

i.e.,

$$M_{C_1} = \begin{bmatrix} T & C & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} : l_2 \oplus l_2 \oplus X_n \rightarrow l_2 \oplus l_2 \oplus X_n,$$

where  $C_1 = \begin{bmatrix} C & 0 \end{bmatrix} : l_2 \oplus X_n \rightarrow l_2$ ,  $S' = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} : l_2 \oplus X_n \rightarrow l_2 \oplus X_n$ .

Then we can claim that

$$0 \in \left( \bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma(M_Q) \right) \setminus (\rho_b(T) \cap \rho_b(S')) \text{ but } 0 \notin \bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma_b(M_Q).$$

Thus,

$$\bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma_b(M_Q) \neq \left( \bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma(M_Q) \right) \setminus (\rho_b(T) \cap \rho_b(S')).$$

In fact:

- (i) Since  $M_Q = \begin{bmatrix} T & Q \\ 0 & S' \end{bmatrix}$  cannot be surjective for any  $Q \in \mathcal{B}(l_2 \oplus X_n, l_2)$ , then  $M_Q$  is not invertible. Thus  $0 \in \bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma(M_Q)$ .
- (ii) Since  $\text{dsc}(T) = \text{asc}(S') = \infty$ , then neither  $S'$  nor  $T$  is a Browder operator. Hence  $0 \notin \rho_b(T) \cap \rho_b(S')$ .
- (iii) It well known that  $\begin{bmatrix} T & C \\ 0 & S \end{bmatrix}$  is unitary, so it is invertible. Moreover, since  $\dim X_n = n < \infty$ , then  $M_{C_1}$  is a Drazin invertible Fredholm operator. Therefore  $0 \notin \bigcap_{Q \in \mathcal{B}(l_2 \oplus X_n, l_2)} \sigma_b(M_Q)$ .

From (iii), we can get that  $\sigma(M_{C_1}) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$  and  $\sigma_b(M_{C_1}) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ . It is easy to prove that

$$\sigma(T) = \sigma_b(T) = \sigma(S') = \sigma_b(S') = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Thus

$$\{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\} = W_1(T, S', C_1) \neq W_3(T, S', C_1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

and

$$\{0\} = \text{iso } \sigma(M_{C_1}) \neq \text{iso}(\sigma(T) \cup \sigma(S')) = \emptyset.$$

### 1.8. Completions of upper triangular matrices to Kato nonsingular operator

Vast research can be found in the literature concerned with the study of bounded linear operators with closed range. Their significance stems from the many applications they have, e.g., in the spectral study of differential operators or in the context of perturbation theory, but also from the important role they play when it comes to purely theoretical considerations. By an old theorem of Banach, for such operators the equation  $Tx = y$  is solvable if for all bounded linear functionals  $g$  there is the implication  $gT = 0 \Rightarrow gy = 0$ . In Hilbert spaces, these are precisely the regular operators (those possessing the Moore-Penrose inverse). S. Goldberg (see [84]), among others, considered the set  $\sigma_c(T)$  consisting of all  $\lambda \in \mathbb{C}$  such that  $\mathcal{R}(T - \lambda)$  is not closed. However,  $\sigma_c(T)$  fails to have some basic properties expected from of a decent spectrum. For example, it may be empty and does not satisfy the spectral mapping theorem (not even for polynomials). On the other hand, M. Mbekhta (see [140]–[145]) defines  $\sigma_h(T)$  as the set of all  $\lambda \in \mathbb{C}$  such that

there are a neighborhood  $U$  of 0 in  $\mathbb{C}$  and a holomorphic function  $H : U \rightarrow \mathcal{X}$  such that  $T - \lambda = (T - \lambda)H(\lambda)(T - \lambda)$  for all  $\lambda \in U$ , and he shows that, in a Hilbert space, this set is always nonempty, closed, and subject to the spectral mapping theorem.

A useful characterization of closed range operators was given by T. Kato (see [114]) who introduced the reduced minimum modulus

$$\gamma(T) = \inf\{\|Tx\| : \text{dist}(x, \mathcal{N}(T)) = 1\}$$

and proved that  $\mathcal{R}(T)$  is closed if and only if  $\gamma(T) > 0$ . Related to this is the study of the function  $z \rightarrow \gamma(T - z)$  which, although not continuous in general, has nice continuity properties. For example we have the following result.

**THEOREM 1.92 ([146]).** *Let  $\mathcal{X}$  be a Banach space and let  $T \in \mathcal{B}(\mathcal{X})$  be with closed range. The following conditions are equivalent:*

- (1) *The function  $z \rightarrow \gamma(T - z)$  is continuous at  $z = 0$ .*
- (2)  $\mathcal{N}(T) \subseteq \bigcap_{n \in \mathbb{N}} \mathcal{R}(T^n)$ .
- (3)  $\bigcup_{n \in \mathbb{N}} \mathcal{N}(T^n) \subseteq \mathcal{R}(T)$ .
- (4)  $\bigcup_{n \in \mathbb{N}} \mathcal{N}(T^n) \subseteq \bigcap_{n \in \mathbb{N}} \mathcal{R}(T^n)$ .

The equivalence between (2), (3), and (4) is of a purely algebraic nature and does not require the assumption of closedness of the range. An operator  $T \in \mathcal{B}(\mathcal{X})$  satisfying the condition (2) of the previous theorem is said to be hyperexact (for more details see [101]). Various equivalents of hyperexactness and related concepts can be found in the excellent paper of R. E. Harte [100]. Hyperexact operators with closed range are called Kato nonsingular operators. For many interesting properties of this class of operators see [3, 73, 148].

There are only a few results that are concerned with completion problems to Kato operators. M. Barraa et al. [14] considered the completion problem to a Kato nonsingular  $M_C$  and gave some sufficient conditions for the existence of such operators  $C$ . These conditions always assumed, among other things, Kato nonsingularity of  $A$  or injectivity of  $B$ . Later Y. N. Dou et al. [67] generalized some of their results. In this section, we completely solve the problem of completion to Kato nonsingularity in each of the following seven cases:

- Case 1: if one of the operators  $A$  or  $B$  is Kato nonsingular,
- Case 2: if  $B$  is injective,
- Case 3: if  $A$  is with dense range,
- Case 4: if  $B$  is with finite ascent,
- Case 5: if  $A$  is with finite descent,
- Case 6: if  $0 \notin \text{int}(\sigma_p(B))$ ,
- Case 7: if  $0 \notin \text{int}(\sigma_{cp}(A))$ .

Throughout the section  $\mathcal{H}, \mathcal{K}$  are infinite-dimensional separable complex Hilbert spaces. For  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ , we use the following notation:

$$S_K(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : M_C \text{ is Kato nonsingular}\}.$$

We also set  $\mathcal{R}^\infty(A) = \bigcap_{n \in \mathbb{N}} \mathcal{R}(A^n)$ .

We begin by listing the results that will be made use of later in the section. The next two are well-known useful results.

**LEMMA 1.93.** *If  $\mathcal{X}$  is a closed and  $\mathcal{X}_0$  a finite-dimensional subspace of a Banach space, then  $\mathcal{X} + \mathcal{X}_0$  is a closed subspace.*

Using elementary techniques one readily obtains the result below.

LEMMA 1.94. *Let  $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ ,  $B \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ , and  $C \in \mathcal{B}(\mathcal{K}_1, \mathcal{H}_2)$  be given operators. If  $\overline{\mathcal{R}(A)} = \mathcal{H}_2$  and  $\mathcal{R}(M_C)$  is closed, then  $\mathcal{R}(B)$  is closed.*

Throughout this section we make extensive use of a variant of a particular result from [67] on the completion problem to closed range operator matrices. One readily checks that both the statement and the proof of Theorem 2.5 from [67] remain valid if  $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  and  $B \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ , for Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1$ , and  $\mathcal{K}_2$ , where neither  $\mathcal{H}_1$  and  $\mathcal{H}_2$  nor  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are assumed to be the same. More precisely, we have the following result.

THEOREM 1.95. *Let  $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  and  $B \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ . If  $\mathcal{R}(A)$  is not closed and  $\mathcal{R}(B)$  is closed, then  $\mathcal{R}(M_C)$  is closed for some  $C \in \mathcal{B}(\mathcal{K}_1, \mathcal{H}_2)$  if and only if  $\alpha(B) = \infty$ .*

Now, we list some results concerning properties of Kato nonsingular operators, the proofs of which can be found in [148].

LEMMA 1.96. *If  $A \in \mathcal{B}(\mathcal{H})$  is (hyperexact) Kato nonsingular, then so is  $A^n$  for all  $n \in \mathbb{N}$ .  $A$  is Kato nonsingular if and only if  $A^*$  is such.*

THEOREM 1.97. *If  $T$  is Kato nonsingular, then  $T - z$  is Kato nonsingular for all  $z$  in a neighborhood of  $0 \in \mathbb{C}$ .*

LEMMA 1.98 ([147]). *Let  $T \in \mathcal{B}(\mathcal{H})$  be Kato nonsingular and set  $U = \{z \in \mathbb{C} : |z| < \gamma(T)\}$ . Then for every  $\lambda \in U$  and  $x \in \mathcal{N}(T - \lambda)$  there exists an analytic function  $f : U \rightarrow \mathcal{H}$  such that  $(T - z)f(z) = 0$  for all  $z \in U$  and  $f(\lambda) = x$ .*

The next proposition plays an important role in the proof of the main result.

PROPOSITION 1.99. *If  $M_C$  is hyperexact for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ , then*

$$(1.75) \quad \text{codim}_{\mathcal{N}(B^k)}(\mathcal{R}(B^n) \cap \mathcal{N}(B^k)) \leq \beta(A^k),$$

for all  $n, k \in \mathbb{N}$ .

PROOF. In view of Lemma 1.96, if we prove the statement above for  $k = 1$ , then we are done, as  $M_C^m = \begin{bmatrix} A^m & * \\ 0 & B^m \end{bmatrix}$  and

$$\text{codim}_{\mathcal{N}(B^m)}(\mathcal{R}(B^n) \cap \mathcal{N}(B^m)) \leq \text{codim}_{\mathcal{N}(B^m)}(\mathcal{R}(B^{n+1}) \cap \mathcal{N}(B^m)),$$

for all  $n, m \in \mathbb{N}$ .

