

Contents

List of Figures	ix
List of Tables	xv
Preface	xix
Introduction	1
0.1. Topological modular forms	1
0.2. (Co-)homology and complex bordism of tmf	3
0.3. The Adams E_2 -term for S	5
0.4. The Adams differentials for S	6
0.5. The Adams E_2 -term for tmf	8
0.6. The Adams differentials for tmf	11
0.7. The graded homotopy ring of tmf	15
0.8. Duality	19
0.9. The sphere spectrum	21
0.10. Finite coefficients	22
0.11. Odd primes	23
0.12. Adams charts	23
Part 1. The Adams E_2-term	43
Chapter 1. Minimal resolutions	45
1.1. The Adams E_2 -term for S	45
1.2. The Adams E_2 -term for tmf	64
1.3. Steenrod operations in $E_2(tmf)$	76
1.4. The Adams E_2 -term for $tmf/2$, tmf/η and tmf/ν	81
Chapter 2. The Davis–Mahowald spectral sequence	97
2.1. Ext over a pair of Hopf algebras	97
2.2. A dual formulation	99
2.3. A filtered cobar complex	103
2.4. Multiplicative structure	107
2.5. The spectral sequence for $A(1)$	112
2.6. Real, quaternionic and complex K -theory spectra	114
Chapter 3. Ext over $A(2)$	123
3.1. The Davis–Mahowald E_1 -term for $A(2)$	123
3.2. Syzygies and Adams covers	126
3.3. A comparison of $A(1)_*$ -comodule algebras	132
3.4. The d_1 -differential for $A(2)$	137

3.5. The Shimada–Iwai presentation	148
Chapter 4. Ext with coefficients	159
4.1. Coefficients in M_1	159
4.2. Adams periodicity	164
4.3. Coefficients in M_2	169
4.4. Coefficients in M_4	174
Part 2. The Adams differentials	183
Chapter 5. The Adams spectral sequence for tmf	185
5.1. The E_2 -term for tmf	185
5.2. The d_2 -differentials for tmf	187
5.3. The d_3 -differentials for tmf	192
5.4. The d_4 -differentials for tmf	197
5.5. The E_∞ -term for tmf	208
Chapter 6. The Adams spectral sequence for $tmf/2$	219
6.1. The E_2 -term for $tmf/2$	219
6.2. The d_2 -differentials for $tmf/2$	221
6.3. The d_3 -differentials for $tmf/2$	223
6.4. The d_4 -differentials for $tmf/2$	228
6.5. The E_∞ -term for $tmf/2$	236
Chapter 7. The Adams spectral sequence for tmf/η	247
7.1. The E_2 -term for tmf/η	247
7.2. The d_2 -differentials for tmf/η	251
7.3. The d_3 -differentials for tmf/η	252
7.4. The E_∞ -term for tmf/η	256
Chapter 8. The Adams spectral sequence for tmf/ν	269
8.1. The E_2 -term for tmf/ν	269
8.2. The d_2 -differentials for tmf/ν	272
8.3. The d_3 -differentials for tmf/ν	274
8.4. The d_4 -differentials for tmf/ν	280
8.5. The E_∞ -term for tmf/ν	291
Part 3. The abutment	303
Chapter 9. The homotopy groups of tmf	305
9.1. Algebra generators for the E_∞ -term	307
9.2. Hidden extensions	314
9.3. The image of $\pi_*(tmf)$ in modular forms	329
9.4. Algebra generators for $\pi_*(tmf)$	334
9.5. Relations in $\pi_*(tmf)$	343
9.6. The algebra structure of $\pi_*(tmf)$	366
Chapter 10. Duality	377
10.1. Pontryagin duality in the B -power torsion of $\pi_*(tmf)$	377
10.2. Torsion submodules and divisible quotients	380
10.3. Brown–Comenetz duality	381

