.17. *Let (X, d_X) , (Σ, d_{Σ}) be metric spaces and $\mathbb{X} = X \times \Sigma$ with the product metric. Consider Θ a semigroup in Σ and \mathcal{K} a cocycle relative to Θ . If Θ has a global attractor \mathcal{S} in Σ and \mathcal{K} is bounded dissipative and asymptotically compact, then the cocycle \mathcal{K} has a compact uniform attractor \mathcal{A} , and if \mathbb{A} is the global attractor for the associated skew-product semigroup Π , then $\mathcal{A} = \pi_X(\mathbb{A})$. Furthermore*

$$\mathcal{A} = \bigcup_{\eta} A_{\eta}(0),$$

where the union is taken over all bounded global solutions η for Θ .

This result is a straightforward application of Theorem 6.5. Just note that if η is a bounded global solution for Θ , then so is $\eta_r(\cdot) = \eta(\cdot + r)$, for each $r \in \mathbb{R}$, and we can conclude that $A_{\eta_r}(0) = A_{\eta}(r)$.

Notes. The theories of pullback, cocycle, skew-product, and uniform attractors have been developed in the literature as more or less different subjects and frameworks. This is why we can find many papers with a single one of these approaches, independently of the others. Related to the *existence of attractors* we can find [41, 51, 78, 150] for pullback attractors, [89, 153] for cocycle attractors and [96–98, 206] for uniform attractors. Note that all the theories could have been written without any reference to cocycle attractors, but only to pullback, uniform and attractors for skew-product semigroups. A nice set of results has also proved the finiteness of the box-counting dimension of pullback attractors; see, for instance, [28, 78, 160]. The relationship among all of these concepts is newer, and can be found in [29, 30, 50, 50]. We can also find nice papers on applications of

the theory of pullback attractors to non-autonomous delay equations ([151, 169]), differential equations without uniqueness of solutions, or even the more general case of multivalued dynamical systems ([45, 49, 107, 205]). Applications to models of differential or partial differential equations systems (also considering delays or nonuniqueness of solutions) has also received a lot of attention in the last years, and, for instance, we can refer to [32, 46, 51, 99, 154, 154, 196, 199, 207] for applications to PDEs for the pullback attractor theory, or to [100, 117, 128, 165, 198] for the case of uniform attractors applied to different phenomena from Physics, Chemistry or Biology. Finally, some papers have focused on the relation and differences between pullback and forward behavior on invariant sets for non-autonomous evolution processes and cocycles ([46, 50, 56, 90, 149, 155, 162, 188]), a crucial insight for a correct understanding of the rich dynamics of this kind of systems. More recently, interesting papers have gone to the definition of exponential pullback attractors ([87, 110, 161]), and the description of attractors for non-autonomous equations with random terms ([53, 109, 203]). In all of these papers it is clear that the concept of pullback attractor can be generalized for families of sets which are unbounded, but in this case they are not related to the attractor of the associated skew-product semigroup (that may not exist in this scenario).

Non-autonomous Morse-Smale Dynamical Systems

This chapter is devoted to the proof that, under suitable conditions, a *small* non-autonomous perturbation of a Morse-Smale semigroup is an evolution process which has a pullback attractor with the same structure of connections. We will also show this property in the skew-product framework.

Consider X a metric space, $\Lambda = [0, 1]$ the space of parameters, and $\mathcal{S}_\lambda = \{S_\lambda(t, s) : (t, s) \in \mathcal{P}\}$ an evolution process in X for each $\lambda \in \Lambda$. In Theorem 8.14 we have proved that if \mathcal{S}_{λ_0} is a dynamically gradient semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ (that is, \mathcal{S}_{λ_0} is an autonomous evolution process and $T(t) = S_\lambda(t + \tau, \tau)$ for each $t \geq 0$ and $\tau \in \mathbb{R}$) with a global attractor \mathcal{A} and a disjoint collection of *unitary* isolated invariant sets (equilibria) $\mathbf{E} = \{e_1^*, \dots, e_n^*\}$, then the *small non-autonomous perturbation* \mathcal{S}_λ (of \mathcal{T}), which has a pullback attractor $\hat{A}_\lambda = \{A_\lambda(t)\}_{t \in \mathbb{R}}$ with a disjoint collection of *unitary* isolated invariant families (isolated global solutions) $\mathbf{E}_\lambda = \{\xi_{1,\lambda}^*, \dots, \xi_{n,\lambda}^*\}$, that approaches \mathbf{E} as $\lambda \rightarrow \lambda_0$ and satisfy a uniform *topological hyperbolicity property* (see condition (d) of Theorem 8.14), is a dynamically gradient evolution process.

This is saying that any global solution of \mathcal{S}_λ in the pullback attractor \hat{A}_λ connects (from $-\infty$ to ∞) two of the isolated global solutions of \mathbf{E}_λ , and that there are no homoclinic structures in \hat{A}_λ .

The aim of this chapter is to show that, assuming that \mathcal{T} is a Morse-Smale semigroup together with some additional technical conditions, the connections between equilibria in the global attractor \mathcal{A} of \mathcal{T} are also seen in the pullback attractor \hat{A}_λ of \mathcal{S}_λ .

The crucial additional condition is the *transversality of unstable and stable manifolds along a connection for the unperturbed problem*. This condition ensures that a connection existing between two equilibria of the limiting autonomous Morse-Smale semigroup \mathcal{T} will also appear, under smooth non-autonomous perturbation, in the associated nearby isolated global solutions in \mathbf{E}_λ of the perturbed evolution process \mathcal{S}_λ . Also, having connections between two global isolated solutions $\xi_{i,\lambda}^*$ and $\xi_{j,\lambda}^*$ of \mathbf{E}_λ for all λ suitably close to λ_0 ensures, according to Proposition 10.6, that there exists a sequence of connections between elements of \mathbf{E} starting at e_i^* and ending at e_j^* . Now, as a consequence of the λ -lemma, we have proved in Proposition 5.9 that there is a direct connection between e_i^* and e_j^* .

This chapter is dedicated to precisely state the conditions ensuring that the above reasoning holds, that is, that the diagram of connections between elements of \mathbf{E}_λ and \mathbf{E} is the same.

A Morse-Smale semigroup (see Definition 5.2) is a reversible semigroup \mathcal{T} with a global attractor \mathcal{A} , a finite set of hyperbolic fixed points and normally hyperbolic periodic solutions, all with finite-dimensional local unstable manifolds, and such

that the intersection between the local unstable manifold and the stable manifold of two different critical elements - fixed points or periodic orbits - is transversal.

