Preface

Perverse sheaves were discovered in the fall of 1980 by Beilinson–Bernstein–Deligne–Gabber [**24**], sitting at the confluence of two major developments of the 1970s: the intersection homology theory of Goresky–MacPherson [**85**, **86**], and the Riemann–Hilbert correspondence, due to Kashiwara [**121**] and Mebkhout [**175**]. Those same two ingredients had already been combined a few months prior for a breakthrough in representation theory: the proof of the Kazhdan–Lusztig conjecture on Lie algebra representations [**26**, **50**, **129**, **130**]. From today's perspective, the Kazhdan–Lusztig conjecture may be seen as the spectacular first application of perverse sheaves. Ever since, perverse sheaves have been a powerful tool of fundamental importance in geometric representation theory.

This is partly due to the diversity of perspectives from which one may approach this subject. Perverse sheaves have close connections (especially in their computational aspects) to topics in classical algebraic topology, including fundamental groups, covering spaces, and singular cohomology. On the other hand, perverse sheaves (at least with field coefficients) have algebraic features reminiscent of modules over an artinian ring: every perverse sheaf has a composition series, and one can classify the simple perverse sheaves.

But in my opinion, the most significant reason for the usefulness of perverse sheaves is the following secret known to experts: perverse sheaves are *easy*, in the sense that most arguments come down to a rather short list of tools, such as proper base change, smooth pullback, and open–closed distinguished triangles. In practice, one can reason and compute with perverse sheaves just using a list of these tools, much as calculus students might use a table of integrals. One does not have to dig into the details of flabby resolutions or sheafification any more than a calculus student needs to revisit Riemann sums to integrate a polynomial. In this book, I have tried to emphasize this perspective with computational exercises and with the **Quick Reference** pages near the end of the book.

Organization and prerequisites. This book is divided into two parts: the first six chapters develop the general theory of constructible sheaves on complex algebraic varieties, and the last four chapters give brief introductions to selected applications of perverse sheaves in representation theory. The prerequisites for the first six chapters are: familiarity with the language of derived and triangulated categories; familiarity with introductory algebraic topology; and (starting from Chapter 2) some minimal familiarity with complex algebraic varieties. For the applications in Chapters 7–10, some knowledge of Lie theory is required.

Chapter 1 covers the foundations of sheaf theory on topological spaces, including the definitions of the six basic sheaf operations, and a number of natural compatibilities between them, such as the proper base change theorem and the projection formula. This chapter also contains material on local systems and fundamental groups. Much of this material can be found in many other textbook-level sources, so a number of proofs in this chapter are merely sketched, or sometimes omitted entirely.

In **Chapter 2** we begin the study of constructible sheaves on complex algebraic varieties. Some highlights of results proved in this chapter include Artin's vanishing theorem, the Verdier duality theorem, and the "constructibility theorem" (which says that in the algebraic setting, all six sheaf operations preserve constructibility). We also show that in the setting of constructible sheaves, the external tensor product functor and the extension-of-scalars functor commute with all sheaf operations. The chapter ends with a selection of other topics related to constructible sheaves, including hyperbolic localization, Borel–Moore homology, and fundamental classes.

In **Chapter 3**, we begin the study of perverse sheaves, including the important special case of intersection cohomology complexes. Key results in this chapter describe the behavior of perverse sheaves with respect to push-forward along affine morphisms, and pullback along smooth morphisms. The former is very closely related to Artin's vanishing theorem. In the context of the latter, we prove that perverse sheaves satisfy "smooth descent"—that is, a perverse sheaf can be recovered from its pullback along a smooth surjective morphism. The chapter concludes with a discussion of two of the deepest results about perverse sheaves with coefficients in $\mathbb{Q}$: the decomposition theorem and the hard Lefschetz theorem.

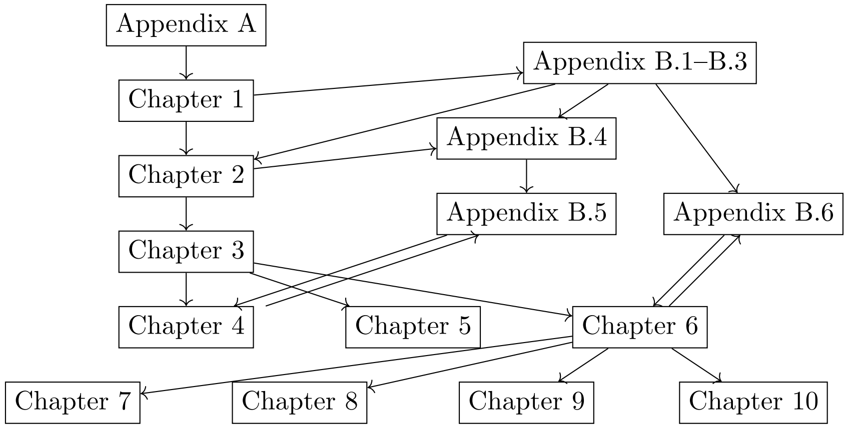
Chapter 4 discusses the nearby cycles functor. The definition of this functor requires leaving the algebraic setting (it involves the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$), so most results from Chapter 2 cannot be applied directly. Nevertheless, we prove that this functor preserves constructibility; that it takes perverse sheaves to perverse sheaves; and that it is compatible with Verdier duality and the extension of scalars. As an application, we prove Beilinson's theorem, which says that the derived category of perverse sheaves (with coefficients in a field) is equivalent to the constructible derived category.