10.4.	Anderson duality	385
10.5.	Explicit formulas	387
Chapter 11.	The Adams spectral sequence for the sphere	401
11.1.	H_∞ ring spectra	402
11.2.	Steenrod operations in $E_2(S)$	419
11.3.	The Adams d - and e -invariants	428
11.4.	Some d_2 -differentials for S	437
11.5.	Some d_3 -differentials for S	444
11.6.	Some d_4 -differentials for S	451
11.7.	Collapse at E_5	457
11.8.	Some homotopy groups of S	459
11.9.	A hidden η -extension	481
11.10.	The tmf -Hurewicz homomorphism	486
11.11.	The tmf -Hurewicz image	497
Chapter 12.	Homotopy of some finite cell tmf -modules	503
12.1.	Homotopy of $tmf/2$	503
12.2.	Homotopy of tmf/η	515
12.3.	Homotopy of tmf/ν	523
12.4.	Homotopy of tmf/B	532
12.5.	Homotopy of $tmf/(2, B)$	543
12.6.	Modified Adams spectral sequences	557
Chapter 13.	Odd primes	575
13.1.	The tmf -module Steenrod algebra and its dual	577
13.2.	The Adams E_2 -term	581
13.3.	The Adams differentials	583
13.4.	The graded ring $\pi_*(tmf)$	583
13.5.	Brown–Comenetz and Anderson duality	590
13.6.	Explicit formulas	591
13.7.	The tmf -Hurewicz image	593
Appendix A.	Calculation of $E_r(tmf)$ for $r = 3, 4, 5$	597
A.1.	Calculation of $E_3(tmf) = H(E_2(tmf), d_2)$	597
A.2.	Calculation of $E_4(tmf) = H(E_3(tmf), d_3)$	603
A.3.	Calculation of $E_5(tmf) = H(E_4(tmf), d_4)$	608
Appendix B.	Calculation of $E_r(tmf/2)$ for $r = 3, 4, 5$	617
B.1.	Calculation of $E_3(tmf/2) = H(E_2(tmf/2), d_2)$	617
B.2.	Calculation of $E_4(tmf/2) = H(E_3(tmf/2), d_3)$	622
B.3.	Calculation of $E_5(tmf/2) = H(E_4(tmf/2), d_4)$	628
Appendix C.	Calculation of $E_r(tmf/\eta)$ for $r = 3, 4$	637
C.1.	Calculation of $E_3(tmf/\eta) = H(E_2(tmf/\eta), d_2)$	637
C.2.	Calculation of $E_4(tmf/\eta) = H(E_3(tmf/\eta), d_3)$	644
Appendix D.	Calculation of $E_r(tmf/\nu)$ for $r = 3, 4, 5$	651
D.1.	Calculation of $E_3(tmf/\nu) = H(E_2(tmf/\nu), d_2)$	651
D.2.	Calculation of $E_4(tmf/\nu) = H(E_3(tmf/\nu), d_3)$	656
D.3.	Calculation of $E_5(tmf/\nu) = H(E_4(tmf/\nu), d_4)$	663

Bibliography	675
Index	683

List of Figures

0.1	($E_2(tmf), d_2$) for $t - s \leq 48$	24
0.2	($E_3(tmf), d_3$) for $t - s \leq 48$	25
0.3	($E_4(tmf), d_4$) for $t - s \leq 48$	26
0.4	$E_5(tmf) = E_\infty(tmf)$ for $t - s \leq 48$	27
0.5	$E_2^{s,t}(S) = \text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $t \leq 200$	28
0.6	$E_2^{s,t}(tmf) = \text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 200$	29
0.7	$E_\infty^{s,t}(tmf) \implies \pi_{t-s}(tmf)$ for $0 \leq t - s \leq 200$	30
0.8	$E_\infty^{s,t}(tmf)$ for $0 \leq t - s \leq 96$ and $96 \leq t - s \leq 192$	31
0.9	$\pi_n(tmf)$ for $0 \leq n \leq 200$	32
0.10	$\pi_n(tmf)$ for $0 \leq n \leq 96$ and $96 \leq n \leq 192$	33
0.11	$E_2^{s,t}(tmf/2) = \text{Ext}_{A(2)}^{s,t}(M_1, \mathbb{F}_2)$ for $t - s \leq 200$	34
0.12	$E_\infty^{s,t}(tmf/2) \implies \pi_{t-s}(tmf/2)$ for $0 \leq t - s \leq 200$	35
0.13	$E_\infty^{s,t}(tmf/2)$ for $0 \leq t - s \leq 96$ and $96 \leq t - s \leq 192$	36
0.14	$E_2^{s,t}(tmf/\eta) = \text{Ext}_{A(2)}^{s,t}(M_2, \mathbb{F}_2)$ for $t - s \leq 200$	37
0.15	$E_\infty^{s,t}(tmf/\eta) \implies \pi_{t-s}(tmf/\eta)$ for $0 \leq t - s \leq 200$	38
0.16	$E_\infty^{s,t}(tmf/\eta)$ for $0 \leq t - s \leq 96$ and $96 \leq t - s \leq 192$	39
0.17	$E_2^{s,t}(tmf/\nu) = \text{Ext}_{A(2)}^{s,t}(M_4, \mathbb{F}_2)$ for $t - s \leq 200$	40
0.18	$E_\infty^{s,t}(tmf/\nu) \implies \pi_{t-s}(tmf/\nu)$ for $0 \leq t - s \leq 200$	41
0.19	$E_\infty^{s,t}(tmf/\nu)$ for $0 \leq t - s \leq 96$ and $96 \leq t - s \leq 192$	42
1.1	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	46
1.2	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	47
1.3	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $48 \leq t - s \leq 72$	48
1.4	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $72 \leq t - s \leq 96$	49
1.5	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $96 \leq t - s \leq 120$	50
1.6	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $120 \leq t - s \leq 144$	51
1.7	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $144 \leq t - s \leq 168, t \leq 200$	52
1.8	$\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $168 \leq t - s \leq 200, t \leq 200$	53
1.9	Indecomposables in $\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	56
1.10	Indecomposables in $\text{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	57