Let  $P_{\mathcal{K}} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{K})$  be the orthogonal projector onto  $\mathcal{K}$  and set  $\mathcal{S} = P_{\mathcal{K}}[\mathcal{N}(M_C)]$ . If  $y \in \mathcal{S}$ , then  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C)$  for some  $x \in \mathcal{H}$ , so for each  $n \in \mathbb{N}$  we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = M_C^n \begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} A^n p_n + X_n q_n \\ B^n q_n \end{bmatrix},$$

for some  $\begin{bmatrix} p_n \\ q_n \end{bmatrix} \in \mathcal{H} \oplus \mathcal{K}$ , where  $X_n = \sum_{i=1}^n A^{n-i} C B^{i-1}$ . Hence  $\mathcal{S} \subseteq \mathcal{R}(B^n)$ , for all  $n \in \mathbb{N}$ . Actually,  $\mathcal{S} \subseteq \mathcal{R}(B^n) \cap \mathcal{N}(B)$ , so

$$\text{codim}_{\mathcal{N}(B)}(\mathcal{R}(B^n) \cap \mathcal{N}(B)) \leq \text{codim}_{\mathcal{N}(B)} \mathcal{S}.$$

Pick subspaces  $\mathcal{S}_0 \subseteq \mathcal{N}(B)$  and  $\mathcal{L} \subseteq \mathcal{H}$  such that  $\mathcal{S} \oplus \mathcal{S}_0 = \mathcal{N}(B)$  and  $\mathcal{R}(A) \oplus \mathcal{L} = \mathcal{H}$  and let  $P_{\mathcal{L}} : \mathcal{H} \rightarrow \mathcal{H}$  be the idempotent linear operator with range  $\mathcal{L}$  and null space

$\mathcal{R}(A)$ . Then  $P_{\mathcal{L}}C$  takes  $\mathcal{S}_0$  injectively into  $\mathcal{L}$ . Indeed, let  $y_0 \in \mathcal{S}_0$  be nonzero. Then  $y_0 \in \mathcal{N}(B)$  but  $y_0 \notin \mathcal{S} = \mathcal{N}(B) \cap C^{-1}[\mathcal{R}(A)]$ , so  $Cy_0 \notin \mathcal{R}(A)$ ; i.e.,  $P_{\mathcal{L}}Cy_0 \neq 0$ . This proves that  $\text{codim}_{\mathcal{N}(B)}\mathcal{S} = \dim \mathcal{S}_0 \leq \dim \mathcal{L} = \beta(A)$ .  $\square$

Using Lemma 1.96 and taking adjoints in Proposition 1.99 we directly obtain the following result.

PROPOSITION 1.100. If  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ , then

$$(1.76) \quad \text{codim}_{\mathcal{N}((A^*)^k)}(\mathcal{R}((A^*)^n) \cap \mathcal{N}((A^*)^k)) \leq \beta((B^*)^k),$$

for all  $n, k \in \mathbb{N}$ .

Now, we consider completion to Kato nonsingular operator in the following seven cases:

**Case 1:  $A$  or  $B$  is Kato nonsingular.** In this case we address the question of completion of  $M_C$  to Kato nonsingularity under the additional assumption that one of the operators  $A$  or  $B$  is Kato nonsingular. As we will see in Cases 3 and 4 below, for certain important classes of operators (such as the compact ones, those with finite descent, quasinilpotent operators, ...) Kato nonsingularity of the operator  $A \in \mathcal{B}(\mathcal{H})$  is a necessary condition for the existence of some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular.

Some particular cases of the completion problem to Kato nonsingularity in the case  $A$  is Kato nonsingular were already considered in [14] and [67]. The following two theorems, which we often use in the sequel, were proved:

THEOREM 1.101 ([14]). *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be given operators. If  $A$  is Kato nonsingular,  $\mathcal{R}(B)$  is closed, and  $\alpha(B) \leq d(A)$ , then there exists  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular.*

In the sequel we present a proof of the following theorem given in [67] which will be substantially shortened and at the same time will describe a way to produce operators in  $S_K(A, B)$ .

THEOREM 1.102 ([67]). *Let  $A \in \mathcal{B}(\mathcal{H})$  be Kato nonsingular and let  $B \in \mathcal{B}(\mathcal{K})$  be an arbitrary operator. If  $d(A) = \infty$ , then there exists  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular.*

PROOF. If  $d(A) = \infty$ , there exists  $C_3 \in \mathcal{B}(\mathcal{N}(B), \mathcal{R}(A)^\perp)$  left invertible such that  $d(C_3) = \infty$ . For such  $C_3$  we get that there exists  $C_4 \in \mathcal{B}(\mathcal{N}(B)^\perp, \mathcal{R}(A)^\perp)$  such that  $S = \begin{bmatrix} C_3 & C_4 \\ 0 & B_1 \end{bmatrix}$  has a closed range, where  $B = \begin{bmatrix} 0 & B_1 \end{bmatrix} : \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp \rightarrow \mathcal{K}$  (if  $\mathcal{R}(B)$  is closed take  $C_4 = 0$ ; otherwise, use Theorem 1.95). Now, for  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  given by

$$(1.77) \quad C = \begin{bmatrix} 0 & 0 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix},$$

$M_C$  has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B)^\perp \\ \mathcal{R}(B) \end{bmatrix}.$$

Since  $\mathcal{R}(A)$  and  $\mathcal{R}(S)$  are closed it follows that  $\mathcal{R}(M_C)$  is closed. Also, by the injectivity of  $B_1$  and  $C_3$ ,  $\mathcal{N}(S) = \{0\}$ . So,  $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus \mathcal{N}(S) = \mathcal{N}(A) \subseteq \mathcal{R}(A^n) \subseteq \mathcal{R}(M_C^n)$ . Hence,  $M_C$  is Kato nonsingular.  $\square$

It is interesting to notice that for  $C_3$  and  $C_4$  chosen as in the proof above and for arbitrary  $C_1$  and  $C_2$  from the appropriate spaces, the operator  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  given by

$$(1.78) \quad C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

belongs to  $S_K(A, B)$ .

In the two theorems below, which are the main result of this part, we completely answer the question posed.

**THEOREM 1.103.** *Let  $A \in \mathcal{B}(\mathcal{H})$  be Kato nonsingular and let  $B \in \mathcal{B}(\mathcal{K})$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following two conditions is satisfied:*

- (1)  $d(A) = \infty$ .
- (2)  $d(A) < \infty$ ,  $\mathcal{R}(B^n)$  is closed for all  $n \in \mathbb{N}$  and (1.75) holds for all  $n, k \in \mathbb{N}$ .

**THEOREM 1.104.** *Let  $B \in \mathcal{B}(\mathcal{K})$  be Kato nonsingular and let  $A \in \mathcal{B}(\mathcal{H})$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following two conditions is satisfied:*

- (1)  $\alpha(B) = \infty$ .
- (2)  $\alpha(B) < \infty$ ,  $\mathcal{R}(A^n)$  is closed for all  $n \in \mathbb{N}$  and (1.76) holds for all  $n, k \in \mathbb{N}$ .

The second theorem obviously follows by taking adjoints in Theorem 1.103 and using Lemma 1.96.

By virtue of Theorem 1.102, to prove Theorem 1.103 it remains to find necessary and sufficient conditions for  $M_C$  to be Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ , under the assumption that  $A$  is Kato nonsingular and  $d(A) < \infty$ . It turns out that this case is much more complicated to deal with. We first prove an auxiliary result in which some sufficient conditions are given for the set  $S_K(A, B)$  to be nonempty.

**PROPOSITION 1.105.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be given operators. If  $A$  is Kato nonsingular,  $d(A) < \infty$ ,  $\mathcal{R}(B)$  is closed, and (1.75) holds for all  $n, k \in \mathbb{N}$ , then there is some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular provided that either*

- (i)  $\mathcal{R}^\infty(B)$  is closed or
- (ii)  $\text{codim}_{\mathcal{N}(B)}(\mathcal{R}(B) \cap \mathcal{N}(B)) = d(A)$ .

**PROOF.** For each  $k \in \mathbb{N}$  set  $S_k = \mathcal{R}^\infty(B) \cap \mathcal{N}(B^k)$ ; we have that

$$\text{codim}_{\mathcal{N}(B^k)}(\mathcal{R}(B^n) \cap \mathcal{N}(B^k)) \leq \beta(A^k) \leq k\beta(A) < \infty$$

for all  $n \in \mathbb{N}$ , by Lemma 1.93, so there must be some  $m_k \in \mathbb{N}$  such that  $S_k = \mathcal{R}(B^{m_k}) \cap \mathcal{N}(B^k) = \mathcal{R}(B^l) \cap \mathcal{N}(B^k)$  for all  $l \geq m_k$ .

We claim that for each  $k \in \mathbb{N}$  and  $x \in S_k$  there is some  $x_1 \in S_{k+1}$  such that  $x = Bx_1$ . Indeed,  $x \in S_k$  implies that  $x = B^{m_{k+1}+1}x_2$  for some  $x_2 \in \mathcal{K}$ . This means that  $Bx_1 = x$  for  $x_1 = B^{m_{k+1}}x_2 \in \mathcal{N}(B^{k+1})$ . But  $x_1 \in \mathcal{R}(B^{m_{k+1}}) \cap \mathcal{N}(B^{k+1}) = S_{k+1}$ .