In Section 1 (see Corollary 5.12) we have proved that a reversible semigroup, with finitely many equilibria (all hyperbolic) and finitely many periodic orbits (all normally hyperbolic) having unstable and stable manifolds that intersect transversally along connections, is Morse-Smale if and only if it is dynamically gradient.

The results of Section 1 are synthesised next for easy reference.

THEOREM 10.1. *Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a semigroup with a global attractor \mathcal{A} .*

(a) *If \mathcal{T} is a Morse-Smale semigroup with nonwandering set*

$$\mathbf{E} = \{e_1^*, \dots, e_n^*, \xi_1^*(\mathbb{R}), \dots, \xi_m^*(\mathbb{R})\},$$

where each e_i^ , $1 \leq i \leq n$, is a hyperbolic equilibrium point and $\xi_j^* : \mathbb{R} \rightarrow X$, $1 \leq j \leq m$, is a normally hyperbolic periodic orbit, then \mathbf{E} is a disjoint collection of isolated invariant sets and \mathcal{T} is a generalized dynamically gradient semigroup with respect to \mathbf{E} .*

(b) *If \mathcal{T} is a reversible dynamically gradient semigroup associated to the disjoint collection of isolated invariant sets*

$$\mathbf{E} = \{e_1^*, \dots, e_n^*, \xi_1^*(\mathbb{R}), \dots, \xi_m^*(\mathbb{R})\},$$

where each e_i^ , $1 \leq i \leq n$, is a hyperbolic equilibrium point and $\xi_j^* : \mathbb{R} \rightarrow X$, $1 \leq j \leq m$, is a normally hyperbolic periodic orbit, with $\dim W_{loc}^u(E) < \infty$ whenever $E \in \mathbf{E}$, and $W^u(E) \cap W_{loc}^s(\tilde{E}) = \emptyset$ whenever $E, \tilde{E} \in \mathbf{E}$, then \mathcal{T} is a Morse-Smale semigroup with nonwandering set \mathbf{E} .*

As we have mentioned before, we already know that a small non-autonomous perturbation of a dynamically gradient semigroup is a dynamically gradient evolution process (the so-called *topological structural stability*). With the concept of Morse-Smale semigroups at hand, we can go one step further and prove that *the structure of connections remains the same*, a property that here we will refer to as **geometrical structural stability**.

THEOREM 10.2. *Let $\Lambda = [0, 1]$ and let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a family of evolution processes satisfying the following conditions:*

- (a) *S_λ has a pullback attractor \hat{A}_λ and is reversible for each $\lambda \in \Lambda$;*
- (b) *$\overline{\bigcup_{\lambda \in \Lambda} \bigcup_{t \in \mathbb{R}} A_\lambda(t)}$ is compact;*
- (c) *for each compact set $K \subset \mathbb{R}^+ \times X$ we have*

$$\sup_{s \in \mathbb{R}} \sup_{(t,x) \in K} \|S_\lambda(t+s, s)x - S_{\lambda_0}(t+s, s)x\|_X \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0;$$

(d) *for each compact set $J \in \mathbb{R}^+$ we have*

$$\sup_{s \in \mathbb{R}} \sup_{t \in J} \sup_{z \in A(s)} \|S'_\lambda(t+s, s)(z) - S'_{\lambda_0}(t+s, s)(z)\|_{\mathcal{L}(X)} \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0;$$

(e) *$S_{\lambda_0}(t, s) = T(t-s)$ for all $t \geq s$, where $\mathcal{T} = \{T(t) : t \geq 0\}$ is a Morse-Smale semigroup in X , with global attractor \mathcal{A} and set of isolated stationary solutions $\mathbf{E} = \{e_1^*, \dots, e_n^*\}$.*

Then

(i) \mathcal{S}_λ has a disjoint set of isolated hyperbolic global solutions

$$\mathbf{E}_\lambda = \{\xi_{1,\lambda}^*, \dots, \xi_{n,\lambda}^*\} \quad \text{for each } \lambda \in \Lambda,$$

with

$$\lim_{\lambda \rightarrow \lambda_0} \max_{1 \leq i \leq n} \text{dist}_H(\xi_{i,\lambda}^*(\mathbb{R}), z_i^*) = 0;$$

- (ii) there exist $\delta > 0$ and $\epsilon_0 > 0$ such that, if $|\lambda - \lambda_0| < \epsilon_0$, $\xi_\lambda: \mathbb{R} \rightarrow X$ is a global solution of \mathcal{S}_λ in \hat{A}_λ , $t_0 \in \mathbb{R}$ and $\text{dist}_H(\xi_\lambda(t), \xi_{i,\lambda}^*(\mathbb{R})) \leq \delta$ for all $t \leq t_0$ ($t \geq t_0$), then $\text{dist}_H(\xi_\lambda(t), \xi_{i,\lambda}^*(t)) \rightarrow 0$ as $t \rightarrow \infty$ ($t \rightarrow -\infty$);
- (iii) there exists $\epsilon_1 \in (0, \epsilon_0)$ such that \mathcal{S}_λ is a Morse-Smale evolution process (see Definition 10.3) for all $|\lambda - \lambda_0| < \epsilon_1$ and there exists a phase diagram isomorphism between \mathcal{S}_λ and \mathcal{T} .

The rest of the chapter will be dedicated to giving precise meaning to the term *Morse-Smale evolution process*, to the proof of Theorem 10.2, and to present some applications. We also write the associated results for skew-product semigroups and cocycles and their respective attractors (global and uniform).

The approach of this chapter is taken from our work, together with Geneviève Raugel [31]. In a finite-dimensional framework, the structural stability of Morse-Smale maps under small time-dependent perturbations has been proved by Franks in [121], and by Kryzhevich and Pliss in [158]. The study of non-autonomous perturbations of a Morse-Smale semigroup in the infinite-dimensional case has been done in [31], where we show that if the nonwandering set of a Morse-Smale semigroup (having a compact global attractor) is reduced to a finite number of hyperbolic equilibria, then the resulting evolution process is still of Morse-Smale type (in a sense to be specified in Definition 10.3) and there is a phase diagram isomorphism (see Definition 10.4) between the dynamical systems defined by the limiting semigroup and the perturbed evolution processes (see Theorem 10.2).