1.11	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	66
1.12	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	66
1.13	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $48 \leq t - s \leq 72$	67
1.14	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $72 \leq t - s \leq 96$	67
1.15	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $96 \leq t - s \leq 120$	68
1.16	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $120 \leq t - s \leq 144$	68
1.17	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $144 \leq t - s \leq 168$	69
1.18	$\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $168 \leq t - s \leq 192$	69
1.19	Indecomposables in $\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	71
1.20	Indecomposables in $\text{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	71
1.21	Part of a chain map $E_* \rightarrow C_*$, showing that $\iota'(w_1) = v_1^4$	73
1.22	A 3-fold extension K_* representing c_0	79
1.23	Chain map $D_0: C_* \rightarrow K_* \otimes K_*$ covering \mathbb{F}_2	80
1.24	$\text{Ext}_{A(2)}^{s,t}(M_1, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	90
1.25	$\text{Ext}_{A(2)}^{s,t}(M_1, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	90
1.26	$\text{Ext}_{A(2)}^{s,t}(M_1, \mathbb{F}_2)$ for $48 \leq t - s \leq 72$	91
1.27	$\text{Ext}_{A(2)}^{s,t}(M_1, \mathbb{F}_2)$ for $72 \leq t - s \leq 96$	91
1.28	$\text{Ext}_{A(2)}^{s,t}(M_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	92
1.29	$\text{Ext}_{A(2)}^{s,t}(M_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	92
1.30	$\text{Ext}_{A(2)}^{s,t}(M_2, \mathbb{F}_2)$ for $48 \leq t - s \leq 72$	93
1.31	$\text{Ext}_{A(2)}^{s,t}(M_2, \mathbb{F}_2)$ for $72 \leq t - s \leq 96$	93
1.32	$\text{Ext}_{A(2)}^{s,t}(M_4, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	94
1.33	$\text{Ext}_{A(2)}^{s,t}(M_4, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	94
1.34	$\text{Ext}_{A(2)}^{s,t}(M_4, \mathbb{F}_2)$ for $48 \leq t - s \leq 72$	95
1.35	$\text{Ext}_{A(2)}^{s,t}(M_4, \mathbb{F}_2)$ for $72 \leq t - s \leq 96$	95
2.1	(E_1, d_1) -term of Davis–Mahowald spectral sequence for $A(1)$	115
2.2	$E_2 = E_\infty$ -term of Davis–Mahowald spectral sequence for $A(1)$	115
2.3	E_2 -term of Adams spectral sequence for ko	116
2.4	Spliced $A(1)$ -extensions	118
2.5	E_2 -term of Adams spectral sequence for ksp	119
2.6	E_2 -term of Adams spectral sequence for ku	120
3.1	The syzygy $\Omega_{E(1)_*}^\sigma(\mathbb{F}_2)$ for $\sigma = 3$	127
3.2	The syzygy $\Omega_{A(1)_*}^\sigma(E(\xi_1^2))$ for $\sigma = 3$	128