Take a finite-dimensional subspace  $\mathcal{X}$  of  $\mathcal{N}(B)$  such that  $S_1 \oplus \mathcal{X} = \mathcal{N}(B)$ . Since  $\dim \mathcal{X} \leq d(A)$  by assumption, we can fix a left invertible  $C_0 \in \mathcal{B}(\mathcal{X}, \mathcal{R}(A)^\perp)$ . If (i) holds, let  $\mathcal{X}_1 = \mathcal{R}^\infty(B)$ , and if (ii) holds, let  $\mathcal{X}_1 = \mathcal{R}(B)$ . Note that if (ii) holds, then it must be  $S_1 = \mathcal{R}(B) \cap \mathcal{N}(B)$ . In any case  $\mathcal{X}_1 \cap \mathcal{X} = \{0\}$  so  $\mathcal{X}_1$  and  $\mathcal{X}$  are two closed complementary subspaces of the closed subspace  $\mathcal{Y} = \mathcal{X}_1 \oplus \mathcal{X} = \mathcal{X}_1 + \mathcal{N}(B)$  of  $\mathcal{K}$  (recall that  $\dim \mathcal{X} < \infty$ ). Thus we can define a  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  by

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C_0 \end{bmatrix} : \begin{bmatrix} \mathcal{Y}^\perp \\ \mathcal{X}_1 \\ \mathcal{X} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

We show that  $M_C$  is Kato nonsingular. First use left invertibility of  $C_0$  to verify that  $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus S_1$ . Next, for a given  $n \in \mathbb{N}$  we show that  $\mathcal{N}(M_C) \subseteq \mathcal{R}(M_C^n)$ . So let  $\begin{bmatrix} x \\ y_1 \end{bmatrix} \in \mathcal{N}(A) \oplus S_1$ . By an earlier observation we can pick recursively  $y_i \in S_i$  for  $2 \leq i \leq n+1$  such that  $By_i = y_{i-1}$ . Since  $y_i \in \mathcal{R}^\infty(B) \subseteq \mathcal{X}_1$  we have  $Cy_i = 0$ , for  $1 \leq i \leq n+1$ .  $A$  is Kato nonsingular and  $x \in \mathcal{N}(A)$  so  $x = A^n p$  for some  $p \in \mathcal{H}$ . Thus from  $y_1 = B^n y_{n+1}$  and  $(\sum_{i+j=n-1} A^i C B^j) y_{n+1} = 0$  we see that

$$\begin{bmatrix} x \\ y_1 \end{bmatrix} = M_C^n \begin{bmatrix} p \\ y_{n+1} \end{bmatrix}.$$

Since  $C$  is a finite-rank operator, we have that  $\mathcal{R}(M_C)$  is closed if and only if  $\mathcal{R}(M_0)$  is closed (Lemma 1.64), and the latter indeed is such, both  $\mathcal{R}(A)$  and  $\mathcal{R}(B)$  being closed by assumption.  $\square$

Now we can complete the proof of Theorem 1.103.

**THEOREM 1.106.** *Let  $B \in \mathcal{B}(\mathcal{K})$  and let  $A \in \mathcal{B}(\mathcal{H})$  be Kato nonsingular and  $d(A) < \infty$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if  $\mathcal{R}(B^n)$  is closed for all  $n \in \mathbb{N}$  and (1.75) holds for all  $n, k \in \mathbb{N}$ .*

**PROOF.** By Proposition 1.105 and Proposition 1.99, we only need to show that  $\mathcal{R}(B^n)$  is closed for all  $n \in \mathbb{N}$  provided that  $A$  is Kato nonsingular,  $d(A) < \infty$ , and that  $M_C$  is Kato nonsingular for some  $C$ .

Given  $n \in \mathbb{N}$  we have  $(M_C^*)^n = \begin{bmatrix} B^{*n} & * \\ 0 & A^{*n} \end{bmatrix}$ . By Lemma 1.96 we know that the ranges of both  $(M_C^*)^n$  and  $A^{*n}$  are closed. Should  $\mathcal{R}(B^{*n})$  not be closed, Theorem 1.95 would then imply that  $\infty = \alpha(A^{*n}) = d(A^n) \leq n \cdot d(A) < \infty$ , a contradiction.  $\square$

**Cases 2–5:  $B$  injective or with finite ascent;  $A$  with dense range or with finite descent.** In this case for given operators  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  we discuss the completion problem of  $M_C$  under the hypothesis that  $B$  is with finite ascent or under the more restrictive one that it is an injective operator. In both of these cases we solve the problem completely.

First we deal with the case when  $B$  is with finite ascent. We note that in this case the closedness of the range of  $A$  turns out to be a necessary condition. Furthermore,  $A$  must be a Kato nonsingular operator.

We begin by establishing the algebraic result given below.

**THEOREM 1.107.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be such that  $\text{asc}(B) < \infty$ . If  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  is such that  $M_C$  is hypereexact, then:*

- (1)  $A$  is hypereexact.
- (2)  $\mathcal{N}(M_C^k) = \mathcal{N}(A^k)$  for all  $k \in \mathbb{N}$ .
- (3)  $\alpha(B) \leq \beta(A)$ .

**PROOF.** Let  $k_0 = \text{asc}(B)$ .

(1) Let  $x \in \mathcal{N}(A)$  and  $m \in \mathbb{N}$ . Since  $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{N}(M_C)$  and  $M_C$  is Kato nonsingular, there are  $p \in \mathcal{H}$ ,  $q \in \mathcal{K}$  such that

$$\begin{cases} x = A^{m+k_0}p + \left( \sum_{i=0}^{m+k_0-1} A^{m+k_0-1-i}CB^i \right)q, \\ 0 = B^{m+k_0}q. \end{cases}$$

From here it follows that  $q \in \mathcal{N}(B^{m+k_0}) = \mathcal{N}(B^{k_0})$  and thus

$$x = A^m \left( A^{k_0}p + \left( \sum_{i=0}^{k_0-1} A^{k_0-1-i}CB^i \right)q \right) \in \mathcal{R}(A^m).$$

(2) This is proved by induction. For  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C)$  it follows that  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{R}(M_C^n)$  for every  $n \in \mathbb{N}$  which implies that  $\begin{bmatrix} x \\ y \end{bmatrix} = M_C^{k_0} \begin{bmatrix} p \\ q \end{bmatrix}$  for some  $p \in \mathcal{H}$ ,  $q \in \mathcal{K}$ . So  $q \in \mathcal{N}(B^{k_0+1}) = \mathcal{N}(B^{k_0})$  and thus  $y = B^{k_0}q = 0$ . But this implies that  $Ax = 0$ ; i.e.,  $\mathcal{N}(M_C) = \mathcal{N}(A)$ .

Assume now that  $\mathcal{N}(M_C^k) = \mathcal{N}(A^k)$  for some  $k$  and let  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C^{k+1})$ . Then

$$\begin{bmatrix} Ax + Cy \\ By \end{bmatrix} \in \mathcal{N}(M_C^k) = \mathcal{N}(A^k),$$

so  $By = 0$  and  $Ax + Cy \in \mathcal{N}(A^k) \subseteq \mathcal{R}(A)$  (by item (1)). From this we have

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

for some  $x_1 \in \mathcal{H}$ ; i.e.,  $\begin{bmatrix} x_1 \\ y \end{bmatrix} \in \mathcal{N}(M_C) = \mathcal{N}(A)$ . Hence  $y = 0$  and consequently  $x \in \mathcal{N}(A^{k+1})$ . We have thus shown  $\mathcal{N}(M_C^{k+1}) = \mathcal{N}(A^{k+1})$ .

(3) Let  $\mathcal{S}$ ,  $\mathcal{S}_0$ ,  $\mathcal{L}$ , and  $P_{\mathcal{L}}$  be as in the proof of Proposition 1.99; we know that  $P_{\mathcal{L}}C$  maps  $\mathcal{S}_0$  injectively into  $\mathcal{L}$ . By (1) we have  $\mathcal{S} = \{0\}$ , so  $\mathcal{S}_0 = \mathcal{N}(B)$  and thus  $\alpha(B) = \dim(\mathcal{N}(B)) \leq \dim \mathcal{L} = \beta(A)$ .  $\square$

In the next result we show that if  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  with the additional condition that  $\mathcal{N}(M_C) = \mathcal{N}(A)$ , then  $\mathcal{R}(A)$  must be closed.

**PROPOSITION 1.108.** *Let  $A \in \mathcal{B}(\mathcal{H})$ ,  $B \in \mathcal{B}(\mathcal{K})$  be such that there exists  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular and  $\mathcal{N}(M_C) = \mathcal{N}(A)$ . Then  $\mathcal{R}(A)$  must be closed.*

PROOF. Suppose that  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  given by (1.78). In that case,  $M_C$  has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B)^\perp \\ \overline{\mathcal{R}(B)} \end{bmatrix},$$

where  $A_1, B_1$  are injections with dense ranges.

By Lemma 1.94, using the fact that  $B_1^*$  has a dense range, it follows that  $S = \begin{bmatrix} A_1 & C_1 \\ 0 & C_3 \end{bmatrix}$  has a closed range. Since  $\mathcal{N}(M_C) = \mathcal{N}(A)$ , we conclude that  $S$  is injective. Hence  $S$  is left invertible, so  $A_1$  is left invertible; i.e.,  $\mathcal{R}(A)$  is closed.  $\square$

In the following theorem, we resolve the completion problem in the case  $B$  is with finite ascent. In addition to that, we give a description of the set  $S_K(A, B)$  whenever it is nonempty.

**THEOREM 1.109.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be such that  $\text{asc}(B) < \infty$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following conditions is satisfied:*

(i)  *$A$  is Kato nonsingular,  $\mathcal{R}(B)$  is closed, and  $\alpha(B) \leq d(A)$ . In this case,*

$$S_K(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}, \right. \\ \left. C_3 \text{ is left invertible} \right\}.$$

(ii)  *$A$  is Kato nonsingular,  $\mathcal{R}(B)$  is not closed, and  $d(A) = \infty$ . In this case,*

$$S_K(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}, \right. \\ \left. C_3 \text{ is left invertible, } \mathcal{R}(B^*) + \mathcal{R}(C_4^*(I - P_{\mathcal{R}(C_3)})) \text{ is closed} \right\}.$$

PROOF. By Proposition 1.108 and Theorems 1.95, 1.101, 1.102, and 1.107 it immediately follows that  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if (i) or (ii) holds. It remains to describe the set  $S_K(A, B)$  when one of the conditions (i) or (ii) is satisfied.

Assume for the moment that  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  given by (1.78). We know that  $\mathcal{N}(M_C) = \mathcal{N}(A)$  as well as that  $\mathcal{R}(A)$  is closed. In that case,  $M_C$  has a matrix representation

$$M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A)^\perp \\ \mathcal{N}(A) \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B)^\perp \\ \overline{\mathcal{R}(B)} \end{bmatrix},$$

where  $A_1$  is invertible and  $B_1$  is an injection with dense range.

By  $\mathcal{N}(M_C) = \mathcal{N}(A)$ , we have that  $S = \begin{bmatrix} A_1 & C_1 \\ 0 & C_3 \end{bmatrix}$  is injective which, using the invertibility (surjectivity to be precise) of  $A_1$ , implies that  $C_3$  is injective.



Now, using twice Lemma 1.94, we first conclude that  $P = \begin{bmatrix} C_3 & C_4 \\ 0 & B_1 \end{bmatrix}$  has closed range and then that  $\mathcal{R}(C_3)$  is closed. So,  $C_3$  must be left invertible.