1. Morse-Smale evolution processes

In this section, given a Morse-Smale semigroup, we explore the structure's behavior of its global attractor under a small non-autonomous perturbation. More precisely, we verify that the *phase diagrams* (see Definition 10.4) between the Morse-Smale semigroup and the perturbed non-autonomous evolution processes are *isomorphic*, that is, the connections and their orientations between the isolated invariant sets of the semigroup and the isolated invariant families of the perturbed non-autonomous evolution process are the same. We will restrict ourselves, however, to the case of a Morse-Smale semigroup with nonwandering set containing only hyperbolic equilibria, that is, without normally hyperbolic periodic orbits.

Next we ask the reader to recall the notions of *dynamically gradient evolution processes*, given in Definition 8.10 (see Chapter 8 - Section 3), and of *dynamically gradient semigroups*, given in Definition 3.12 (see Chapter 3 - Section 2). With these definitions, we know that a small non-autonomous perturbation of a dynamically gradient semigroup is a dynamically gradient evolution process, that is, for a dynamically gradient semigroup, conditions (G1) and (G2) of Definition 3.12 are stable under non-autonomous perturbations. This is what we called *topological structural stability*, and it was proved in Theorem 8.14.

Moreover, inspired by Theorem 8.14, we define *Morse-Smale evolution processes*.

DEFINITION 10.3. Let \mathcal{S} be an evolution process in a Banach space X with a pullback attractor \hat{A} . We say that \mathcal{S} is a **Morse-Smale evolution process** if:

- (i) \mathcal{S} is a dynamically gradient evolution process with respect to a disjoint collection of isolated hyperbolic global solutions $\mathbf{E} = \{\xi_1, \dots, \xi_n\}$;
- (ii) \mathcal{S} is a reversible evolution process, that is, $S(t, s)|_{A(s)}: A(s) \rightarrow A(t)$ is injective, $S(t, s)$ is differentiable in $A(s)$ and $S'(t, s)(z): X \rightarrow X$ is an injective bounded linear operator, for each $t \geq s \in \mathbb{R}$;
- (iii) $W_{loc}^u(\xi_i)$ is finite dimensional, for all $i = 1, \dots, n$;
- (iv) given $\xi_i, \xi_j \in \mathbf{E}$ such that $W^u(\xi_i)(t_0) \cap W_{loc}^s(\xi_j)(t_0) \neq \emptyset$ for some $t_0 \in \mathbb{R}$, there exists a point of transversal intersection in $W^u(\xi_i)(t_0) \cap W_{loc}^s(\xi_j)(t_0)$.

Before proving our main result, we need the definition of a *phase diagram isomorphism* between two evolution processes, and also some auxiliary results.

DEFINITION 10.4. Let \mathcal{S} be a dynamically gradient evolution process in a metric space X associated with the disjoint collection of isolated invariant sets $\mathbf{E} = \{\xi_1, \dots, \xi_n\}$, where $\xi_i: \mathbb{R} \rightarrow X$ is an isolated bounded global solution of \mathcal{S} for each $1 \leq i \leq n$. The *phase diagram* of \mathcal{S} is the disjoint collection $\mathbf{E} = \{\xi_1, \dots, \xi_n\}$ together with all the bounded global solutions $\xi: \mathbb{R} \rightarrow X$ of \mathcal{S} .

DEFINITION 10.5. Let $\mathcal{R} = \{R(t, s): (t, s) \in \mathcal{P}\}$ and $\mathcal{S} = \{S(t, s): (t, s) \in \mathcal{P}\}$ be two dynamically gradient evolution processes in a metric space (X, d) associated with the collections of isolated invariant bounded solutions $\mathbf{E}_{\mathcal{R}} = \{\xi_1, \dots, \xi_n\}$ and $\mathbf{E}_{\mathcal{S}} = \{\psi_1, \dots, \psi_m\}$, respectively. We say that there exists a **phase diagram isomorphism** between \mathcal{R} and \mathcal{S} if there exists a bijection $\mathcal{B}: \mathbf{E}_{\mathcal{R}} \rightarrow \mathbf{E}_{\mathcal{S}}$ (in particular, $m = n$) satisfying the following: there exists a global solution ξ of \mathcal{R} with

$$\lim_{t \rightarrow -\infty} d(\xi_i(t), \xi(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi_j(t), \xi(t)) = 0,$$

if and only if there exists a global solution ψ of \mathcal{S} with

$$\lim_{t \rightarrow -\infty} d(\mathcal{B}(\xi_i)(t), \psi(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\mathcal{B}(\xi_j)(t), \psi(t)) = 0.$$

PROPOSITION 10.6. *Suppose that all the hypotheses of Theorem 8.14 are satisfied. Assume also that $\{\lambda_k\} \subset \Lambda \setminus \{\lambda_0\}$, $\lambda_k \rightarrow \lambda_0$ as $k \rightarrow \infty$ and, for each $k \in \mathbb{N}$, ξ_k is a global solution of \mathcal{S}_k with*

$$\lim_{t \rightarrow -\infty} d(\xi_k(t), \xi_{i, \lambda_k}^*(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} d(\xi_k(t), \xi_{j, \lambda_k}^*(t)) = 0.$$

Then there exist a finite sequence $\{\xi^l\}_{l=1}^m$ (with $m \leq n$) of global solutions of \mathcal{T} , and equilibria $\{e^l\}_{l=1}^m$ with $e^1 = e_i^$, $e^m = e_j^*$ (for some $1 \leq i, j \leq n$) such that*

$$e^l \xleftarrow{t \rightarrow -\infty} \xi^l(t) \xrightarrow{t \rightarrow \infty} e^{l+1}, \quad \text{for each } 1 \leq l < m.$$

PROOF. Taking subsequences, if necessary, we may assume that

$$\lim_{k \rightarrow \infty} d(\xi_k(t), \xi^1(t)) = 0, \quad \text{uniformly in compact subsets of } \mathbb{R}.$$

