Preface

Completion problems for operator matrices are concerned with the question of whether a partially specified operator matrix can be completed to form an operator of a desired type. The research devoted to this topic constitutes an excellent means to investigate the structure of operators, which is one of the most important problems in operator theory.

This book provides an overview of completion problems dealing with completions to invertible, injective, Fredholm, Kato, Browder, and Fredholm consistent operators and can be considered as a sequel to the results presented in the monographs (see [**82**, **83**]), which are mostly concerned with matrix completions. The study of operator matrices arises naturally from a well-known fact that if $\mathcal{Z}$ is a Banach space, decomposed as a direct sum of two closed and complementary subspaces $\mathcal{X}$ and $\mathcal{Y}$, then each bounded linear operator $T : \mathcal{Z} \to \mathcal{Z}$ can be expressed in the operator matrix form

$$T = \begin{bmatrix} A & C \\ D & B \end{bmatrix},$$

with respect to that particular space decomposition, where $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{Y} \to \mathcal{Y}$, $C : \mathcal{Y} \to \mathcal{X}$, and $D : \mathcal{X} \to \mathcal{Y}$ are bounded linear operators. Studying an operator through these component operators can simplify its analysis.

A particular version of the completion problem, in which a matrix is to be completed so that it becomes positive definite, has received considerable attention due, in part, to the role it plays in a number of applications in probability and statistics, image enhancement, systems engineering, geophysics, etc., but also due to the fact that it is closely linked to certain other completion problems, related to spectral norm contractions and Euclidean distance matrices, both of which are important in the molecular conformation problem in chemistry (see [**6**, **7**, **15**, **16**]). Another way to view the positive definite completion problem is as a mechanism for addressing the following fundamental problem in Euclidean geometry: Which potential geometric configurations of vectors (i.e., configurations with angles between some vectors specified) are realizable in Euclidean space (see [**8**, **126**, **158**])? It is also worth mentioning that unitary completions of complex symmetric and skew symmetric matrices are exploited in atomic physics, in so-called phase-integral halfway-house variational continuum distorted wave theory (PIVCDW) (see [**159**]).

A good theoretical background for this book can be found in the monographs (see [**1**, **2**, **99**]). Besides applications of completion techniques in theoretical research (see [**150**], [**168**]), completion problems occur also in a variety of research areas of applied origin, such as the following: computer science, in particular in the problem of data compression, recommender systems (see [**109**, **153**]), recovery of missing data in electricity distribution systems (see [**80**]), in urban systems (see [**81**]) and others (see [**33**, **85**, **138**, **166**]), in statistics for missing data analysis and entropy methods (see [**25**–**27**, **33**, **37**, **38**, **131**, **154**]), for different prediction methods (see [**110**, **112**, **139**, **157**]), in chemistry (the molecular conformation problem), systems theory, discrete optimization (relaxation methods), etc. For an extensive reference list of matrix completion problems see [**129**].

The completions problems have long been a subject of research for many authors, and several relevant problems have already been resolved. However, there still remain problems that have been either only partially solved or remain open to this day. The present book aims at reviewing and summarizing these results (scattered in various literature sources) and at pointing out to the reader possible directions for further research in this area of mathematics as well as indicating their potential applications in different types of problems.

The problem of completion of operator matrices might be considered for the following types of matrices:

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

$$M_{(X\ Y)} = \begin{bmatrix} A & B \\ X & Y \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}.$$

With respect to the first type of matrices, the following questions can be posed:

Question 1: Is there an operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is of a certain type (invertible, right invertible, left invertible, regular, ...)?

Question 2: For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, can we compute $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_*(M_C)$, where $\sigma_*(M_C)$ is any type of spectra of M_C, such as the following: essential, left, point, residual, right, Weyl, Browder, essential approximation spectrum, etc.?

Question 3: For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, is there an operator $C' \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\sigma_*(M_{C'}) = \bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_*(M_C),$$

where again $\sigma_*(.)$ is any type of spectra of an argument operator?

Clearly, analogous questions can be posed with respect to the other two types of matrices listed above, i.e., M_X and $M_{(X\ Y)}$. These questions play a focal role in the present book and will be discussed thoughtfully on the next pages.

The book consists of five chapters, among which three concern completion problems for three different types of operator matrices. The additional two chapters are concerned with the problem of completing the operator $A + CX$ to different types of operators and with applications of completion results, respectively.

The first chapter gives an exposition of completion problems for upper triangular operator matrices. The study of upper triangular operator matrices arises from the fact that if $A \in \mathcal{B}(\mathcal{H})$ and S is a closed, complemented A-invariant subspace of $\mathcal{H}$, then A has a representation in the form of an upper triangular operator matrix. Special emphasis is put on injectivity and boundedness below, invertibility, regularity, Drazin and generalized Drazin invertibility, Fredholmness, generalized left and right Weylness, the Browder property, Kato nonsingularity, and Fredholm consistency. Also, some related results for the spectra of upper triangular operator matrices are presented.

In the second chapter, we shift our attention from the problem of the completion of operator matrices to that of the completion by X of the operator $A + CX$, for given operators A, C, to different types of operators. The first to ever address the question of the existence of $X \in \mathcal{B}(\mathcal{H})$ such that $A + CX$ is of a desired type, for given $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{H})$, was K. Takahashi, who considered the invertibility of the sum. More precisely, he established a complete characterization of situations in which, for given $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{H})$, there exists $X \in \mathcal{B}(\mathcal{H})$ such that $A + CX$ is invertible. Based on this result, Takahashi derived a solution to a different, though related, problem, concerned with the completion of the operator matrix M_X, by an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ (see [**160**]). Therefore, Takahashi established a link between the invertibility completion problem and the existence of X such that $A + CX$ is invertible. For given operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$, we consider the existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $A + CX$ is injective, surjective, with dense range, with closed range, or Fredholm.

The next chapter deals with the problems concerned with completion of operator matrices of the form M_X to invertibility, right and left invertibility, injectivity, Fredholmness as well as to an operator with nondense or closed ranges and right and left semi-Fredholm operators.

The fourth chapter concerns the third type of completions, namely of operator matrices $M_{(X,Y)}$. The chapter covers such issues as completions of $M_{(X,Y)}$ to invertibility, right and left invertibility. Furthermore, the existence of operators X, Y such that a given operator $M_{(X,Y)}$ is consistent in invertibility and Fredholmness is considered as well.

In the fifth chapter necessary and sufficient conditions for the sum $A + B$ of two operators $A, B \in \mathcal{B}(\mathcal{H})$ to belong to a certain class of operators are identified. The task is achieved by exploiting the results on completion problems of the upper triangular operator matrices M_C.

The present book is the outcome of long-lasting research on this subject by the author. Nevertheless, an extensive impact on its content originated from works of several of my colleagues and coauthors. I am thankful for all of the inspirational contributions and partnerships, which affected the realization of this project.

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