From the discussion above it follows that:

- (a) If  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  given by (1.78), then  $S$ ,  $C_3$ , and  $P$  are left invertible.

Also, the following fact can easily be verified:

- (b) If  $\mathcal{R}(A)$  is closed, then  $\mathcal{R}(M_C)$  is closed if and only if  $\mathcal{R}(P)$  is closed.

Suppose (i) holds: Denote the set from the right side of the equality in (i) by  $\mathcal{S}_1$ . By (a), we immediately have  $S_K(A, B) \subseteq \mathcal{S}_1$ . On the other hand, take  $C \in \mathcal{S}_1$ . Since  $C_3$  is injective it can be checked that  $\mathcal{N}(M_C) = \mathcal{N}(A)$  which by hyperexactness of  $A$  implies that  $M_C$  is hyperexact. Since  $A_1$  and  $B_1$  are invertible and  $C_3$  is left invertible, it is not difficult to verify that  $\mathcal{R}(M_C)$  is closed (use left invertibility of  $A_1$ ,  $B_1$ , and  $C_3$ ). Hence,  $C \in S_K(A, B)$ . Suppose (ii) holds: Let us denote the set from the right side of the equality in (ii) by  $\mathcal{S}_2$  and let  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  be such that  $M_C$  is Kato nonsingular. Then by (a), we have that  $C_3$  and  $P$  are left invertible. By Theorem 1.15, it follows that  $\begin{bmatrix} (I - P_{\mathcal{R}(C_3)})C_4 \\ B_1 \end{bmatrix}$  is left invertible. As  $B_1$  is injective, this is equivalent to saying that  $\mathcal{R}(\begin{bmatrix} C_4^*(I - P_{\mathcal{R}(C_3)}) & B_1^* \end{bmatrix})$  is closed. Hence  $C \in \mathcal{S}_2$ . Conversely, let  $C \in \mathcal{S}_2$ . By injectivity of  $C_3$  we have  $\mathcal{N}(M_C) = \mathcal{N}(A)$ , so  $M_C$  is hyperexact since  $A$  is. By Theorem 2.5 of [47] and injectivity of  $B_1$ , we have that  $P$  is left invertible. Thus  $\mathcal{R}(P)$  is closed. From (b) it now follows that  $\mathcal{R}(M_C)$  is closed. Hence,  $C \in S_K(A, B)$ .  $\square$

In the special case when  $B$  is a nilpotent operator, we have the following result:

**COROLLARY 1.110.** Let  $A \in \mathcal{B}(\mathcal{H})$  and let  $B \in \mathcal{B}(\mathcal{K})$  be nilpotent. Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if  $A$  is Kato nonsingular and  $d(A) = \infty$ .

**PROOF.** By Theorem 1.109 we only need to show that if  $M_C$  is Kato nonsingular for some  $C$ , then  $d(A) = \infty$ . But then, as  $A$  is Kato nonsingular, so is  $M_C^k = \begin{bmatrix} A^k & * \\ 0 & 0 \end{bmatrix}$ , where  $k \in \mathbb{N}$  is such that  $B^k = 0$ . Using Theorem 1.107 once again we conclude that  $k d(A) \geq d(A^k) \geq \min\{\dim(\mathcal{N}(B^k)), \infty\} = \infty$ .  $\square$

Dually to Theorem 1.109 one has the following result.

**THEOREM 1.111.** Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be such that  $\text{dsc}(A) < \infty$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following conditions is satisfied:

- (i)  $B$  is Kato nonsingular,  $\mathcal{R}(A)$  is closed, and  $d(A) \leq \alpha(B)$ . In this case,

$$S_K(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}, \right. \\ \left. C_3 \text{ is right invertible} \right\}.$$

(ii)  $B$  is Kato nonsingular,  $\mathcal{R}(A)$  is not closed, and  $\alpha(B) = \infty$ . In this case,

$$S_K(A, B) = \left\{ C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{bmatrix}, \right. \\ \left. C_3 \text{ is right invertible, } \mathcal{R}(A) + \mathcal{R}(C_1(I - P_{\mathcal{R}(C_3^\perp)})) \text{ is closed} \right\}.$$

PROOF. Notice that  $\text{dsc}(A) < \infty$  implies that  $\text{asc}(A^*) < \infty$ , so the proof follows directly by Theorem 1.109 and Lemma 1.96.  $\square$

The problem of existence of  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular in the case when  $B$  is injective was almost completely covered by Theorems 2.3 and 2.4 from [14]. Below we complete these results and additionally provide a description of the set  $S_K(A, B)$ .

**THEOREM 1.112.** *Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  be an injective operator. Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following conditions is satisfied:*

(i)  $A$  is Kato nonsingular and  $\mathcal{R}(B)$  are closed. In this case,

$$S_K(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(ii)  $A$  is Kato nonsingular,  $\mathcal{R}(B)$  is nonclosed, and  $d(A) = \infty$ . In this case,

$$(1.79) \quad S_K(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(B^*) + \mathcal{R}(C^*P_{\mathcal{R}(A)^\perp}) \text{ is closed}\}.$$

PROOF. From Theorems 2.3 and 2.4 from [14] and Theorem 1.109 it follows that  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if (i) or (ii) holds. Also that  $S_K(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H})$ , provided (i) holds, was proved in Theorem 2.4 from [14]. We only need to establish the equality (1.79) if the conditions in (ii) are satisfied. But this directly follows from Theorem 1.109 since  $\mathcal{N}(B) = \{0\}$  implies that  $C_3 = 0$ .  $\square$

Taking adjoints in the previous theorem, we find necessary and sufficient conditions for the existence of  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular in the case when  $A$  is with dense range.

**THEOREM 1.113.** *Let  $A \in \mathcal{B}(\mathcal{H})$  be with dense range and  $B \in \mathcal{B}(\mathcal{K})$ . Then  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  if and only if one of the following conditions is satisfied:*

(i)  $B$  is Kato nonsingular and  $\mathcal{R}(A)$  is closed. In this case,

$$S_K(A, B) = \mathcal{B}(\mathcal{K}, \mathcal{H}).$$

(ii)  $B$  is Kato nonsingular,  $\mathcal{R}(A)$  is nonclosed, and  $\alpha(B) = \infty$ . In this case,

$$S_K(A, B) = \{C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) : \mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)}) \text{ is closed}\}.$$

**Cases 6–7:**  $0 \notin \text{int}(\sigma_p(B))$  or  $0 \notin \text{int}(\sigma_{cp}(A))$ . We have seen that when  $\text{asc}(B) < \infty$  or  $\text{dsc}(A) < \infty$ , one of the operators of  $A$  and  $B$  must be Kato nonsingular in order for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  to exist such that  $M_C$  is Kato nonsingular. However it can happen that  $M_C$  is Kato nonsingular without  $\mathcal{R}(B)$  or  $\mathcal{R}(A)$  being closed (see Example 3 of [107]). In what follows we describe yet one more situation when, for the completion problem to Kato nonsingularity to have an affirmative answer, it is necessary that one of the operators  $A$  or  $B$  is Kato nonsingular.

PROPOSITION 1.114. Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ . If  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  and  $\text{acc}(\mathbb{C} \setminus \sigma_p(B)) \cap K(0, \gamma(M_C)) \neq \emptyset$ , then  $\mathcal{N}(A) = \mathcal{N}(M_C)$ .

PROOF. We know that  $\mathcal{N}(A) \subseteq \mathcal{N}(M_C)$ . Towards a contradiction suppose there is some  $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{N}(M_C) \setminus \mathcal{N}(A)$ . Then

$$\begin{cases} Ax + Cy = 0, \\ By = 0, \end{cases}$$

and  $y \neq 0$ . Put  $r = \gamma(M_C)$ . By Lemma 1.98 there is an analytic function  $f : K(0, r) \rightarrow \mathcal{H} \oplus \mathcal{K}$  such that  $f(0) = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $f(\lambda) \in \mathcal{N}(M_C - \lambda)$  for all  $\lambda \in K(0, r)$ . The assumption that  $\text{acc}(\mathbb{C} \setminus \sigma_p(B)) \cap K(0, \gamma(M_C)) \neq \emptyset$  implies that for the analytic function  $z \mapsto P_{\mathcal{K}}f(z)$ ,  $z \in K(0, r)$ , we have  $P_{\mathcal{K}}f(z) = 0$  for infinitely many points  $z$  belonging to a closed disc contained in the open connected set  $K(0, r)$ . Thus  $P_{\mathcal{K}}f(z) = 0$  for all  $z \in K(0, r)$ . But then  $y = P_{\mathcal{K}}f(0) = 0$ , a contradiction.  $\square$

For our next result we need the following characterization of Kato nonsingularity from [147]:

THEOREM 1.115 ([147]). Let  $T \in \mathcal{B}(\mathcal{H})$  be an operator with closed range. The following conditions are equivalent:

- (1)  $T$  is Kato nonsingular.
- (2)  $\mathcal{N}(T) \subseteq \bigvee_{z \neq 0} \mathcal{N}(T - z)$ .

THEOREM 1.116. Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ . If  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  and  $0 \notin \text{int}(\sigma_p(B))$ , then:

- (1)  $\mathcal{N}(A^k) = \mathcal{N}(M_C^k)$  for all  $k \in \mathbb{N}$ .
- (2)  $A$  is Kato nonsingular.

PROOF. Observe that if  $0 \notin \sigma_p(B)$ , then both conclusions follow from Theorem 1.107 and Proposition 1.108. Thus in the sequel we assume that  $0 \in \sigma_p(B)$ .

(1) Since  $\sigma_p(q(B)) = \{q(\lambda) : \lambda \in \sigma_p(B)\}$ , for all polynomials  $q$ , we have  $0 \notin \text{int}(\sigma_p(B^k))$  for all  $k \in \mathbb{N}$ . Thus, using Lemma 1.96, we need only prove the assertion under (1) for  $k = 1$ . But this case is covered by Proposition 1.114 since  $0 \in \sigma_p(B) \setminus \text{int}(\sigma_p(B))$  and  $0 \in K(0, \gamma(M_C))$ .

(2) By Proposition 1.108 and what we have just proved under (1) we already know that  $\mathcal{R}(A)$  must be closed.

Put  $r = \gamma(M_C)$ . By Theorems 1.92 and 1.97, there is some  $\epsilon \in (0, r/2)$  such that  $M_C - zI$  is Kato nonsingular and  $\gamma(M_C - zI) > r/2$  for all  $z \in K(0, \epsilon)$ .