Since $\lim_{t \rightarrow -\infty} d(\xi_k(t), \xi_{i, \lambda_k}^*(t)) = 0$ and $\text{dist}_H(\xi_{i, \lambda}^*(\mathbb{R}), e_i^*) \rightarrow 0$ as $k \rightarrow \infty$ we have $\lim_{t \rightarrow -\infty} d(\xi^1(t), e_i^*) = 0$, and we set $e^1 = e_i^*$. Now (G1) implies that there exists an equilibrium e^2 of \mathcal{T} (which, from (G2), is different from e^1) such that $\lim_{t \rightarrow \infty} d(\xi^1(t), e^2) = 0$. If $e^2 = e_j^*$ the proof is complete. Otherwise there exists $t'_\ell > t_\ell$ such that $d(\xi_{k_\ell}(t), e^2) < \delta$ for all $t \in [t_\ell, t'_\ell]$ and $d(\xi_{k_\ell}(t'_\ell), e^2) = \delta$. Taking

subsequences, if necessary, let $\xi^2(t) = \lim_{\ell \rightarrow \infty} \xi_{k_\ell}(t + t_\ell)$. Then, since $t'_\ell - t_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$, $d(\xi^2(t), e^2) \leq \delta$ for all $t \leq 0$ and consequently $\lim_{t \rightarrow -\infty} d(\xi^2(t), e^2) = 0$. Furthermore, from (G1) and (G2), $\lim_{t \rightarrow \infty} d(\xi^2(t), e^3) = 0$ for some equilibrium e^3 of \mathcal{T} (different from e^1 and e^2). This reasoning can be done as long as e^l is different from e_j^* , and since we have just a finite number of equilibrium points of \mathcal{T} , we conclude the result. \square

Now we are able to present the proof of Theorem 10.2. We recall that the auxiliary results concerning perturbation of stable and unstable sets are included in Chapter 9 - Subsection 6.2.

PROOF OF THEOREM 10.2. From Theorem 8.14, we know that there exists $\epsilon_0 > 0$ such that \mathcal{S}_λ is a dynamically gradient evolution process associated with the disjoint collection of isolated global solutions \mathbf{E}_λ for $|\lambda - \lambda_0| < \epsilon_0$. Also ϵ_0 can be taken sufficiently small so that each global solution in \mathbf{E}_λ is hyperbolic (see Theorem 9.24). Now, define $\mathcal{B}: \mathbf{E} \rightarrow \mathbf{E}_\lambda$ by

$$\mathcal{B}(e_i^*) = \xi_{i,\lambda}^*, \text{ for } i = 1, \dots, n.$$

Then \mathcal{B} is clearly a bijection, and we can assume by the openness of the transversality property that ϵ_0 is small enough so that, if $W^u(e_i^*) \cap W_{loc}^s(e_j^*) \neq \emptyset$, then $W^u(\xi_{i,\lambda}^*)(t) \cap W_{loc}^s(\xi_{j,\lambda}^*)(t) \neq \emptyset$ for all $t \in \mathbb{R}$, and moreover, there is a point in which the intersection is transversal (this holds since, from the results in Section 6.2, the unstable and stable manifolds behave continuously in the C^1 topology).

This shows that if there exists a connection between e_i^* and e_j^* , then there exists a connection between $\xi_{i,\lambda}^*$ and $\xi_{j,\lambda}^*$. It remains to show the converse.

Assume that we have a sequence $\lambda_k \rightarrow \lambda_0$ and global solutions ξ_k of S_{λ_k} for each k such that

$$\lim_{t \rightarrow -\infty} \|\xi_{i,\lambda}^*(t) - \xi_k(t)\| = 0 \text{ and } \lim_{t \rightarrow \infty} \|\xi_{j,\lambda}^*(t) - \xi_k(t)\| = 0.$$

Then, by Proposition 10.6, there exist a finite sequence $\{\xi^l\}_{l=1}^m$ ($m \leq n$) of global solutions of \mathcal{T} and equilibria $\{e_l\}_{l=1}^m$ in \mathbf{E} with $e_1 = e_i^*$, $e_m = e_j^*$ such that

$$e_l \xleftarrow{t \rightarrow -\infty} \varphi_l(t) \xrightarrow{t \rightarrow \infty} e_{l+1} \text{ for all } l = 1, \dots, m - 1.$$

But Proposition 5.9 ensures the existence of a global solution ξ of \mathcal{T} such that

$$e_i^* \xleftarrow{t \rightarrow -\infty} \varphi(t) \xrightarrow{t \rightarrow \infty} e_j^*,$$

and completes the proof. \square

2. Geometrical structural stability for skew-product semigroups

In this section we prove the geometrical structural stability for global attractors of skew-product semigroups. As a corollary, we also obtain the structural stability of uniform attractors for cocycles.

We begin by considering the initial value problem:

$$(10.1) \quad \begin{cases} \dot{u} = f_\lambda(t, u), & t > \tau \geq 0, \\ u(\tau) = u_0 \in X, \end{cases}$$

where X is a Banach space and $f_\lambda: \mathbb{R}^+ \times D \subset \mathbb{R} \times X \rightarrow X$ is a map belonging to some complete metric space (\mathcal{C}, ρ) for each $\lambda \in \Lambda = [0, 1]$. We assume that for each $g \in \mathcal{C}$, $\tau \in \mathbb{R}$ and $u_0 \in X$ the problem:

$$(10.2) \quad \begin{cases} \dot{u} = g(t, u), & t > \tau \geq 0, \\ u(\tau) = u_0 \in X \end{cases}$$

is globally well posed, that is:

- (i) for each $u_0 \in X$, $\tau \in \mathbb{R}^+$ and $g \in \mathcal{C}$, there exists a unique function $[\tau, \infty) \ni t \mapsto u(t, \tau, g, u_0) \in X$ satisfying (10.2), in a suitable sense;
- (ii) $\{(t, \tau, u_0) \in \mathbb{R}^+ \times \mathbb{R}^+ \times X : t \geq \tau\} \ni (t, \tau, u_0) \mapsto u(t, \tau, g, u_0) \in X$ is continuous.

We are interested in the case when f_{λ_0} is independent of time and $\rho(f_\lambda, f_{\lambda_0}) \rightarrow 0$ as $\lambda \rightarrow \lambda_0$. In particular, we are interested in the case where the dynamical system associated to (10.1), with $\lambda = \lambda_0$, is a Morse-Smale semigroup $\mathcal{T}_0 = \{T_0(t) : t \geq 0\}$ in the sense of Definition 5.2. For λ suitably close to λ_0 , the dynamical system associated to (10.1) will be a non-autonomous perturbation of a Morse-Smale semigroup, and hence, a Morse-Smale evolution process (Definition 10.3). We prove that the structure of connections of a Morse-Smale semigroup remains the same under non-autonomous perturbations, that is, we have geometrical structural stability.