3.3	E_2 -term of Adams spectral sequence for $ku\langle\sigma\rangle$ for $\sigma = 3$	129
3.4	$\phi^1: \bar{R}^1 \rightarrow \Sigma^3\Omega_{A(1)*}^1(E(\xi_1^2))$ and its cokernel ψ^1	133
3.5	$\phi^2: \bar{R}^2 \rightarrow \Sigma^6\Omega_{A(1)*}^2(E(\xi_1^2))$ and its cokernel ψ^2	135
3.6	The Adams chart $G\langle\sigma\rangle^{*,*}$ for $\sigma = 3$	135
3.7	$(\bar{E}_1^{\sigma,*,*}, d_1^\sigma)$ and $\bar{E}_2^{\sigma,*,*}$ for $0 \leq \sigma \leq 3$	140
3.8	$(\bar{E}_1^{\sigma,*,*}, d_1^\sigma)$ and $\bar{E}_2^{\sigma,*,*}$ for $4 \leq \sigma \leq 7$	141
3.9	Schematic view of the Davis–Mahowald (E_1, d_1) -term	146
3.10	Connecting homomorphism $\delta: \bar{E}_2^{\sigma-4,*,*}\{x_7^4\} \rightarrow \bar{E}_2^{\sigma+1,*,*}$	147
3.11	$\mathbb{F}_2[w_1, x_7^8]$ -basis for $E_2^{*,*,*} = E_\infty^{*,*,*} \implies \text{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$	147
3.12	R_0 -module generators of $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t - s \leq 24$	155
3.13	R_0 -module generators of $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ for $24 \leq t - s \leq 48$	155
3.14	The (first) Mahowald–Tangora wedge in $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$	158
4.1	R_0 -module generators of $\text{Ext}_{A(2)}(M_1, \mathbb{F}_2)$	165
4.2	R_0 -module generators of $\text{Ext}_{A(2)}(M_2, \mathbb{F}_2)$	175
4.3	R_0 -module generators of $\text{Ext}_{A(2)}(M_4, \mathbb{F}_2)$	182
5.1	$E_5(tmf) = E_\infty(tmf)$ for $0 \leq t - s \leq 24$	214
5.2	$E_5(tmf) = E_\infty(tmf)$ for $24 \leq t - s \leq 48$	214
5.3	$E_5(tmf) = E_\infty(tmf)$ for $48 \leq t - s \leq 72$	215
5.4	$E_5(tmf) = E_\infty(tmf)$ for $72 \leq t - s \leq 96$	215
5.5	$E_5(tmf) = E_\infty(tmf)$ for $96 \leq t - s \leq 120$	216
5.6	$E_5(tmf) = E_\infty(tmf)$ for $120 \leq t - s \leq 144$	216
5.7	$E_5(tmf) = E_\infty(tmf)$ for $144 \leq t - s \leq 168$	217
5.8	$E_5(tmf) = E_\infty(tmf)$ for $168 \leq t - s \leq 192$	217
6.1	$E_5(tmf/2) = E_\infty(tmf/2)$ for $0 \leq t - s \leq 24$	242
6.2	$E_5(tmf/2) = E_\infty(tmf/2)$ for $24 \leq t - s \leq 48$	242
6.3	$E_5(tmf/2) = E_\infty(tmf/2)$ for $48 \leq t - s \leq 72$	243
6.4	$E_5(tmf/2) = E_\infty(tmf/2)$ for $72 \leq t - s \leq 96$	243
6.5	$E_5(tmf/2) = E_\infty(tmf/2)$ for $96 \leq t - s \leq 120$	244
6.6	$E_5(tmf/2) = E_\infty(tmf/2)$ for $120 \leq t - s \leq 144$	244
6.7	$E_5(tmf/2) = E_\infty(tmf/2)$ for $144 \leq t - s \leq 168$	245
6.8	$E_5(tmf/2) = E_\infty(tmf/2)$ for $168 \leq t - s \leq 192$	245
7.1	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $0 \leq t - s \leq 24$	262
7.2	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $24 \leq t - s \leq 48$	262
7.3	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $48 \leq t - s \leq 72$	263
7.4	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $72 \leq t - s \leq 96$	263
7.5	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $96 \leq t - s \leq 120$	264
7.6	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $120 \leq t - s \leq 144$	264

7.7	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $144 \leq t - s \leq 168$	265
7.8	$E_4(tmf/\eta) = E_\infty(tmf/\eta)$ for $168 \leq t - s \leq 192$	265
8.1	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $0 \leq t - s \leq 24$	298
8.2	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $24 \leq t - s \leq 48$	298
8.3	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $48 \leq t - s \leq 72$	299
8.4	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $72 \leq t - s \leq 96$	299
8.5	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $96 \leq t - s \leq 120$	300
8.6	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $120 \leq t - s \leq 144$	300
8.7	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $144 \leq t - s \leq 168$	301
8.8	$E_5(tmf/\nu) = E_\infty(tmf/\nu)$ for $168 \leq t - s \leq 192$	301
9.1	\mathbb{Z}_2 -algebra generators of $\pi_*(tmf)$	305
9.2	Δ' on E_2 -classes, connecting the η - and ν -families	313
9.3	Δ' on homotopy classes, connecting the η - and ν -families	314
9.4	Products of the ν_i , with ν_i chosen independently	345
9.5	Products of the ν_i , with specified ν_5 and ν_6 and $s \in \{\pm 1\}$	345
9.6	$\pi_n(tmf)$ for $0 \leq n \leq 24$	368
9.7	$\pi_n(tmf)$ for $24 \leq n \leq 48$	368
9.8	$\pi_n(tmf)$ for $48 \leq n \leq 72$	369
9.9	$\pi_n(tmf)$ for $72 \leq n \leq 96$	369
9.10	$\pi_n(tmf)$ for $96 \leq n \leq 120$	370
9.11	$\pi_n(tmf)$ for $120 \leq n \leq 144$	370
9.12	$\pi_n(tmf)$ for $144 \leq n \leq 168$	371
9.13	$\pi_n(tmf)$ for $168 \leq n \leq 192$	371
10.1	The self-dual submodule $\Theta\pi_n(tmf)$ for $4 \leq n, 170 - n \leq 46$	378
10.2	The self-dual submodule $\Theta\pi_n(tmf)$ for $52 \leq n, 170 - n \leq 94$	379
11.1	Factorization of power operation $\alpha^*(y)$	411
11.2	Delayed and ordinary Adams spectral sequences for $\pi_*(\Sigma^n P_n^{n+1})$	415
11.3	Maps from two hemispheres	419
11.4	Delayed Adams spectral sequence for $\pi_*(\Sigma^7 P_7^\infty)$	424
11.5	Adams spectral sequence for $\pi_*(\Sigma^7 P_7^\infty)$	425
11.6	Delayed E_∞ -term for $\pi_*(\Sigma^8 P_8^\infty)$	427
11.7	Adams spectral sequence for $\pi_*(\Sigma^8 P_8^\infty)$	427
11.8	$E_2(Cp) = E_\infty(Cp)$ for $t - s \leq 16$, with vanishing and periodicity range	432
11.9	$(E_2(j), d_2)$ for $t - s \leq 24$	437
11.10	$(E_2(S), d_2)$ for $t - s \leq 48$	438
11.11	$(E_3(S), d_3)$ for $t - s \leq 48$	439
11.12	$(E_4(S), d_4)$ for $t - s \leq 48$	440