Now we prove that  $\mathcal{N}(M_C - zI) = \mathcal{N}(A - zI)$  for all  $z \in K(0, \epsilon)$ . To see this fix  $z \in K(0, \epsilon)$ . Here  $0 \in \sigma_p(B) \setminus \text{int}(\sigma_p(B))$  gives

$$-z \in \sigma_p(B - zI) \setminus \text{int}(\sigma_p(B - zI)).$$

It follows that

$$-z \in \text{acc}(\mathbb{C} \setminus \sigma_p(B - zI)) \cap K(0, r/2),$$

which in turn, by Kato nonsingularity of  $M_C - zI$  and Proposition 1.114, yields  $\mathcal{N}(M_C - zI) = \mathcal{N}(A - zI)$ , since  $r/2 < \gamma(M_C - zI)$ .

Now let  $x \in \mathcal{N}(A)$  be arbitrary. Since  $x \in \mathcal{N}(M_C)$  by Lemma 1.98 there is an analytic function  $f : K(0, \epsilon) \rightarrow \mathcal{H} \oplus \mathcal{K}$  with  $f(0) = x$  and  $f(z) \in \mathcal{N}(M_C - zI)$  for all  $z \in K(0, \epsilon)$ . We have

$$x = f(0) = \frac{1}{2\pi i} \int_{|z|=\epsilon/2} \frac{f(z)}{z} dz \in \overline{\bigvee_{|z|=\epsilon/2} \mathcal{N}(M_C - zI)}.$$

Since  $\mathcal{N}(M_C - zI) = \mathcal{N}(A - zI)$ , we thus have  $x \in \overline{\bigvee_{z \neq 0} \mathcal{N}(A - zI)}$ , and consequently  $A$  must be Kato nonsingular by Theorem 1.115. □

Combining Theorems 1.103 and 1.116 the reader can easily give the answer to the completion problem considered, in the case  $0 \notin \text{int}(\sigma_p(B))$ .

**COROLLARY 1.117.** Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ . If  $A$  is not Kato nonsingular, then there is no  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  such that  $M_C$  is Kato nonsingular whenever  $\text{int}(\sigma_p(B)) = \emptyset$ , thus in particular in each of the following cases:

- $B$  is a compact operator,
- $B$  is a quasinilpotent operator,
- $B$  is an algebraic operator.

Easily, we can conclude the following:

**THEOREM 1.118.** Let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ . If  $M_C$  is Kato nonsingular for some  $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  and  $0 \notin \text{int}(\sigma_{cp}(A))$ , then  $B$  must be Kato nonsingular.

### 1.9. Fredholm consistency of upper triangular operator matrices

Let  $\mathcal{S}(\mathcal{H})$  denote a subset of  $\mathcal{B}(\mathcal{H})$ . An operator  $A \in \mathcal{B}(\mathcal{H})$  is said to be  $\mathcal{S}(\mathcal{H})$  consistent, or consistent in  $\mathcal{S}(\mathcal{H})$ , if

$$AB \in \mathcal{S}(\mathcal{H}) \Leftrightarrow BA \in \mathcal{S}(\mathcal{H}),$$

for every  $B \in \mathcal{B}(\mathcal{H})$ .

A characterization of operators consistent in invertibility (CI operators) and a characterization of CI operators invariant under compact perturbations are given in [86]. It is worth mentioning the very interesting paper of D. S. Djordjević et al. [63] where the property of being  $\mathcal{S}(\mathcal{H})$  consistent is related with the single-valued extension property (SVEP) for a variety of choices of the subset  $\mathcal{S}(\mathcal{H})$  of  $\mathcal{B}(\mathcal{H})$ . Recall that  $A \in \mathcal{B}(\mathcal{H})$  has the SVEP at a point  $\lambda_0 \in \mathbb{C}$  if for every open neighborhood  $U_\lambda$  of  $\lambda_0$  the only analytic function  $f : U_\lambda \rightarrow \mathcal{H}$  satisfying  $(A - \lambda)f(\lambda) = 0$  for every  $\lambda \in U_\lambda$  is the function  $f = 0$ . Evidently,  $A$  has the SVEP at every point of the resolvent set  $\rho(A)$  of  $A$ . The SVEP provides a simple sufficient condition for determining  $\mathcal{S}(\mathcal{H})$  consistent operators  $A \in \mathcal{B}(\mathcal{H})$  for a variety of choices of the subset  $\mathcal{S}(\mathcal{H})$  of  $\mathcal{B}(\mathcal{H})$ .

The work presented in this section is a continuation of the work of G. Hai et al. [91] in which the problem of completions of the upper triangular operator matrix

$$(1.80) \quad \begin{bmatrix} A & ? \\ 0 & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$$

to consistent invertibility was considered, where  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$  are given operators on separable Hilbert spaces.

(2) First suppose that  $\mathcal{R}(R)$  is closed. Then  $\mathcal{M} = S^{-1}[\mathcal{R}(R)]$  is closed. In that case operators  $S$  and  $R$  have the following representations:

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(R)^\perp \\ \mathcal{N}(R) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix},$$

where  $S_3$  is injective and  $R_1$  is invertible. Since  $\mathcal{R}(R)$  is infinite dimensional, there exists a left invertible  $T : \mathcal{M} \rightarrow \mathcal{R}(R)$ .

In the case when  $\mathcal{R}(S_3)$  is closed, since  $S_3$  is injective for

$$X = \begin{bmatrix} R_1^{-1}(T - S_1) & -R_1^{-1}S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(R)^\perp \\ \mathcal{N}(R) \end{bmatrix},$$

we have that

$$S + RX = \begin{bmatrix} T & 0 \\ 0 & S_3 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix};$$

i.e.,  $S + RX$  is left invertible. In the case when  $\mathcal{R}(S_3)$  is nonclosed, take  $T_0 : \mathcal{M} \rightarrow \mathcal{R}(R)$  to be a left invertible operator such that  $d(T_0) = \infty$ . By Theorem 1.18 we know that there exists an operator  $Y \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{R}(R))$  such that

$$\begin{bmatrix} T_0 & Y \\ 0 & S_3 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix}$$

has closed range. Let

$$X = \begin{bmatrix} R_1^{-1}(T_0 - S_1) & R_1^{-1}(Y - S_2) \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(R)^\perp \\ \mathcal{N}(R) \end{bmatrix}.$$

It can be checked that  $S + RX = \begin{bmatrix} T_0 & Y \\ 0 & S_3 \end{bmatrix}$  and that this is a left invertible operator.

If  $\mathcal{R}(R)$  is nonclosed, then there exists a closed infinite-dimensional subspace  $\mathcal{M}$  of  $\mathcal{H}$  such that  $\mathcal{R}(R|_{\mathcal{M}})$  is closed and infinite dimensional. Let  $T \in \mathcal{B}(\mathcal{H})$  be such that  $\mathcal{R}(T) = \mathcal{M}$ . Then  $\mathcal{R}(RT)$  is closed and  $RT$  is a noncompact operator, so we can apply the first part of the proof.  $\square$

If we compare Theorems 2.10 and 2.27 in the case when  $R$  is a compact operator, it is not so evident that if  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  is right invertible and  $S$  is Weyl, the condition  $\text{codim}_{\mathcal{R}(R)}(\mathcal{R}(S) \cap \mathcal{R}(R)) \geq \dim(\mathcal{N}(S))$  from Theorem 2.27(1) is satisfied. Indeed, suppose that  $T$  is a subspace such that  $\mathcal{R}(R) = \mathcal{R}(R) \cap \mathcal{R}(S) \oplus T$ . Then  $\text{codim}_{\mathcal{R}(R)}(\mathcal{R}(S) \cap \mathcal{R}(R)) = \dim T$  and since  $\mathcal{R}(S) + \mathcal{R}(R) = \mathcal{H}$  we have that  $\mathcal{R}(S) + T = \mathcal{H}$ . Hence  $\dim T \geq \dim(\mathcal{R}(S)^\perp)$ . Since  $S$  is Weyl it follows that  $\dim T \geq \dim(\mathcal{N}(S))$ .

## 2.6. Fredholmness of an operator $A + CX$

We start this section with a characterization of all pairs of operators  $(S, R)$  for which the operator  $S + RX$  is left semi-Fredholm for some  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ .

**THEOREM 2.28.** *Let  $S \in \mathcal{B}(\mathcal{H})$  and  $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ . The following hold:*

- (1) *If  $R$  is compact, then there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is left semi-Fredholm if and only if  $S$  is left semi-Fredholm.*

- (2) If  $R$  is noncompact, then there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is left semi-Fredholm.

PROOF. (1) If  $R$  is compact and there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is left semi-Fredholm, then  $S$  is left semi-Fredholm. If  $S$  is left semi-Fredholm, take  $X = 0$ .

(2) If  $R$  is noncompact, then by Theorem 2.27, there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is left invertible; i.e., it is left semi-Fredholm.  $\square$

Now, we consider the case of right semi-Fredholmness which is a very complicated one. Notice that the right semi-Fredholmness of  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  is clearly a necessary condition for the existence of  $X$  such that  $S + RX$  is right semi-Fredholm.

THEOREM 2.29. *Let  $S \in \mathcal{B}(\mathcal{H})$  and  $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ . The following hold:*

- (1) *If  $R$  is compact, then there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm if and only if  $S$  is right semi-Fredholm.*
- (2) *If  $R$  is noncompact, then there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm if and only if  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  is right semi-Fredholm and  $\mathcal{N}(S|R)$  contains a noncompact operator.*

PROOF. (1) If  $R$  is compact and there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm, then  $S$  is right semi-Fredholm. If  $S$  is right semi-Fredholm, take  $X = 0$ .