Define

$$H_\lambda = \overline{\{f_\lambda(s + \cdot, \cdot) : s \in \mathbb{R}^+\}}^\rho \quad \text{and} \quad M = \bigcup_{\lambda \in \Lambda} \overline{H_\lambda}^\rho.$$

Clearly $M \subset \mathcal{C}$ and we define the (driving) semigroup $\Theta = \{\Theta_t : t \geq 0\}$ in M by $\Theta_t h(\cdot, \cdot) = h(t + \cdot, \cdot)$ for each $h \in M$. For each $h \in M$ and $u_0 \in X$ we consider the initial value problem

$$\begin{cases} \dot{u} = h(t, u), & t > 0, \\ u(0) = u_0 \in X \end{cases}$$

and let $\mathbb{R}^+ \ni t \mapsto \mathcal{K}(t, h)u_0 \in X$ be its only solution. With this, we have defined the cocycle

$$\mathbb{R}^+ \times M \ni (t, h) \mapsto \mathcal{K}(t, h) \in C(X),$$

and the skew-product semigroup $\Pi_M = \{\Pi_M(t) : t \geq 0\}$ in $\mathbb{X} = X \times M$ by

$$\Pi_M(t)(u, h) = (\mathcal{K}(t, h)u, \Theta_t h) \text{ for } t \geq 0.$$

We assume that Π_M has a global attractor \mathbb{A}_M in \mathbb{X}_M , and as we know, this implies in particular that Θ has a global attractor \mathcal{S}_M in M .

2.1. Skew-product semigroups and evolution processes. For each $h_\lambda \in H_\lambda$ and $u_0 \in X$ consider the initial value problem:

$$\begin{cases} \dot{u} = h_\lambda(t, u), & t > 0, \\ u(0) = u_0 \in X \end{cases}$$

and let $\mathbb{R}^+ \ni t \mapsto \mathcal{K}(t, h_\lambda)u_0 \in X$ be its only solution. With this, we have defined the cocycle

$$\mathbb{R}^+ \times H_\lambda \ni (t, h_\lambda) \mapsto \mathcal{K}(t, h_\lambda) \in C(X),$$

and the skew-product semigroup $\Pi_\lambda = \{\Pi_\lambda(t) : t \geq 0\}$ in $\mathbb{X}_\lambda = X \times H_\lambda$ by

$$\Pi_\lambda(t)(u, h_\lambda) = (\mathcal{K}(t, h_\lambda)u, \Theta_t h_\lambda) \text{ for } t \geq 0.$$

If f_{λ_0} is time-independent, $H_{\lambda_0} = \{f_{\lambda_0}\}$, $\mathbb{X}_{\lambda_0} = X \times \{f_{\lambda_0}\}$, $\Theta_t f_{\lambda_0} = f_{\lambda_0}$ for each $t \geq 0$, and setting $T_0(t) = \mathcal{K}(t, f_{\lambda_0})$ for each $t \geq 0$, $\mathcal{T}_0 = \{T_0(t) : t \geq 0\}$ defines a semigroup in X and

$$\Pi_{\lambda_0}(t)(u, f_{\lambda_0}) = (\mathcal{K}(t, f_{\lambda_0})u, f_{\lambda_0}) = (T_0(t)u, f_{\lambda_0}) \text{ for } t \geq 0.$$

It is clear that the skew-product semigroup Π_λ has a global attractor \mathbb{A}_λ in $\mathbb{X}_\lambda = X \times H_\lambda$ and, if Θ^λ denotes the restriction of the semigroup Θ to H_λ , it has a global attractor \mathcal{S}_λ in H_λ .

Given a bounded global solution η_λ of Θ_λ through h_λ , we consider the problem:

$$\begin{cases} \dot{u} = \eta_\lambda(\tau)(t - \tau, u), & t > \tau, \\ u(\tau) = u_0 \in X. \end{cases}$$

From the previous assumptions, this problem generates an evolution process $\mathcal{S}_{\eta_\lambda}$ given by $S_{\eta_\lambda}(t, \tau) = \mathcal{K}(t - \tau, \eta_\lambda(\tau))$ for each $t \geq \tau$, and $\mathcal{S}_{\eta_\lambda}$ has a pullback attractor \hat{A}_{η_λ} .

We have seen in Theorem 8.14 the robustness of the dynamically gradient structure of pullback attractors for evolution processes, i.e., there exists an $\epsilon > 0$ such that $\mathcal{S}_{\eta_\lambda}$ is a dynamically gradient evolution process, for all $\lambda \in \Lambda$ such that $|\lambda - \lambda_0| < \epsilon$, when \mathcal{T}_0 is a dynamically gradient semigroup. Now if $\epsilon > 0$ is as in Theorem 8.14 and, for $|\lambda - \lambda_0| < \epsilon$, we define

$$\mathbb{M}_{\lambda,i} = \{(\xi_{i,\lambda}^*(0), \eta_\lambda(0)) : \eta_\lambda \text{ is a bounded global solution of } \Theta^\lambda \text{ in } H_\lambda\},$$

the following result holds as a corollary of the proof of Theorem 8.14.

PROPOSITION 10.7. *Suppose the assumptions of Theorem 8.14 hold, $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$, and there exist global solutions $\pi_{\lambda_n} : \mathbb{R} \rightarrow \mathbb{A}_{\lambda_n}$ of Π_{λ_n} in \mathbb{X}_{λ_n} with*

$$\mathbb{M}_{\lambda_n,i} \xrightarrow{t \rightarrow -\infty} \pi_{\lambda_n}(t) \xrightarrow{t \rightarrow \infty} \mathbb{M}_{\lambda_n,j} \quad \text{for some } 1 \leq i, j \leq n.$$

Then there exist global solutions $\pi_{0,k} : \mathbb{R} \rightarrow \mathbb{A}_0$ of Π_{λ_0} in \mathbb{X}_{λ_0} , for $1 \leq k \leq n_0 \leq n$ such that

$$\mathbb{M}_{0,i_{k-1}} \xrightarrow{t \rightarrow -\infty} \pi_{0,k}(t) \xrightarrow{t \rightarrow \infty} \mathbb{M}_{0,i_k} \quad \text{for } k = 1, \dots, n_0,$$

where $i_0 = i$ and $i_{n_0} = j$.