11.13	$E_5(S) = E_\infty(S)$ for $t - s \leq 48$	441
11.14	$\pi_n(S)$ for $n \leq 48$	442
11.15	$(E_2(C\sigma), d_2)$ for $12 \leq t - s \leq 24$	447
11.16	$E_2(C\nu)$ for $32 \leq t - s \leq 44$, with some d_2 -differentials	449
11.17	$E_2(\Sigma^8 C(2\sigma))$ for $20 \leq t - s \leq 32$, with some d_2 -differentials	453
11.18	$E_2(C\sigma \cup_{2\sigma} e^{16})$ for $20 \leq t - s \leq 40$, with some d_2 - and d_3 -differentials	455
11.19	$E_2(C\sigma)$ for $36 \leq t - s \leq 40$, with some d_3 -differentials	456
11.20	$E_2(C\eta)$ for $36 \leq t - s \leq 40$, with some d_4 -differentials	474
11.21	$(E_2(C\sigma), d_2)$ for $40 \leq t - s \leq 44$	476
11.22	$(E_2(C\eta), d_2)$ for $12 \leq t - s \leq 24$	481
11.23	$(E_3(C\eta), d_3)$ for $12 \leq t - s \leq 24$	483
11.24	$(E_2(C\eta \wedge C\nu), d_2)$ for $12 \leq t - s \leq 24$	485
11.25	$E_2(C\eta \wedge C\nu)$ for $52 \leq t - s \leq 56$, $8 \leq s \leq 16$, with one d^2 -differential	486
11.26	$(E_2(tmf/S), d_2)$ for $44 \leq t - s \leq 52$, $12 \leq s \leq 16$	487
11.27	$(E_2(tmf/S), d_2)$ for $t - s \leq 48$	488
11.28	$(E_3(tmf/S), d_3)$ for $t - s \leq 48$	489
11.29	$(E_4(tmf/S), d_4)$ for $t - s \leq 48$	490
11.30	$E_5(tmf/S) = E_\infty(tmf/S)$ for $t - s \leq 48$	491
12.1	$E_\infty(tmf/2)$ for $0 \leq t - s \leq 24$, with hidden extensions	504
12.2	$E_\infty(tmf/2)$ for $24 \leq t - s \leq 48$, with hidden extensions	504
12.3	$E_\infty(tmf/2)$ for $48 \leq t - s \leq 72$, with hidden extensions	505
12.4	$E_\infty(tmf/2)$ for $72 \leq t - s \leq 96$, with hidden extensions	505
12.5	$E_\infty(tmf/2)$ for $96 \leq t - s \leq 120$, with hidden extensions	506
12.6	$E_\infty(tmf/2)$ for $120 \leq t - s \leq 144$, with hidden extensions	506
12.7	$E_\infty(tmf/2)$ for $144 \leq t - s \leq 168$, with hidden extensions	507
12.8	$E_\infty(tmf/2)$ for $168 \leq t - s \leq 192$, with hidden extensions	507
12.9	$E_\infty(tmf/\eta)$ for $0 \leq t - s \leq 24$, with hidden extensions	516
12.10	$E_\infty(tmf/\eta)$ for $24 \leq t - s \leq 48$, with hidden extensions	516
12.11	$E_\infty(tmf/\eta)$ for $48 \leq t - s \leq 72$, with hidden extensions	517
12.12	$E_\infty(tmf/\eta)$ for $72 \leq t - s \leq 96$, with hidden extensions	517
12.13	$E_\infty(tmf/\eta)$ for $96 \leq t - s \leq 120$, with hidden extensions	518
12.14	$E_\infty(tmf/\eta)$ for $120 \leq t - s \leq 144$, with hidden extensions	518
12.15	$E_\infty(tmf/\eta)$ for $144 \leq t - s \leq 168$, with hidden extensions	519
12.16	$E_\infty(tmf/\eta)$ for $168 \leq t - s \leq 192$, with hidden extensions	519
12.17	$E_\infty(tmf/\nu)$ for $0 \leq t - s \leq 24$, with hidden extensions	524
12.18	$E_\infty(tmf/\nu)$ for $24 \leq t - s \leq 48$, with hidden extensions	524
12.19	$E_\infty(tmf/\nu)$ for $48 \leq t - s \leq 72$, with hidden extensions	525
12.20	$E_\infty(tmf/\nu)$ for $72 \leq t - s \leq 96$, with hidden extensions	525
12.21	$E_\infty(tmf/\nu)$ for $96 \leq t - s \leq 120$, with hidden extensions	526