(2) Suppose that  $R$  is noncompact and that there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm. Since

$$S + RX = \begin{bmatrix} S & R \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix},$$

it follows that  $\begin{bmatrix} S & R \end{bmatrix}$  is a right semi-Fredholm operator and that  $\mathcal{R}\left(\begin{bmatrix} I \\ X \end{bmatrix}\right) + \mathcal{N}\left(\begin{bmatrix} S & R \end{bmatrix}\right)$  is a subspace of finite codimension in  $\mathcal{H} \oplus \mathcal{K}$ . Now, let  $\begin{bmatrix} G \\ H \end{bmatrix} : \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K}$  be such that

$$(2.4) \quad \mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} S & R \end{bmatrix}\right).$$

Now, we have that  $W = \begin{bmatrix} I & G \\ X & H \end{bmatrix}$  is right semi-Fredholm. If  $\mathcal{N}(S|R)$  contains only compact operators, then  $G$  is a compact operator, so  $W = \begin{bmatrix} I & 0 \\ X & H \end{bmatrix}$  is right semi-Fredholm. This implies that  $H$  is right semi-Fredholm, so there exists  $T \in \mathcal{B}(\mathcal{H})$  such that  $HT = I + K$ , for some compact operator  $K \in \mathcal{B}(\mathcal{H})$ . Since  $SG + RH = 0$ , we have that  $SGT + R + RK = 0$ ; i.e.,  $R$  is a compact operator. Thus we get a contradiction, so  $\mathcal{N}(S|R)$  contains a noncompact operator.

For the converse, if  $\begin{bmatrix} S & R \end{bmatrix}$  is right semi-Fredholm and  $\mathcal{N}(S|R)$  contains a noncompact operator, the proof follows K. Takahashi's proof (see [160]) for the case of invertibility, but we present it here for the sake of completeness. First remark that if  $\mathcal{N}(S|R)$  contains a noncompact operator, then  $\dim(\mathcal{N}\left(\begin{bmatrix} S & R \end{bmatrix}\right)) = \infty$ .

Let  $\begin{bmatrix} G \\ H \end{bmatrix} : \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K}$  be left invertible such that

$$(2.5) \quad \mathcal{R} \left( \begin{bmatrix} G \\ H \end{bmatrix} \right) = \mathcal{N}([S \ R]).$$

We consider 3 cases:

Case 1:  $R$  is left invertible: Evidently, since  $R_l^{-1}SG + H = 0$ , we get that  $G$  is left invertible. Now, there exists an operator  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $H - XG$  is invertible (for example let  $X = G_l^{-1} - R_l^{-1}S$ ). Now, by

$$\begin{bmatrix} I & G \\ X & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & G \\ 0 & H - XG \end{bmatrix},$$

we have that  $\begin{bmatrix} I & G \\ X & H \end{bmatrix}$  is invertible. From

$$[S \ R] \begin{bmatrix} I & G \\ X & H \end{bmatrix} = [S + RX \ 0],$$

we get that  $S + RX$  is right semi-Fredholm. Hence the case when  $R$  is left invertible is solved.

Case 2:  $S$  is not left semi-Fredholm: Let  $E$  be the spectral measure of  $R^*R$ . For  $x \in \mathcal{K}$ , let  $E_x$  be the positive measure given by  $E_x(S) = (E(S)x, x)$  for a Borel subset  $S$  of  $\mathbb{C}$ . First we prove that there exists  $\epsilon > 0$  such that  $\mathcal{K}_\epsilon = \mathcal{R}(E(\epsilon, \infty))$  is infinite dimensional. Let  $R_\epsilon = RE(\epsilon, \infty)$ . Then

$$\|(R - R_\epsilon)x\|^2 = \int \lambda \chi_{[0, \epsilon]} dE_{x_0}(\lambda) \leq \epsilon \|x\|^2,$$

where  $x_0 = E[0, \epsilon]x$ . If for each  $\epsilon > 0$ ,  $\mathcal{K}_\epsilon$  is finite dimensional, then we have that  $R$  is compact as a limit of finite-dimensional operators  $R_\epsilon$ . Hence there exists  $\epsilon_0 > 0$  such that  $\mathcal{K}_{\epsilon_0} = \mathcal{R}(E(\epsilon_0, \infty))$  is infinite dimensional. Since the set of right semi-Fredholm operators is open in the set of bounded linear operators, we get that for small enough  $\epsilon > 0$  such that  $\epsilon < \epsilon_0$ , the operator  $[S \ R_\epsilon]$  is right semi-Fredholm; i.e.,  $[S \ R]_{\mathcal{K}_\epsilon}$  is right semi-Fredholm, where  $\mathcal{K}_\epsilon$  is an infinite-dimensional closed subspace since  $\mathcal{K}_{\epsilon_0} \subseteq \mathcal{K}_\epsilon$ . Also, we have that

$$\|R_\epsilon x\|^2 = (R^*R x_0, x_0) = \int \lambda \chi_{(\epsilon, \infty)} dE_{x_0}(\lambda) \geq \epsilon \|x_0\|^2,$$

where  $x_0 = E(\epsilon, \infty)x$ , implying that  $R|_{\mathcal{K}_\epsilon}$  is left invertible. Furthermore, we have  $\dim \mathcal{N}([S \ R]_{\mathcal{K}_\epsilon}) = \infty$ . In fact, if  $\dim(\mathcal{N}([S \ R]_{\mathcal{K}_\epsilon})) < \infty$ , then  $[S \ R]_{\mathcal{K}_\epsilon}$  is left semi-Fredholm, which contradicts the assumption that  $S$  is not left semi-Fredholm. Indeed, from  $\dim(\mathcal{N}([S \ R]_{\mathcal{K}_\epsilon})) < \infty$  we can conclude that  $\dim(\mathcal{N}(S)) < \infty$  and that  $\dim(\mathcal{R}(S) \cap \mathcal{R}(R|_{\mathcal{K}_\epsilon})) < \infty$  which, by the fact that  $[S \ R]_{\mathcal{K}_\epsilon}$  is right semi-Fredholm, implies that  $\mathcal{R}(S)$  is closed. Hence  $S$  is left semi-Fredholm. Now, by Case 1 of this proof there exists  $F_1 \in \mathcal{B}(\mathcal{L}, \mathcal{K}_\epsilon)$  such that  $S + (R|_{\mathcal{K}_\epsilon})F_1$  is invertible. Define  $F = JF_1$  where  $J : \mathcal{K}_\epsilon \rightarrow \mathcal{K}$  is the inclusion mapping. Then  $S + RF$  is right semi-Fredholm.

Case 3:  $S$  is left semi-Fredholm: It is sufficient to show that there exists a closed infinite-dimensional subspace  $\mathcal{K}_1$  of  $\mathcal{K}$  such that  $R_1 = R|_{\mathcal{K}_1}$  is left invertible and  $\begin{bmatrix} S & R_1 \end{bmatrix}$  is a right semi-Fredholm operator with  $\dim(\mathcal{N}(\begin{bmatrix} S & R_1 \end{bmatrix})) = \infty$ . By assumption there exists noncompact  $D \in \mathcal{N}(S|R)$ ; i.e.,  $SD + RT = 0$ , for some  $T \in \mathcal{B}(\mathcal{K})$ . Thus  $\mathcal{R}\left(\begin{bmatrix} D \\ T \end{bmatrix}\right) \subseteq \mathcal{N}(\begin{bmatrix} S & R \end{bmatrix})$  which by (2.5) implies that  $\mathcal{R}(D) \subseteq \mathcal{R}(G)$ , so  $G$  is noncompact. Also, from (2.5) we have that  $SG + RH = 0$  which, together with the fact that  $S$  is left semi-Fredholm, implies that  $RH$  is noncompact. Now similarly as in the previous case using the spectral measure of  $(RH)^*(RH)$ , there exists an infinite-dimensional subspace  $\mathcal{K}_0$  of  $\mathcal{K}$  such that  $RH|_{\mathcal{K}_0}$  is left invertible. Let  $\mathcal{K}_1 = H[\mathcal{K}_0] \oplus \mathcal{R}(H)^\perp$ . To prove that  $\mathcal{K}_1$  is closed it is sufficient to prove that  $H[\mathcal{K}_0]$  is closed. Since  $RH[\mathcal{K}_0]$  is closed it follows that  $R^{-1}[RH[\mathcal{K}_0]] = H[\mathcal{K}_0] \oplus \mathcal{N}(R)$  is closed as an inverse image of a closed subspace. Hence  $H[\mathcal{K}_0]$  is a closed subspace. Now we have that  $\mathcal{K}_1$  is closed. Let us show that it satisfies the required conditions. First, we prove that  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp} : \mathcal{R}(H)^\perp \rightarrow \mathcal{R}(S)^\perp$  is injective and right semi-Fredholm. Suppose that for  $h \in \mathcal{R}(H)^\perp$ ,  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}Rh = 0$ .

Then  $Rh \in \mathcal{R}(S)$ , so there exists  $t \in \mathcal{K}$  such that  $Rh = St$ . Now  $\begin{bmatrix} t \\ -h \end{bmatrix} \in \mathcal{N}(\begin{bmatrix} S & R \end{bmatrix})$  implying by (2.5) that  $h \in \mathcal{R}(H)$ . Hence,  $h = 0$  and we have that  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp} : \mathcal{R}(H)^\perp \rightarrow \mathcal{R}(S)^\perp$  is injective. Right semi-Fredholmness of  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp} : \mathcal{R}(H)^\perp \rightarrow \mathcal{R}(S)^\perp$  is established as follows: Since  $\begin{bmatrix} S & R \end{bmatrix}$  is right semi-Fredholm we have that  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R$  is of finite codimension in  $\mathcal{R}(S)^\perp$ . Since  $\mathcal{R}(R|_{\mathcal{R}(H)}) \subseteq \mathcal{R}(S)$  (follows by (2.5)) we have  $\mathcal{R}(P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp}) = \mathcal{R}(P_{\mathcal{R}(S)^\perp}^{\text{rst}}R)$  implying that  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp} : \mathcal{R}(H)^\perp \rightarrow \mathcal{R}(S)^\perp$  is right semi-Fredholm.

Now, by the right semi-Fredholmness of  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp} : \mathcal{R}(H)^\perp \rightarrow \mathcal{R}(S)^\perp$ , having in mind that  $\mathcal{R}(R|_{\mathcal{R}(H)^\perp}) \subseteq \mathcal{R}(R_1)$  and that  $\begin{bmatrix} S & R \end{bmatrix}$  is right semi-Fredholm, we get that  $\begin{bmatrix} S & R_1 \end{bmatrix}$  is right semi-Fredholm, where  $R_1 = R|_{\mathcal{K}_1}$ .