REMARK 10.8 (Abstract phase diagram commutativity result). Under the assumptions of Theorem 8.14, to ensure that the structure of connections between the perturbed and limiting attractors remains the same, we need to show the following:

- (a) if there is an *indirect connection* between two isolated invariant sets $\mathbb{M}_{\lambda_0,i}$ and $\mathbb{M}_{\lambda_0,j}$ of the limiting problem, then there is a direct connection between $\mathbb{M}_{\lambda_0,i}$ and $\mathbb{M}_{\lambda_0,j}$ (which is achieved using the transversality property together with the λ -lemma);
- (b) if there is a connection between two isolated invariant sets $\mathbb{M}_{\lambda_0,i}$ and $\mathbb{M}_{\lambda_0,j}$ of the limiting problem, then there will be a connection between the associated isolated invariant sets $\mathbb{M}_{\lambda,i}$ and $\mathbb{M}_{\lambda,j}$ of the perturbed problem (which is achieved using the transversality property).

2.2. Geometrical structural stability. Now we are ready to show the phase diagram commutativity under non-autonomous perturbations for Morse-Smale semigroups (see [195]).

THEOREM 10.9 (cf. [31]). *Assume that Π_{λ_0} is a C^1 -reversible Morse-Smale semigroup with global attractor \mathbb{A}_0 on $X \times \{f_{\lambda_0}\}$ and that its nonwandering set consists of a finite number of hyperbolic equilibria $\Omega(\Pi_{\lambda_0}) = \{(z_1^*, f_{\lambda_0}), \dots, (z_n^*, f_{\lambda_0})\}$. Assume also that:*

- (a) Π_λ has a global attractor \mathbb{A}_λ in \mathbb{X}_λ for each $\lambda \in \Lambda$, and $\mathbb{K} = \overline{\bigcup_{\lambda \in (0,1]} \mathbb{A}_\lambda}$ is compact;
- (b) $\Pi_\lambda(t)(x, h_\lambda) \rightarrow \Pi_{\lambda_0}(t)(x, f_{\lambda_0})$ in \mathbb{X}_M as $\lambda \rightarrow \lambda_0$, uniformly for $h_\lambda \in H_\lambda$ and for (t, x) in compact subsets of $\mathbb{R}^+ \times X$;
- (c) for each compact set $J \in \mathbb{R}^+$ we have

$$\|DK(t, h_\lambda)(z) - DK(t, f_{\lambda_0})(z)\|_{\mathcal{L}(X)} \rightarrow 0 \text{ as } \lambda \rightarrow \lambda_0,$$

uniformly for $t \in J$, $h_\lambda \in H_\lambda$ and $z \in \pi_X(\mathbb{K})$.

Then there exists $\epsilon_0 > 0$ such that for $|\lambda - \lambda_0| < \epsilon_0$ we have:

- (i) for any bounded global solution η_λ of Θ_λ in H_λ , the evolution process $\mathcal{S}_{\eta_\lambda}$ has a disjoint collection of isolated hyperbolic global solutions

$$\{\xi_{1,\eta_\lambda}^*, \dots, \xi_{n,\eta_\lambda}^*\}$$

with

$$\lim_{\lambda \rightarrow \lambda_0} \sup_{\eta_\lambda \in \mathcal{S}_\lambda} \max_{1 \leq i \leq n} \text{dist}_H^X(\xi_{i,\eta_\lambda}^*(\mathbb{R}), z_i^*) = 0,$$

- (ii) for any bounded global solution η_λ of Θ_λ in H_λ , $W_{loc}^u(\xi_{i,\eta_\lambda}^*)$ is finite dimensional, and

$$W^u(\xi_{i,\eta_\lambda}^*)(t_0) \bar{\cap} W_{loc}^s(\xi_{j,\eta_\lambda}^*)(t_0),$$

for some $t_0 \in \mathbb{R}$, for each $1 \leq i, j \leq n$;

- (iii) if $(\xi_\lambda, \eta_\lambda)$ is a bounded global solution of Π_λ , $t_0 \in \mathbb{R}$ and

$$\text{dist}_H^H(\xi_\lambda(t), \xi_{i,\eta_\lambda}^*(t)) \leq \delta \text{ for all } t \leq t_0 \text{ (resp. } t \geq t_0),$$

then

$$\text{dist}_H^X(\xi_\lambda(t), \xi_{i,\eta_\lambda}^*(t)) \rightarrow 0 \text{ as } t \rightarrow -\infty \text{ (resp. } t \rightarrow \infty);$$

- (iv) Π_λ is dynamically gradient relative to the disjoint collection of isolated invariant sets $\{\mathbb{M}_{\lambda,1}, \dots, \mathbb{M}_{\lambda,n}\}$, $\mathbb{M}_{\lambda,i} = \bigcup_{\eta_\lambda} \{(\xi_{i,\eta_\lambda}^*(0), \eta_\lambda(0))\}$, where the union is taken over all bounded global solutions of Θ_λ in H_λ ;
- (v) there exists a phase diagram isomorphism between \mathcal{S}_λ and \mathcal{T}_0 , and between Π_λ and Π_{λ_0} .

This theorem can also be written in the context of uniform attractors for cocycles, with the same hypotheses, using the notion of lifted invariant sets and projecting the isolated invariant sets on the first coordinate to obtain the lifted invariant isolated invariant sets. Before stating this result, we need the following definition.

DEFINITION 10.10. Let $(\mathcal{K}, \Theta)_{(X,\Sigma)}$ be a non-autonomous dynamical system in a Banach space X , with a uniform attractor \mathcal{A} and such that the associated skew-product semigroup Π has a global attractor \mathbb{A} in \mathbb{X}_Σ . We say that $(\mathcal{K}, \Theta)_{(X,\Sigma)}$ (or simply, \mathcal{K}) is a **Morse-Smale non-autonomous dynamical system** if for each bounded global solution η of Θ in Σ , the associated evolution process \mathcal{S}_η , given by $S_\eta(t, s) = \mathcal{K}(t - s, \eta(s))$ for each $t \geq s$, is a Morse-Smale evolution process with nonwandering set given as a finite number of hyperbolic global solutions $\mathbf{E}_\eta = \{\xi_{\eta,1}, \dots, \xi_{\eta,n}\}$.

