Preface

Let $A, B \in \mathbb{C}[x]$ be two coprime polynomials, $\deg A = 10$, $\deg B = 15$, and consider the difference $A^3 - B^2$. Both A^3 and B^2 are of degree 30. How many higher coefficients of $A^3 - B^2$ can be made equal to zero? It is trivial to make the leading coefficient of $A^3 - B^2$ vanish; it is also simple to make the second coefficient disappear. Can we do more than that? What is the minimum degree of the difference $R = A^3 - B^2$ that can be attained?

It turns out that $\min \deg R = 6$, which means that, by a clever choice of A and B, we can make 24 higher coefficients of the difference $R = A^3 - B^2$ vanish, and it is impossible to do better.

The question, in a general form, concerning the pairs of polynomials (A, B) such that $\deg A = 2k$ and $\deg B = 3k$, so that $\deg A^3 = \deg B^2 = 6k$, was raised in 1965 [**BCHS-65**]. The above example corresponds to $k = 5$. For this case there exist four essentially different solutions (A, B) (the exact definition of what does it mean to be "essentially different" will be given later). For two of them, the coefficients of the polynomials are rational; for the other two, they belong to an imaginary quadratic field. It is incredibly difficult to compute these solutions. All the four pairs (A, B) were found only 40 years later. One of the solutions, defined over $\mathbb{Q}$, was already computed in 1965 in the original paper [**BCHS-65**]; the second one, also over $\mathbb{Q}$, was found 35 years later in [**Elk-00**]; finally, the pair of solutions over a quadratic field was found in 2005 in [**Shi-05**]; it turned out the the field in question (to which the coefficients of the polynomials belong) is $\mathbb{Q}(\sqrt{-3})$.

But there exists a wizardly method of proving all the above statements (except the fact that there is -3 under the square root) without any computation. This method consists in drawing very simple pictures: so simple that they are usually called *dessins d'enfants*. The latter means, in French, "children's drawings". This half-joking term was coined by Alexandre Grothendieck in his unpublished notes "Esquisse d'un Programme" (1984). Later on, these notes were published in [**Gro-84**] (both the French original and an English translation). Based on an earlier work of Belyĭ [**Bel-79**], Grothendieck pointed out that there are profound relations between (a) Galois theory; (b) Belyĭ functions (meromorphic functions with at most three critical values); and (c) a class of pictures drawn on Riemann surfaces. Nowadays, the theory of dessins d'enfants is an active (and, we dare say, fashionable) domain of research. The books [**JoWo-16**] and [**GiGo-12**] are dedicated to this subject. A vast bibliography is collected in [**SiVo-14**]. The book [**BoSt-04**], as well as our book, may be considered as particular chapters of this theory. The French expression *dessins d'enfants* is also currently used in English language literature. It has also been made into one of the entries of the AMS Subject Classification Index: 11G32.

Our book is among the most elementary ones dedicated to the subject: all our dessins are planar and "tree-like", and the meromorphic functions in question are just ordinary rational functions. However, this elementary character notwithstanding, the book preserves the most attractive aspect proper to the entire theory, namely, an intricate entanglement of combinatorics, the theory of polynomials, symbolic computations, special functions, Galois theory, number theory, and group theory.

The general class of polynomials we study in the book is as follows: they have the multiplicities of their roots fixed in advance. For example, the multiplicities of the roots of $P = A^3$ are all equal to 3 (or they may be multiples of 3), while the multiplicities of the roots of $Q = B^2$ are 2 (or multiples of 2). In the general setting, the root multiplicities will form a pair of partitions $\alpha, \beta \vdash n$ of the number n which is the common degree of the polynomials P and Q:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p), \quad \beta = (\beta_1, \beta_2, \dots, \beta_q), \quad \sum_{i=1}^{p} \alpha_p = \sum_{j=1}^{q} \beta_j = n.$$

When, for a given pair of partitions (α, β), the degree $\deg(P - Q)$ attains its minimum, we call the pair of polynomials (P, Q) a *Davenport–Zannier pair* or, in short, a *DZ-pair*. There are many researchers who contributed to the study of these polynomials, so we might as well call them Birch–Chowla–Hall–Schinzel–Davenport–Stothers–Boccara–Zannier–Beukers–Stewart… polynomials (maybe the readers will be kind enough to add our own names to the list?), but for the sake of brevity we have chosen the two names which seem to us the most appropriate. The pair of partitions (α, β) is called the *passport* of the corresponding DZ-pair.