The left invertibility of  $RH|_{\mathcal{K}_0}$  implies that of  $R|_{H[\mathcal{K}_0]} : H[\mathcal{K}_0] \rightarrow \mathcal{R}(S)$ , which together with the left invertibility of  $P_{\mathcal{R}(S)^\perp}^{\text{rst}}R|_{\mathcal{R}(H)^\perp}$  implies that  $R_1$  is left invertible. Since

$$\mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}|_{\mathcal{K}_0}\right) \subseteq \mathcal{N}(\begin{bmatrix} S & R_1 \end{bmatrix})$$

and since  $\mathcal{K}_0$  is infinite dimensional and  $\begin{bmatrix} G \\ H \end{bmatrix}$  is left invertible, it follows that  $\dim(\mathcal{N}(\begin{bmatrix} S & R_1 \end{bmatrix})) = \infty$ .  $\square$

In the next theorem, we present necessary and sufficient conditions for the existence of  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is a Fredholm operator:

**THEOREM 2.30.** *Let  $S \in \mathcal{B}(\mathcal{H})$  and  $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ . The following hold:*

- (1) *If  $R$  is compact, there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is Fredholm if and only if  $S$  is a Fredholm operator.*
- (2) *If  $R$  is noncompact, there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is Fredholm if and only if  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  is right semi-Fredholm and  $\mathcal{N}(S|R)$  contains a noncompact operator.*



PROOF. (1) If  $R$  is compact and for some  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ,  $S + RX$  is Fredholm, then we have that  $S$  is a Fredholm operator. In the other direction take  $X = 0$ .

(2) Suppose that  $R$  is noncompact,  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  is right semi-Fredholm, and  $\mathcal{N}(S|R)$  contains a noncompact operator. We consider 3 cases:

Case 1:  $\mathcal{R}(R)$ : Then  $\mathcal{M} = S^{-1}[\mathcal{R}(R)]$  is closed and since  $\mathcal{N}(S|R)$  contains a noncompact operator, we can conclude that  $\mathcal{M}$  is infinite dimensional. In that case operators  $S$  and  $R$  have the following representations:

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_3 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix},$$

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(R)^\perp \\ \mathcal{N}(R) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix},$$

where  $S_3$  is injective and  $R_1$  is invertible. Also since  $\begin{bmatrix} S & R \end{bmatrix}$  is right semi-Fredholm, it follows that  $S_3$  is right semi-Fredholm. Hence  $S_3$  is a Fredholm operator. Since  $\mathcal{R}(R)$  is infinite dimensional, there exists an invertible  $T : \mathcal{M} \rightarrow \mathcal{R}(R)$ .

For

$$X = \begin{bmatrix} R_1^{-1}(T - S_1) & -R_1^{-1}S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{N}(R)^\perp \\ \mathcal{N}(R) \end{bmatrix},$$

we have that

$$S + RX = \begin{bmatrix} T & 0 \\ 0 & S_3 \end{bmatrix} : \begin{bmatrix} \mathcal{M} \\ \mathcal{M}^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(R) \\ \mathcal{R}(R)^\perp \end{bmatrix};$$

i.e.,  $S + RX$  is a Fredholm operator.

Case 2:  $S$  is not left semi-Fredholm: As in the proof of Theorem 2.29, let  $E$  be the spectral measure of  $R^*R$ . We know that there exists  $\epsilon > 0$  such that  $\mathcal{K}_\epsilon = \mathcal{R}(E(\epsilon, \infty))$  is closed and infinite dimensional,  $R_\epsilon = RE(\epsilon, \infty)$  is left invertible, and  $\begin{bmatrix} S & R_\epsilon \end{bmatrix}$  is right semi-Fredholm; i.e.,  $\begin{bmatrix} S & R|_{\mathcal{K}_\epsilon} \end{bmatrix}$  is right semi-Fredholm. Furthermore, we have  $\dim \mathcal{N}(\begin{bmatrix} S & R|_{\mathcal{K}_\epsilon} \end{bmatrix}) = \infty$  which, since  $R|_{\mathcal{K}_\epsilon}$  is left invertible, implies that  $\mathcal{M} = S^{-1}[R|_{\mathcal{K}_\epsilon}]$  is closed and infinite dimensional. Now we consider Case 1 where instead of  $R$  we consider  $R|_{\mathcal{K}_\epsilon}$ .

Case 3:  $S$  is left semi-Fredholm: Using Case 1, it is sufficient to show that there exists an infinite-dimensional subspace  $\mathcal{K}_0$  such that  $R|_{\mathcal{K}_0}$  has closed infinite-dimensional range. Since there exists a noncompact  $G \in \mathcal{N}(S|R)$ , we have that  $SG + RH = 0$  for some  $H \in \mathcal{B}(\mathcal{K})$ . Since  $S$  is left semi-Fredholm we have that  $SG$  is noncompact, so  $RH$  is noncompact. This means that there exists a closed infinite-dimensional subspace  $\mathcal{T} \subseteq \mathcal{R}(RH)$ . Let  $\mathcal{K}_1 = R^{-1}[\mathcal{T}]$ . Then clearly  $\mathcal{K}_1$  is closed and infinite dimensional and  $\mathcal{R}(R|_{\mathcal{K}_1}) = \mathcal{T}$  is closed.

The reverse implication follows directly from Theorem 2.29.  $\square$

From Theorems 2.25 and 2.28-2.30, we get the following corollaries:

COROLLARY 2.31. Let  $S \in \mathcal{B}(\mathcal{H})$ , and let  $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  be a noncompact operator. Then the following are equivalent:

- (1) There exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm.
- (2) There exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is Fredholm.

COROLLARY 2.32. Let  $S \in \mathcal{B}(\mathcal{H})$ , let  $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$  be a noncompact operator, and let  $\begin{bmatrix} S & R \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H}$  be right invertible. Then the following are equivalent:

- (1) There exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is invertible.
- (2) There exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right invertible.
- (3) There exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $S + RX$  is right semi-Fredholm.

### 2.7. Dense range of an operator $A + CX$

In [55] the problem of characterization of all the pairs of operators  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$  for which the operator  $A + CX$  is injective for some  $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$  was completely solved. In view of the main result of the previous section, to sort of complete this line of investigation it naturally remains to answer the question for which operators  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$  there exists  $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$  such that the operator  $A + CX$  is with dense range. Unfortunately, we have not been able to fully solve this problem, but we have rather given some partial answers to it. In particular, we show that if  $A \notin \mathcal{F}_+(\mathcal{H}, \mathcal{K})$ , then there always exists such an operator  $X$ . Also, we give necessary and sufficient conditions for the existence of such an operator in the case when  $A \in \mathcal{F}(\mathcal{H}, \mathcal{K})$ .

As before, the claim that there exists an operator  $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$  such that  $A + CX$  is with dense range amounts to saying that  $\mathcal{R}(A + T)$  is dense for some  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  with  $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ . Clearly a necessary condition for this to hold is that the subspace  $\mathcal{R}(A) + \mathcal{R}(C)$  is dense in  $\mathcal{K}$  or, equivalently, that  $\mathcal{N}(A^*) \cap \mathcal{N}(C^*) = \{0\}$ .

A simple observation combined with Theorem 2.19 yields the following proposition.

PROPOSITION 2.33. Let  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $C \in \mathcal{B}(\mathcal{L}, \mathcal{K})$  be such that  $\mathcal{R}(A)$  is closed and  $\mathcal{R}(C)$  is finite dimensional. The following are equivalent:

- (1) There exists an operator  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $\mathcal{R}(T) \subseteq \mathcal{R}(C)$  and  $\mathcal{R}(A + T)$  is dense in  $\mathcal{K}$ .
- (2) There exists an operator  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $\mathcal{R}(T) \subseteq \mathcal{R}(C)$  and  $\mathcal{R}(A + T)$  is surjective.
- (3)  $\mathcal{R}(A) + \mathcal{R}(C) = \mathcal{K}$  and  $d(A) \leq \alpha(A)$ .

PROOF. By Lemma 1.64 the subspace  $\mathcal{R}(A + T)$  is closed for every  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $\mathcal{R}(T) \subseteq \mathcal{R}(C)$ . For this reason (1) and (2) say the same thing. To see that (3) is equivalent to these, in view of Theorem 2.19, it suffices to show that if there exists an operator  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that  $\mathcal{R}(T) \subseteq \mathcal{R}(C)$  and  $\mathcal{R}(A + T)$  is surjective, then it must be that  $d(A) \leq \alpha(A)$ . By Theorem 2.19 the inequality  $\alpha(A) < d(A)$  would imply the existence of an infinite-dimensional subspace  $M \subseteq \mathcal{H}$  such that  $A[M] \subseteq \mathcal{R}(C)$ . Given that  $\alpha(A) < \infty$ , we can infer from this that  $M \cap \mathcal{N}(A)^\perp$  must also be of infinite dimension, which would mean that  $A[M]$ , and thus  $\mathcal{R}(C)$  as well is infinite dimensional—a contradiction.  $\square$

As a second partial result we have the following conclusion which immediately follows from Theorem 2.20.

### 3.6. Completions of operator matrices $M_X$ to closed range operator

Call a triple  $(A, C, B)$  of operators closed range completable if there exists  $X \in \mathcal{B}(\mathcal{H})$  such that  $M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}$  has closed range. The closed range completion problem seeks a characterization of those triples  $(A, C, B)$  which are completable. Now we consider how to solve the closed range completion problem in various special cases.

(A) As an application of Theorem 2.41, we characterize completable triples of the form  $(A, C, I)$ :

**THEOREM 3.33.** *Let  $A, C \in \mathcal{B}(\mathcal{H})$ . There exists an operator  $X \in \mathcal{B}(\mathcal{H})$  such that  $\begin{bmatrix} A & C \\ X & I \end{bmatrix}$  is a closed range operator if and only if one of the following conditions holds:*

- (1)  $\mathcal{R}(A)$  is closed.
- (2)  $C$  is a noncompact operator.
- (3)  $C$  is a compact operator and there exists a closed subspace  $F$  of  $\mathcal{R}(A)$  such that  $\mathcal{R}(A) = F + \mathcal{R}(A) \cap \mathcal{R}(C)$ .

Note that the same result holds for the operator matrix  $\begin{bmatrix} A & C \\ X & B \end{bmatrix}$  in the case when  $B \in \mathcal{B}(\mathcal{H})$  is an invertible operator since

$$\begin{bmatrix} I & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} A & C \\ X & B \end{bmatrix} = \begin{bmatrix} A & C \\ B^{-1}X & I \end{bmatrix}.$$

(B) The next theorem characterizes completable triples of the form  $(A, C, 0)$ .