COROLLARY 10.11 (Structural stability for uniform attractors). *Let the hypotheses in Theorem 10.9 hold. In particular, $(\mathcal{K}(t, h_\lambda), \Theta)_{(X, H_\lambda)}$ is a continuously differentiable non-autonomous dynamical system, for each $\lambda \in \Lambda$, which we will denote briefly by \mathcal{K}_λ . Assume that:*

- (a) \mathcal{K}_λ has a uniform attractor \mathcal{A}_λ for each $\lambda \in \Lambda$, and $\mathcal{G} = \overline{\bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda}$ is compact;
- (b) $\mathcal{T}_0 = \mathcal{K}_{\lambda_0}$ is a Morse-Smale semigroup with its nonwandering set consisting of only hyperbolic equilibria $\mathcal{E}_0 = \{z_{0,1}, \dots, z_{0,n}\}$;
- (c) for each compact set $J \subset \mathbb{R}^+$ we have

$$\|\mathcal{K}(t, h_\lambda)x - T_0(t)x\|_X \rightarrow 0,$$

and

$$\|\mathcal{K}'(t, h_\lambda)(x) - T_0'(t)(x)\|_{\mathcal{L}(X)} \rightarrow 0,$$

as $\lambda \rightarrow \lambda_0$, uniformly for $t \in J$, $h_\lambda \in H_\lambda$ and $x \in \mathcal{G}$;

Then there exists $\epsilon_0 > 0$ such that for $|\lambda - \lambda_0| < \epsilon_0$ we have:

- (i) \mathcal{K}_λ has a disjoint family of isolated \mathcal{K}_λ -lifted invariant sets

$$\mathcal{E}_\lambda = \{E_{\lambda,1}, \dots, E_{\lambda,n}\}$$

such that

$$\max_{1 \leq i \leq n} \text{dist}_H^X(E_{\lambda,i}, E_{\lambda_0,i}) \rightarrow 0, \text{ as } \lambda \rightarrow \lambda_0;$$

- (ii) $(\mathcal{K}(t, h_\lambda), \Theta)_{(X, H_\lambda)}$ is a Morse-Smale non-autonomous dynamical system with $\mathcal{E}_\lambda = \{E_{\lambda,1}, \dots, E_{\lambda,n}\}$ as its nonwandering set.

3. Applications

We now illustrate the theory developed in this chapter with some examples and models, applying Theorem 10.9 (or, equivalently, Corollary 10.11 for the uniform attractor).

3.1. Applications to partial differential equations. In this section we recall the two applications in Chapter 5 - Section 3: parabolic equations and a damped wave equation, for which the previous results in this chapter are satisfied under non-autonomous perturbation.

Application of Theorem 10.9 to reaction-diffusion equations.

Let Ω be a bounded smooth domain in \mathbb{R}^n , $n \geq 1$, and let $f_0: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^k -function, $k \geq 2$. We consider the problem

$$(10.3) \quad \begin{cases} u_t = \Delta u + f(x, u), & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u = u_0, & t = 0, x \in \Omega, \end{cases}$$

and we assume that all the conditions of Chapter 5 - Section 3.2 hold for (5.5).

We now fix a function $f \in \mathcal{G}_{MS}^k$, $k \geq 2$, and consider non-autonomous perturbations of (10.3), that is, we consider the problem:

$$(10.4) \quad \begin{cases} u_t = \Delta u + f(t, x, u), & t > s, x \in \Omega, \\ u = 0, & t > s, x \in \partial\Omega, \\ u = u_s, & t = s, x \in \Omega, \end{cases}$$

where $u_s \in X$ and $f: \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a suitably small non-autonomous perturbation of f_0 , satisfying for all $(t, x, y) \in \mathbb{R} \times \Omega \times \mathbb{R}$ the dissipative conditions:

$$(10.5) \quad f(t, x, y)y < C_0 + \mu y^2 \quad \text{and} \quad F(t, x, y) < C_0 + \frac{1}{2}\mu y^2,$$

where $F(t, x, y) = \int_0^y f(t, x, s)ds$.

It follows that Theorem 10.9 applies for f a suitably C^2 (in the third variable) small perturbation of f_0 uniformly in $x \in \Omega$ and in $t \in \mathbb{R}$ and there is a phase diagram isomorphism between the pullback attractors (10.3) and the global attractor of (10.4).

Application of Theorem 10.9 to the damped wave equation.

Let us recall the application in Chapter 5 - Section 3.3. Consider Ω a bounded smooth domain in \mathbb{R}^n , $n = 1, 2, 3$, $f: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ a C^k -function, $k \geq 2$, a constant $\gamma > 0$ and consider the following damped wave equation:

$$(10.6) \quad \begin{cases} u_{tt} + \gamma u_t - \Delta u = f_0(x, u), & t > 0, x \in \Omega, \\ u(x, t) = 0, & t > 0, x \in \partial\Omega, \\ (u(x, 0), u_t(x, 0)) = (u_0, v_0), \end{cases}$$

where $(u_0, v_0) \in Y$, $Y = H_0^1(\Omega) \times L^2(\Omega)$ if $n = 1$, or $Y = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ if $n = 2, 3$.

In Chapter 5 - Section 3.3 we presented results ensuring that (10.6) generates a Morse-Smale semigroup (for a generic set of nonlinearities f). Assuming that the semigroup associated to (10.6) is Morse-Smale, it follows from Theorem 10.9 that, for all suitably small non-autonomous perturbations f of f_0 (as in the previous example), the evolution process associated to

$$(10.7) \quad \begin{cases} u_{tt} + \gamma u_t - \Delta u = f(t, x, u), & t > 0, x \in \Omega, \\ u(x, t) = 0, & t > 0, x \in \partial\Omega, \\ (u(x, 0), u_t(x, 0)) = (u_0, v_0), \end{cases}$$

is a Morse-Smale evolution process, and there is a phase diagram isomorphism between the pullback attractors (10.7) and the global attractor of (10.6).

3.2. A non-autonomous perturbation of a hyperbolic periodic orbit.

This example exhibits, in a particular case, the object we obtain when we make a non-autonomous perturbation of a periodic orbit (a fixed point in polar coordinates). So far, in the non-autonomous setting, we are not able to make a small non-autonomous perturbation of a hyperbolic periodic orbit in the general case, but we can give a full disclosure of what happens in this particular example.

Consider the autonomous equation (3.8) of Chapter 3. Note that $P_0 = \{(0, 0)\}$ is a hyperbolic point and $P_1 = \{r = 1\}$ and $P_2 = \{r = 2\}$ are normally hyperbolic periodic orbits. Also note that the intersections of $W^u(P_i)$ and $W_{loc}^s(P_j)$ are always transverse, for $0 \leq i, j \leq 2$. Consider, for $\lambda \in \Lambda = [0, 1]$, the perturbed problem:

$$(10.8) \quad \begin{cases} \dot{r} = -r(r-1)(r-2) + f_\lambda(t), & t \in \mathbb{R}, \\ \dot{\theta} = 1 \end{cases}$$

where $f_{\lambda_0} \equiv 0$, $f_\lambda(t) \geq 0$, for all $\lambda \in \Lambda$ and $t \in \mathbb{R}$, and $\sup_{t \in \mathbb{R}} f_\lambda(t) \rightarrow 0$ as $\lambda \rightarrow \lambda_0$.