12.22	$E_\infty(tmf/\nu)$ for $120 \leq t - s \leq 144$, with hidden extensions	526
12.23	$E_\infty(tmf/\nu)$ for $144 \leq t - s \leq 168$, with hidden extensions	527
12.24	$E_\infty(tmf/\nu)$ for $168 \leq t - s \leq 192$, with hidden extensions	527
12.25	Delayed $E_\infty(tmf/(B, M))$ for $0 \leq t - s \leq 24$	534
12.26	Delayed $E_\infty(tmf/(B, M))$ for $24 \leq t - s \leq 48$	534
12.27	Delayed $E_\infty(tmf/(B, M))$ for $48 \leq t - s \leq 72$	535
12.28	Delayed $E_\infty(tmf/(B, M))$ for $72 \leq t - s \leq 96$	535
12.29	Delayed $E_\infty(tmf/(B, M))$ for $96 \leq t - s \leq 120$	536
12.30	Delayed $E_\infty(tmf/(B, M))$ for $120 \leq t - s \leq 144$	536
12.31	Delayed $E_\infty(tmf/(B, M))$ for $144 \leq t - s \leq 168$	537
12.32	Delayed $E_\infty(tmf/(B, M))$ for $168 \leq t - s \leq 192$	537
12.33	Delayed $E_\infty(tmf/(2, B, M))$ for $0 \leq t - s \leq 24$	544
12.34	Delayed $E_\infty(tmf/(2, B, M))$ for $24 \leq t - s \leq 48$	544
12.35	Delayed $E_\infty(tmf/(2, B, M))$ for $48 \leq t - s \leq 72$	545
12.36	Delayed $E_\infty(tmf/(2, B, M))$ for $72 \leq t - s \leq 96$	545
12.37	Delayed $E_\infty(tmf/(2, B, M))$ for $96 \leq t - s \leq 120$	546
12.38	Delayed $E_\infty(tmf/(2, B, M))$ for $120 \leq t - s \leq 144$	546
12.39	Delayed $E_\infty(tmf/(2, B, M))$ for $144 \leq t - s \leq 168$	547
12.40	Delayed $E_\infty(tmf/(2, B, M))$ for $168 \leq t - s \leq 192$	547
12.41	Hastened $(E_r(tmf/B), d_r)$ for $0 \leq t - s \leq 24$	564
12.42	Hastened $(E_r(tmf/B), d_r)$ for $24 \leq t - s \leq 48$	564
12.43	Hastened $(E_r(tmf/B), d_r)$ for $48 \leq t - s \leq 72$	565
12.44	Hastened $(E_r(tmf/B), d_r)$ for $72 \leq t - s \leq 96$	565
12.45	Hastened $(E_r(tmf/B), d_r)$ for $96 \leq t - s \leq 120$	566
12.46	Hastened $(E_r(tmf/B), d_r)$ for $120 \leq t - s \leq 144$	566
12.47	Hastened $(E_r(tmf/B), d_r)$ for $144 \leq t - s \leq 168$	567
12.48	Hastened $(E_r(tmf/B), d_r)$ for $168 \leq t - s \leq 192$	567
12.49	Hastened $(E_r(tmf/(2, B)), d_r)$ for $0 \leq t - s \leq 24$	570
12.50	Hastened $(E_r(tmf/(2, B)), d_r)$ for $24 \leq t - s \leq 48$	570
12.51	Hastened $(E_r(tmf/(2, B)), d_r)$ for $48 \leq t - s \leq 72$	571
12.52	Hastened $(E_r(tmf/(2, B)), d_r)$ for $72 \leq t - s \leq 96$	571
12.53	Hastened $(E_r(tmf/(2, B)), d_r)$ for $96 \leq t - s \leq 120$	572
12.54	Hastened $(E_r(tmf/(2, B)), d_r)$ for $120 \leq t - s \leq 144$	572
12.55	Hastened $(E_r(tmf/(2, B)), d_r)$ for $144 \leq t - s \leq 168$	573
12.56	Hastened $(E_r(tmf/(2, B)), d_r)$ for $168 \leq t - s \leq 192$	573
13.1	$E_2^{s,t}(tmf) \implies_s \pi_{t-s}(tmf)$ at $p = 3$ for $0 \leq t - s \leq 72$	584
13.2	$\pi_n(tmf)$ at $p = 3$ for $0 \leq n \leq 72$	585