We call the dessins used in this book *weighted trees*. Such a tree should in fact be called *weighted bicolored plane tree*. It is an object like the one shown in Figure 0.1.

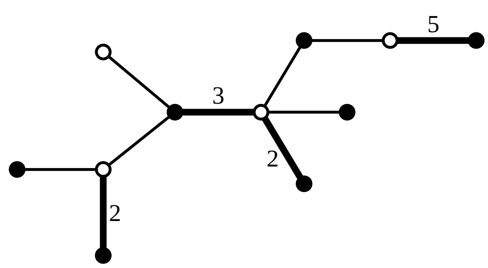

FIGURE 0.1. A weighted bicolored plane tree. The weights which are not explicitly indicated are equal to 1. The passport of this tree is $(5^2 2^3 1^2, 7^1 6^1 4^1 1^1)$.

Namely, it is a tree whose edges are endowed with *weights*, these weights being positive integers. The *degree* of a vertex is the sum of the weights of the edges incident to this vertex. We will also use the term *valency* as a synonym of the term degree. *Bicolored* means that the vertices are colored in black and white in such a way that the ends of each edge have opposite colors. *Plane* means that the cyclic order of branches around a vertex is taken into account: by changing this order we usually change the tree (though of course the new tree may turn out to be isomorphic to the initial one).

This notion will be explained in more detail later. The *passport* of a tree is the pair of partitions (α, β) representing the degrees of black, respectively white, vertices.

Let us, however, disclose our true intention: in fact, an edge of weight k represents a strand of k parallel edges: see Figure 1.1 (page 5). Thus, weighted trees represent a specific class of plane maps. It just happens that technically it is much easier to work with trees than with maps, especially when the number of parallel edges is a parameter which may take arbitrary values.

For the sake of brevity we will call these objects *weighted trees*, or even just *trees*, omitting all the adjectives.

Weighted trees are interesting combinatorial objects in their own right and may give rise to various studies involving enumeration, bijections and so on. We are not completely foreign to this experience: see Chapter 11 where we present some enumerative results. But mainly, for us, weighted trees serve as a remarkably efficient tool for studying DZ-pairs of polynomials. There is a bijection between the (equivalence classes of) DZ-pairs and the (isomorphism classes of) weighted trees having the same passport. Thus, for example, in order to establish the existence of a specific DZ-pair it suffices to draw the corresponding tree. The trees are also helpful in the study of the Galois action on DZ-pairs, but the construction in question needs a more lengthy explanation. Just one example: if, for a given passport, the tree with this passport is unique, then the coefficients of both P and Q are (more exactly, can be made) rational.

The contents of the book are as follows.

Chapter 1 is introductory: we present formal and detailed definitions of the main notions and explain the relations between them.

Chapter 2 is a very brief introduction to the theory of dessins d'enfants.

In Chapter 3 we show that for every "valuable" passport there exists a tree having this passport. In general (that is, for arbitrary maps) the results of this kind may turn out to be very difficult and the conditions of "valuability" may be very intricate. The Euler formula is, of course, a necessary condition, but it is far from being sufficient. However, for the particular case of weighted trees, the condition, which is both necessary and sufficient, is very simple. Namely, the number of vertices must not exceed the upper bound possible for the trees. The proof of the existence theorem is also easy. At the end of this chapter we explain what can be done for the non-valuable passports.

A very short Chapter 4 contains, as is implied by its title, the recapitulation of what has already been done in the previous chapters, and sets the goals for the subsequent ones.

