Preface

This book is about maximal functions and their applications in Sobolev spaces. There are many good texts on the use of maximal functions in harmonic analysis, but we feel that there is room for a source book gathering advances in maximal function methods related to Poincaré and Sobolev inequalities, pointwise estimates and approximation for Sobolev functions, Hardy's inequalities, and partial differential equations. A recurring theme throughout the book is self-improvement of uniform quantitative conditions. Our approach is partially motivated by the theory of analysis on metric measure spaces, but in order to avoid extra complications we restrict our attention only to prototypes in Euclidean space. Nevertheless, the methods applied in analysis on metric measure spaces are useful already in the Euclidean context.

Besides maximal functions and Sobolev spaces, we discuss several related concepts. Capacities are needed for the study of fine properties of Sobolev functions and characterization of Sobolev spaces with zero boundary values. The capacity density condition is applied for Hardy's inequalities and in partial differential equations. In addition to the Hausdorff dimension, we use the Assouad dimension and the lower dimension to characterize density conditions and to give sufficient and necessary conditions for Hardy's inequalities and their generalizations. The distance function appears in Hardy's inequality and also has applications in Sobolev spaces. We study the Muckenhoupt weight properties of distance functions and combine these with general weighted norm inequalities. At the end of the book we discuss the theory of weak solutions to the p-Laplace equation and show how maximal function techniques can be used in this context. The choice of topics is exclusive and reflects the research interests of the authors.

Our style is concise and brief. Instead of lengthy motivations, we give detailed proofs in order to make the arguments flexible and transparent. For this reason some of the proofs are relatively long. We demonstrate interesting techniques, with the idea that the methods will have a wider range of applications beyond the topics covered by this book. This is not always the most direct approach and causes some overlap. Part of the material is rather standard, while some results appear for the first time in book form. One of our goals was to gather material that has been scattered in research papers and make it accessible to a wider audience. Standard references and related research papers are mentioned in the notes at the end of each chapter. We hope that the list of references, which is long but not complete, will make it easier for the reader to further investigate the literature.

We have primarily aimed the book for graduate students, but we also believe that it can be used as a reference by researchers. Some parts of the material have been used by the authors in graduate courses. Most of the book is self-contained, only assuming knowledge of measure and integration theory, in particular the Lebesgue measure and L^p spaces, as well as some functional analysis. We have made an effort to arrange the material so that the chapters can be read relatively independently.

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