List of Tables

1.1	Algebra indecomposables in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ for $t - s \leq 48$	54
1.2	Minimal free A -module resolution (C_*, ∂) of \mathbb{F}_2 for $s \leq 6$ and $t \leq 22$	60
1.3	Algebra indecomposables in $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ (for $t - s \leq 200$)	65
1.4	Minimal free $A(2)$ -module resolution (C_*, ∂) of \mathbb{F}_2 for $s \leq 6$ and $t \leq 22$	74
1.5	$\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ -module generators for $\text{Ext}_{A(2)}(M_1, \mathbb{F}_2)$ (for $t - s \leq 200$)	83
1.6	$\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ -module generators for $\text{Ext}_{A(2)}(M_2, \mathbb{F}_2)$ (for $t - s \leq 200$)	84
1.7	$\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ -module generators for $\text{Ext}_{A(2)}(M_4, \mathbb{F}_2)$ (for $t - s \leq 200$)	85
3.1	$\mathbb{F}_2[w_1]$ -basis for $\bar{E}_2^{*,*,*}$ (x_6^4 -periodic for $\sigma \geq 3$)	143
3.2	$\mathbb{F}_2[w_1, x_7^8]$ -basis for $E_2^{*,*,*}$ (x_6^4 -periodic for $\sigma \geq 7$)	144
3.3	Generators of $SI \cong \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$	148
3.4	Relations in $SI \cong \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$	149
3.5	Gröbner basis for the Shimada–Iwai relations	151
3.6	R_0 -module generators of $SI \cong \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$	153
4.1	Direct sum decompositions of kernel and cokernel of h_0 -multiplication	160
4.2	R_0 -module generators of $\text{Ext}_{A(2)}(M_1, \mathbb{F}_2)$	162
4.3	The non-cyclic R_0 -module summand in $\text{Ext}_{A(2)}(M_1, \mathbb{F}_2)$	164
4.4	Direct sum decompositions of kernel and cokernel of h_1 -multiplication	170
4.5	R_0 -module generators of $\text{Ext}_{A(2)}(M_2, \mathbb{F}_2)$	172
4.6	Direct sum decompositions of kernel and cokernel of h_2 -multiplication	176
4.7	R_0 -module generators of $\text{Ext}_{A(2)}(M_4, \mathbb{F}_2)$	179
5.1	R_0 -module generators of $E_2(tmf)$	186
5.2	R_1 -module generators of $E_3(tmf)$	193
5.3	The non-cyclic R_1 -module summands in $E_3(tmf)$	195
5.4	Algebra generators of $E_3(tmf)$	196
5.5	R_2 -module generators of $E_4(tmf)$	198
5.6	The non-cyclic R_2 -module summands in $E_4(tmf)$	202
5.7	Algebra generators of $E_4(tmf)$	202
5.8	R_2 -module generators of $E_5(tmf)$	208
5.9	The non-cyclic R_2 -module summands in $E_5(tmf)$	212
5.10	Algebra generators of $E_5(tmf) = E_\infty(tmf)$	213

6.1	$E_2(tmf)$ -module generators of $E_2(tmf/2)$	219
6.2	R_0 -module generators of $E_2(tmf/2)$	220
6.3	The non-cyclic R_0 -module summand in $E_2(tmf/2)$	221
6.4	R_1 -module generators of $E_3(tmf/2)$	224
6.5	The non-cyclic R_1 -module summands in $E_3(tmf/2)$	226
6.6	$E_3(tmf)$ -module generators of $E_3(tmf/2)$	226
6.7	R_2 -module generators of $E_4(tmf/2)$	229
6.8	The non-cyclic R_2 -module summands in $E_4(tmf/2)$	233
6.9	$E_4(tmf)$ -module generators of $E_4(tmf/2)$	234
6.10	R_2 -module generators of $E_5(tmf/2)$	237
6.11	The non-cyclic R_2 -module summands in $E_5(tmf/2)$	240
6.12	$E_5(tmf)$ -module generators of $E_5(tmf/2)$	241
7.1	$E_2(tmf)$ -module generators of $E_2(tmf/\eta)$	248
7.2	R_0 -module generators of $E_2(tmf/\eta)$	248
7.3	R_1 -module generators of $E_3(tmf/\eta)$	252
7.4	The non-cyclic R_1 -module summand in $E_3(tmf/\eta)$	254
7.5	R_2 -module generators of $E_4(tmf/\eta) = E_\infty(tmf/\eta)$	256
7.6	The non-cyclic R_2 -module summands in $E_4(tmf/\eta)$	261
7.7	$E_\infty(tmf)$ -module generators of $E_\infty(tmf/\eta)$	266
8.1	$E_2(tmf)$ -module generators of $E_2(tmf/\nu)$	269
8.2	R_0 -module generators of $E_2(tmf/\nu)$	270
8.3	R_1 -module generators of $E_3(tmf/\nu)$	274
8.4	The non-cyclic R_1 -module summands in $E_3(tmf/\nu)$	277
8.5	$E_3(tmf)$ -module generators of $E_3(tmf/\nu)$	277
8.6	R_2 -module generators of $E_4(tmf/\nu)$	280
8.7	The non-cyclic R_2 -module summands in $E_4(tmf/\nu)$	285
8.8	$E_4(tmf)$ -module generators of $E_4(tmf/\nu)$	286
8.9	R_2 -module generators of $E_5(tmf/\nu)$	291
8.10	$E_5(tmf)$ -module generators of $E_5(tmf/\nu)$	296
9.1	Algebra generators of $E_\infty(tmf)$ and $\pi_*(tmf)$	307
9.2	Δ and Δ' on certain decomposable elements of $E_\infty(tmf)$	309
9.3	w_1 -power torsion in $E_\infty(tmf)$	319
9.4	B -power torsion in $\pi_n(tmf)$ for $0 \leq n < 192$	338
9.5	T -module generators of $\pi_*(tmf)$	341
9.6	Preliminary products in $\pi_*(tmf)$. Part 1 of 2: η_i - and ν_i -multiples	359
9.7	Preliminary products in $\pi_*(tmf)$. Part 2 of 2: ϵ_i -, κ_i - and $\bar{\kappa}$ -multiples	362
9.8	Products in $\pi_*(tmf)$. Part 1 of 2: η_i - and ν_i -multiples	373