Chapter 5 is a difficult reading. We establish a complete classification of what we call *unitrees*, that is, trees which are uniquely determined by their passport. Like busy foresters, we plant some trees, graft upon them branches of other trees, choose roots, put weights at various places.... The proof takes almost 30 pages, but we did not find a simpler one. As we have mentioned above, all the unitrees are "defined" over the field $\mathbb{Q}$ of rational numbers. This means that the equivalence class of the corresponding DZ-pairs contains a pair with the coefficients in $\mathbb{Q}$. There exist ten infinite series of unitrees described by some integer-valued parameters, and ten sporadic unitrees which do not belong to any series.

The last section of this chapter contains an example (just one example!) of a series of quadratic orbits, that is, a parametric passport which gives rise to two different trees. The situation is much more intricate than one might suppose.

In Chapter 6 we compute DZ-pairs corresponding to the unitrees. We have already mentioned above how difficult it is to compute a DZ-pair for a given tree. It is incomparably more difficult to compute them for an infinite series. In this chapter we will encounter such topics as Jacobi polynomials, Euler Beta function, Padé approximants, hypergeometric series, etc.

The passport is a Galois invariant, but there are many other invariants as well. The most advanced of them all is the monodromy group of the ramified covering of the Riemann complex sphere by itself corresponding to the rational function $f = P/R$ where $R = P - Q$ and (P, Q) is a DZ-pair. In Chapters 7 and 8 we provide a complete classification of the *primitive* monodromy groups of such coverings. For the definitions and statements, see the text. Not all the DZ-pairs corresponding to these groups are yet computed. To the best of our knowledge, a complete classification of the primitive monodromy groups of coverings of the sphere by itself is not yet achieved. The infinite series are not described, and a complete list of sporadic cases is not presented (maybe it is too large to be written explicitly). There are, however, some partial results, like a complete list of the affine groups appearing in this context which is given in [**MSW-11**]. Our result may be considered a modest contribution to the subject.

Chapter 9 studies some other Galois invariants, like symmetry, composition, self-duality, etc. The most unusual of them, and difficult to detect, is what we call a "megamap invariant". A *megamap* is a dessin which represents the Hurwitz space for a family of coverings of the sphere with *four* ramification points. The property of a dessin to serve as a megamap for such a family is a Galois invariant. Unfortunately, we do not have an algorithm to verify if a given dessin is a megamap for a Hurwitz family; it works only in the opposite direction, from a family to the corresponding megamap. Our examples are found by a kind of "blind search".

Chapter 10 is a case study: a very beautiful one, in fact. Here we consider a particular set of two trees which almost always constitutes a Galois orbit defined over a real quadratic field. However, from time to time this set splits into two Galois orbits, both defined over $\mathbb{Q}$. It turns out that the splitting cases correspond to the solutions of a Pell equation, and are "explained" by these solutions since the latter ensure that the discriminant of the quadratic equation in question is a perfect square. Pell's equation was studied for more than two thousand years. It is amazing to see that it still can tell us something new in the XXI$^{\text{st}}$ century. This example also illustrates the fact that there is absolutely no hope to find an exhaustive set of combinatorial and group-theoretic invariants which would provide us with an "if and only if" criterion of membership of two dessins to the same Galois orbit. Up to now, Diophantine invariants of dessins d'enfants were not yet thoroughly studied. They certainly deserve closer attention.

Chapter 11 is devoted to enumeration of weighted trees. We did not push this subject too far since it does not belong to the mainstream of our interests.

The last (and very short) Chapter 12 formulates some problems for subsequent study.

A few words should be said about the motivation. The first impulse for the study of DZ-pairs had come from number theory, in particular from the question of how close to each other two integer powers can be. But gradually this theory acquired its own intrinsic interest. The worthiness of a mathematical theory is explained by its internal beauty, by an abundance of non-trivial examples, by the difficulty of its main results, an ingenuity of their proofs, and, last but not least, by an interplay of various branches of mathematics. All these aspects are vastly presented in the study of the Davenport–Zannier polynomials.

Acknowledgments

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Many years ago, Umberto Zannier sent to A. Z. his paper [**Zan-95**], and this was the starting point of our interest in the problems described in this book. The second impetus came from a talk given by Cameron Stewart at the University of Bordeaux in 2010.

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