Let $\mathcal{S}_\lambda = \{S_\lambda(t, s) : (t, s) \in \mathcal{P}\}$ be the evolution process generated by (10.8).

Since P_2 is a stable periodic orbit, each region $\{(x, y) \in \mathbb{R}^2 : \alpha_1 \leq x^2 + y^2 \leq \alpha_2\}$ where $1 < \alpha_1 < 4 < \alpha_2 < \infty$ is positively \mathcal{S}_λ -invariant, provided that λ is

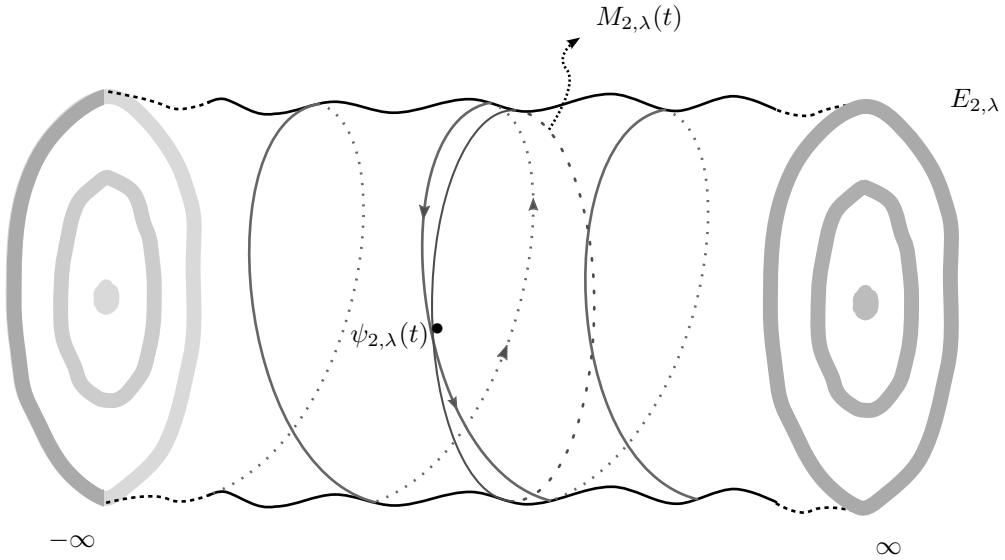


FIGURE 10.1. Solutions in the pullback attractor.

sufficiently close to 0 (and hence $\sup_{t \in \mathbb{R}} f_\lambda(t)$ is small). Therefore, if we consider the restriction of \mathcal{S}_λ to a small annulus $\{(x, y) \in \mathbb{R}^2 : \alpha_1 \leq x^2 + y^2 \leq \alpha_2\}$, where $1 < \alpha_1 < 4 < \alpha_2 < \infty$, it possesses a pullback attractor $\hat{M}_{2,\lambda} = \{M_{2,\lambda}(t)\}_{t \in \mathbb{R}}$.

An analogous reasoning can be used to prove the existence of pullback attractors $\hat{M}_{1,\lambda} = \{M_{1,\lambda}(t)\}_{t \in \mathbb{R}}$ and $\hat{M}_{0,\lambda} = \{M_{0,\lambda}(t)\}_{t \in \mathbb{R}}$ of the evolution process \mathcal{S}_λ restricted to a small annulus around the circle of radius 1 and to a small disc around 0, respectively. In the case of the circle of radius 1, we can see that it is a repeller for the limiting problem, and the argument can be done by simply reversing time.

By the continuity with respect to the parameter, since the solution to the limiting problem (i.e. $\lambda = \lambda_0$) with $r = 1, 2$ is a circle, then we can see that the projection in the phase space of the solutions in $\{M_{i,\lambda}(t)\}_{t \in \mathbb{R}}$ must complete a whole lap, provided that λ is sufficiently close to λ_0 . Thus the pullback and the uniform dynamics around P_2 (the dynamics around 0 and P_1 are analogous) can be described by Figure 10.1.

Since the limiting equation generates a generalized dynamically gradient semi-group with $\{0, P_1, P_2\}$ the family of disjoint isolated invariant sets, Theorem 8.14 implies that each evolution process \mathcal{S}_λ is dynamically gradient with respect to the family $\{\hat{M}_{0,\lambda}, \hat{M}_{1,\lambda}, \hat{M}_{2,\lambda}\}$.

Now, using the results of [75] on perturbation of hyperbolic points, we can assure that there exist hyperbolic global solutions $\xi_{j,\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ for $j = 0, 1, 2$ of the

first equation of (10.8), provided that λ is sufficiently close to λ_0 , such that:

$$(10.9) \quad \sup_{t \in \mathbb{R}} |\xi_{\lambda,0}(t)| \rightarrow 0, \quad \sup_{t \in \mathbb{R}} |\xi_{\lambda,1}(t) - 1| \rightarrow 0 \text{ and } \sup_{t \in \mathbb{R}} |\xi_{\lambda,2}(t) - 2| \rightarrow 0,$$

as $\lambda \rightarrow \lambda_0$. Therefore, (10.8) has infinite normally hyperbolic global solutions, given by

$$P_{i,\lambda,k}: \mathbb{R} \rightarrow \mathbb{R}^2 \\ t \mapsto (\xi_{\lambda,i}(t) \cos(\theta + k), \xi_{\lambda,i}(t) \sin(\theta + k)),$$

for each $i = 0, 1$ and 2 and each $k \in \mathbb{R}$.

Notes. The concept of Morse-Smale evolution process appears in [31, 111]. The structural stability of Morse-Smale maps under small non-autonomous perturbations has been proved by Franks and Kryzhevich and Pliss in [121, 158]. It is shown in [111] that the concept is robust under perturbation. In [31] it is also shown that there exists a phase diagram isomorphism between the limit Morse-Smale semigroup and the evolution process obtained under non-autonomous perturbation. The definition of lifted invariance allows us to define a Morse-Smale characterization for the uniform attractors, and its robustness under perturbation. The theory of local manifolds around a hyperbolic global solution of a given discrete evolution process can be found in [78] and in [142] for the continuous case whereas the continuous dependence with respect to C^1 norms in the non-autonomous case is done in [31].