9.9	Products in $\pi_*(tmf)$. Part 2 of 2: ϵ_i -, κ_i - and $\bar{\kappa}$ -multiples	374
10.1	Duality pairing in ΘN_*	399
11.1	Algebra indecomposables in $E_3(S)$ for $t - s \leq 48$	445
11.2	Algebra indecomposables in $E_4(S)$ for $t - s \leq 48$	452
11.3	Algebra indecomposables in $E_5(S) = E_\infty(S)$ for $t - s \leq 48$	458
11.4	$E_2(S)$ -module generators of $E_2(C\eta)$ for $t - s \leq 24$	482
11.5	$E_3(S)$ -module generators of $E_3(C\eta)$ for $t - s \leq 24$	484
11.6	$E_2(S)$ -module generators of $E_2(tm f/S)$ for $t - s \leq 48$	493
11.7	$E_3(S)$ -module generators of $E_3(tm f/S)$ for $t - s \leq 48$	495
11.8	$E_4(S)$ -module generators of $E_4(tm f/S)$ for $t - s \leq 48$	496
13.1	Algebra generators of $E_\infty(tm f)$ and $\pi_*(tm f)$ at $p = 3$	586
13.2	Products in $\pi_*(tm f)$	589
A.1	Summands in $(E_2(tm f), d_2)$	597
A.2	Summands in $(E_3(tm f), d_3)$	603
A.3	Summands in $(E_4(tm f), d_4)$	608
B.1	Summands in $(E_2(tm f/2), d_2)$	617
B.2	Two-term complexes $a: R_1\{x\} \rightarrow R_1\{y\}$ in $E_2(tm f/2)$	619
B.3	Summands in $(E_3(tm f/2), d_3)$	622
B.4	Two-term complexes $gw_1: R_2/(g^2)\{x\} \rightarrow R_2/(g^2)\{y\}$ in $E_3(tm f/2)$	624
B.5	Summands in $(E_4(tm f/2), d_4)$	628
B.6	Summands of type (B)	633
B.7	Summands of type (K)	635
C.1	Summands in $(E_2(tm f/\eta), d_2)$	637
C.2	Complexes (A1)–(A12) in $E_2(tm f/\eta)$	640
C.3	Nonzero homology of the complexes (A1)–(A12) in $E_2(tm f/\eta)$	640
C.4	Complexes (B1)–(B6) in $E_2(tm f/\eta)$	641
C.5	Complexes (F1)–(F6) in $E_2(tm f/\eta)$	643
C.6	Generators of the homology of the complexes (F1)–(F6) in $E_2(tm f/\eta)$	643
C.7	Complexes (G1)–(G8) in $E_2(tm f/\eta)$	643
C.8	Summands in $(E_3(tm f/\eta), d_3)$	644
C.9	Complexes (C1)–(C4) in $E_3(tm f/\eta)$	648
D.1	Summands in $(E_2(tm f/\nu), d_2)$	651
D.2	Summands of type (A)	653
D.3	Summands of type (B)	654
D.4	Summands in $(E_3(tm f/\nu), d_3)$	657
D.5	Summands of type (C)	660

D.6 Summands of type (E)	661
D.7 Summands in $(E_4(tm\mathcal{f}/\nu), d_4)$	663