Chapter 5 gives an overview of two separate (but conceptually related) topics: mixed ℓ-adic sheaves in the étale topology, and mixed Hodge modules. Both of these theories provide a kind of "enrichment" of perverse sheaves: the objects carry additional structure, most notably the weight filtration. This chapter includes discussions of some related side topics, including the sheaf–function correspondence and the Riemann–Hilbert correspondence. Most theorems in this chapter are stated without proof.

The final chapter in the first part of the book, **Chapter 6**, is devoted to the study of equivariant sheaves. It is straightforward to define the abelian category of equivariant sheaves (or equivariant perverse sheaves), but it is rather nontrivial to define the correct triangulated analogue. (The derived category of the abelian category of equivariant sheaves is usually the "wrong" answer.) We present a solution to this problem following Bernstein–Lunts. We also study compatibilities of sheaf functors with various ways of modifying the group action, such as forgetting, inflation, and averaging. Perhaps the two most useful results from this chapter are the quotient equivalence and the induction equivalence.

The remaining chapters deal with applications in representation theory.

Chapter 7 deals with the study of Borel-equivariant perverse sheaves on the flag variety of a reductive group. This chapter contains a proof that these perverse



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sheaves give a categorification of the Hecke algebra. This fact, which essentially goes back to a 1980 paper of Kazhdan–Lusztig, is one of the ingredients in the Kazhdan–Lusztig conjectures for Lie algebra representations. This chapter also discusses some more recent developments around sheaves on flag varieties, including parity sheaves and Soergel bimodules.

Chapter 8 studies perverse sheaves on the nilpotent cone of the Lie algebra of a reductive group. The starting point for this topic was Springer's discovery in the late 1970s that the stalks of some of these perverse sheaves (with coefficients in $\mathbb{Q}$) carry a natural action of the Weyl group. By the mid-1980s, Lusztig had extended Springer's work to cover all perverse sheaves (still with coefficients in $\mathbb{Q}$); this became the starting point for his theory of character sheaves. This chapter also discusses recent developments on Springer theory for perverse sheaves with coefficients in a field of positive characteristic.

In **Chapter 9**, we study perverse sheaves on the affine Grassmannian of a reductive group G. Results of Lusztig going back to 1983 indicated that these perverse sheaves contained a great deal of information about representations of the Langlands dual group $\check{G}$. In a landmark 2007 paper, Mirković and Vilonen, following an idea of Drinfeld, proved that this can be upgraded to an equivalence of tensor categories, known as the geometric Satake equivalence. We give proofs of the more sheaf-theoretic steps in this theorem, but we will not prove it in full.

Lastly, in **Chapter 10**, we use perverse sheaves on the space of representations of a quiver to construct the canonical basis for a quantum group. The fact that quantum groups are related to (functions on the space of) quiver representations is due to Ringel. The project of upgrading this by replacing functions by sheaves is due to Lusztig.

The book concludes with two appendices. **Appendix A** contains background (mostly without proofs) on category theory and homological algebra. One fact that is proved is the duality theorem for rings of finite global dimension. This result can be seen as a precursor to Verdier duality. **Appendix B** contains a number of calculations involving sheaves on $\mathbb{C}^n$. The results in this appendix are enlightening examples in their own right, but they are also needed for the proofs of a number of theorems in the main body of the book.

Acknowledgments. This book grew out of notes for a mini-course I gave at East China Normal University in July 2015 on the topic of "Perverse sheaves in representation theory." This mini-course was part of a workshop organized by Bin Shu and Weiqiang Wang. I am grateful to them for the opportunity to participate, and especially to Weiqiang Wang for strongly encouraging me to expand my lecture notes into a book.

Over the years, I have had the opportunity to teach a number of graduate courses at Louisiana State University on topics related to this book, including homological algebra, sheaf theory, and Lie theory. In the 2017–2018 academic year, I co-taught a two-semester sequence with my colleague Daniel Sage on sheaves and geometric representation theory. I am grateful to him and to all the students in these courses for the feedback they have given me. These experiences have shaped the presentation of a number of topics in this book.

I learned this subject myself largely from my collaborators. In particular, I have learned a number of explicit examples from Anthony Henderson, Daniel Juteau, Carl Mautner, and Simon Riche. Some of these examples appear as exercises in this book.

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Some notation and conventions. The 1-point topological space is denoted by pt. For any topological space X, the unique continuous map from X to pt is denoted by $\mathrm{a}_X : X \to \mathrm{pt}$.

All rings in this book are unital. Sheaves will almost always have coefficients in a commutative ring, usually denoted by $\Bbbk$. Starting from Chapter 2, the ring $\Bbbk$ is almost always assumed to be noetherian and of finite global dimension. The category of $\Bbbk$-modules is denoted by $\Bbbk$-mod, and the category of finitely generated $\Bbbk$-modules by $\Bbbk\text{-mod}^{\mathrm{fg}}$. However, if π is a group and $\Bbbk[\pi]$ is its group ring, the notation $\Bbbk[\pi]\text{-mod}^{\mathrm{fg}}$ means the category of $\Bbbk[\pi]$-modules that are finitely generated over $\Bbbk$ (and not merely over $\Bbbk[\pi]$).

We write $\mathsf{H}^i(A)$ for the the ith cohomology object of a chain complex A. We write $\mathbf{H}^i(X, \mathcal{F})$ for the ith sheaf (hyper)cohomology of a topological space X with coefficients in a sheaf (or chain complex of sheaves) $\mathcal{F}$.

Sheaf functors such as $f_!$ and f_* are always derived; a separate notation (${}^\circ f_!$, ${}^\circ f_*$) is used for their non-derived counterparts.

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