**THEOREM 3.34.** *Let  $A, C \in \mathcal{B}(\mathcal{H})$ . There exists an operator  $X \in \mathcal{B}(\mathcal{H})$  such that  $\begin{bmatrix} A & C \\ X & 0 \end{bmatrix}$  is a closed range operator if and only if  $\overline{\mathcal{R}(C)} \subseteq \mathcal{R}(A) + \mathcal{R}(C)$ .*

**PROOF.** Suppose first that  $\overline{\mathcal{R}(C)} \subseteq \mathcal{R}(A) + \mathcal{R}(C)$  which is equivalent with  $\overline{\mathcal{R}(C)} = \mathcal{R}(A) \cap \overline{\mathcal{R}(C)} + \mathcal{R}(C)$ . If  $\mathcal{R}(A) + \mathcal{R}(C)$  is closed, then clearly for  $X = 0$  we have that  $M_0$  has a closed range. If  $\mathcal{R}(C)$  is closed, then for any  $X \in \mathcal{B}(\mathcal{H})$ ,

$$M_X = \begin{bmatrix} A_1 & C_1 & 0 \\ A_2 & 0 & 0 \\ X & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{N}(C)^\perp \\ \mathcal{N}(C) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{R}(C)^\perp \\ \mathcal{H} \end{bmatrix},$$

where  $C_1$  is invertible. Evidently for invertible  $X : \mathcal{H} \rightarrow \mathcal{H}$  we get that  $\mathcal{R}(M_X)$  has closed range.

Now suppose that neither one of  $\mathcal{R}(A) + \mathcal{R}(C)$  and  $\mathcal{R}(C)$  is closed. Let  $\mathcal{M} = \{x : Ax \in \overline{\mathcal{R}(C)}\}$ . Evidently  $\mathcal{M}$  is a closed subspace. Since  $\mathcal{R}(C)$  is not closed, we conclude that  $\dim \mathcal{M} = \infty$ . Also, notice that  $\text{codim } \mathcal{M} = \infty$ . If this is not the case, then  $\overline{\mathcal{R}(A)} = \mathcal{R}(A) \cap \overline{\mathcal{R}(C)} + \mathcal{H}_1$ , where  $\mathcal{H}_1 = \{Ax : x \in \mathcal{M}^\perp\}$  is finite dimensional. Since  $\overline{\mathcal{R}(C)} = \mathcal{R}(A) \cap \overline{\mathcal{R}(C)} + \mathcal{R}(C)$  we have  $\overline{\mathcal{R}(C)} + \mathcal{H}_1 = \mathcal{R}(A) + \mathcal{R}(C)$  and we get that  $\mathcal{R}(A) + \mathcal{R}(C)$  is a closed subspace. This is in contradiction to the fact that  $\mathcal{R}(A) + \mathcal{R}(C)$  is not closed.

Let us define  $X \in \mathcal{B}(\mathcal{H})$  as

$$Xx = \begin{cases} 0, & x \in \mathcal{M}, \\ X_1x, & x \in \mathcal{M}^\perp, \end{cases}$$

where  $X_1 : \mathcal{M}^\perp \rightarrow \mathcal{H}$  is invertible. We show that for such  $X$ ,  $\mathcal{R}(M_X)$  is closed.

Now  $M_X$  can be represented by

$$M_X = \begin{bmatrix} A_1 & A_2 & C_1 & 0 \\ A_3 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{M} \\ \mathcal{N}(C)^\perp \\ \mathcal{N}(C) \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(C)} \\ \mathcal{R}(C)^\perp \\ \mathcal{H} \end{bmatrix},$$

where  $X_1$  is invertible and  $A_3$  is injective. Evidently since  $X_1$  is invertible it follows that  $\mathcal{R}(M_X)$  is closed if and only if  $\mathcal{R}(A_2) + \overline{\mathcal{R}(C_1)}$  is closed. Since  $\mathcal{R}(A) \cap \overline{\mathcal{R}(C)} = \mathcal{R}(A_2)$ , it follows that  $\mathcal{R}(A_2) + \overline{\mathcal{R}(C_1)} = \overline{\mathcal{R}(C)}$ .

For the converse, let us suppose that there exists an operator  $X \in \mathcal{B}(\mathcal{H})$  such that  $M_X$  has closed range and let  $y \in \overline{\mathcal{R}(C)}$ . Then  $\begin{bmatrix} y \\ 0 \end{bmatrix} \in \overline{\mathcal{R}(M_X)}$  which implies that  $y \in \mathcal{R}(A) + \mathcal{R}(C)$ . □

**(C)** Using the fact that an operator has closed range if and only if the same is true for its adjoint, we get that  $(0, C, B)$  is closed range completable if and only if  $\overline{\mathcal{R}(C^*)} \subseteq \mathcal{R}(B^*) + \mathcal{R}(C^*)$ .

**(D)** Similarly we characterize completable triples of the form  $(I, C, B)$ .

**(E)**  $(A, I, B)$  is always closed range completable since

$$(3.21) \quad \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \begin{bmatrix} A & I \\ X & B \end{bmatrix} \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ X - BA & 0 \end{bmatrix}.$$

**(F)** Theorems 2.5 and 2.6 from [67] address closed range completability of  $(A, 0, B)$ .

**(G)** Since multiplication of a row or column of  $M_X$  by an invertible operator does not affect whether its range is closed, parts **(B)**, **(C)**, and **(E)** above can be extended to cover all triples  $(A, C, B)$  where at least one of the given components  $A, B, C$  is invertible.

**(H)** Suppose  $C$  is a closed range operator. Then we can represent  $M_X$  as

$$M_X = \begin{bmatrix} A_1 & A_2 & C_1 & 0 \\ A_3 & A_4 & 0 & 0 \\ X_1 & X_2 & B_1 & B_2 \\ X_3 & X_4 & B_3 & B_4 \end{bmatrix},$$

where  $C_1$  is invertible. Subtraction of multiples of the first row from the third and fourth rows and of the third column from the first and second columns reduces the question of range closure to a three-by-three matrix whose first entry is 0, which by blocking is reduced to range closure of a matrix covered by part **(C)**.

Since from this theorem we see that the linear combinations  $\alpha P + \beta Q$  and  $\overline{\alpha}P + \overline{\beta}Q$  are simultaneously left invertible it is easily concluded that  $\alpha P + \beta Q$  is invertible if and only if it is left invertible. This gives the following corollaries:

**COROLLARY 5.37.** Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be given orthogonal projectors and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . Then  $\alpha P + \beta Q$  is invertible if and only if

$$\begin{cases} \mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta \neq 0, \\ \mathcal{R}(P) \oplus \mathcal{R}(Q) = \mathcal{H}, & \alpha + \beta = 0. \end{cases}$$

**COROLLARY 5.38.** Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be given orthogonal projectors and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . Then the following statements are equivalent:

- (i)  $\alpha P + \beta Q$  is left (right) invertible.
- (ii)  $\alpha P + \beta Q$  is invertible.

**COROLLARY 5.39.** Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be given orthogonal projectors and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  such that  $\alpha \neq -\beta$ . The linear combination  $\alpha P + \beta Q$  is invertible, independent of the choice of constants  $\alpha$  and  $\beta$ .

**REMARK 5.40.** In Corollary 4.3 from [120] it was proven that the sum of orthogonal projectors  $P$  and  $Q$  is invertible if and only if

$$\mathcal{N}(P) \cap \mathcal{N}(Q) = \mathcal{R}(P) \cap \mathcal{R}(Q(I - P)) = \{0\} \quad \text{and} \quad P + Q \text{ has closed range.}$$

This result is equivalent to Corollary 5.37 (for the case of sums of orthogonal projectors). Indeed, if  $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$ , we have that  $\overline{\mathcal{R}(P) + \mathcal{R}(Q)} = \mathcal{H}$ . Furthermore, if  $\mathcal{R}(P + Q)$  is closed, from Lemma 2.4 in [119], we have that  $\mathcal{R}(P) + \mathcal{R}(Q)$  is closed as well and thus  $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ . Conversely, if  $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$ , we have that  $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$  and, using Lemma 2.4 from [119] again, we have that  $P + Q$  has closed range.

Some interesting results can be obtained for right invertibility of linear combinations of oblique projectors as well:

**THEOREM 5.41.** Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be given projectors and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . Then  $\alpha P + \beta Q$  is right invertible if and only if  $\mathcal{R}(P) + \mathcal{R}(Q) = \mathcal{H}$  and

$$\begin{cases} \mathcal{R}(P) = \mathcal{R}(P(I - Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)), & \alpha \neq -\beta, \\ \mathcal{R}(P) = \mathcal{R}(P(I - Q)), & \alpha = \beta. \end{cases}$$

### 5.3. Fredholmness of the sum of two operators

In this section we address the question of the Fredholmness of the sum of two given operators  $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ . Let us recall that if  $\mathcal{M}$  is a closed subspace of  $\mathcal{H}$ , we use the symbol  $P_{\mathcal{M}}^{\text{rst}}$  to denote an operator from  $\mathcal{B}(\mathcal{H}, \mathcal{M})$  defined by  $P_{\mathcal{M}}^{\text{rst}}x = P_{\mathcal{M}}x$ , for all  $x \in \mathcal{H}$ , where  $P_{\mathcal{M}}$  is the orthogonal projector onto  $\mathcal{M}$ .

**THEOREM 5.42.** Let  $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  be given operators. Let  $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{P}$  and  $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{Q}$ . Then the operator  $A + B$  is Fredholm if and only if the following conditions hold:

- (i)  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$  and  $A|_{\mathcal{N}(B)}$  has closed range.
- (ii)  $\dim(\mathcal{N}(A') \cap \mathcal{N}(B')) < \infty$  and  $P_{\mathcal{Q}, \overline{\mathcal{R}(A)}}B$  has closed range.

- (iii)  $\dim(\mathcal{P} \cap \mathcal{N}(A+B)) < \infty$ ,  $\dim(\mathcal{R}(A|_{\mathcal{N}(B)}) \cap \mathcal{R}((A+B)|_{\mathcal{P}})) < \infty$  and  $\dim \overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) + \overline{(\mathcal{R}(A) \cap \mathcal{R}((A+B)|_{\mathcal{P}}))}) < \infty$ .

Furthermore, if (i)–(iii) hold, then

$$\begin{aligned} \alpha(A+B) &= \dim(\mathcal{N}(A) \cap \mathcal{N}(B)) + \dim(\mathcal{P} \cap \mathcal{N}(A+B)) \\ &\quad + \dim(\mathcal{R}(A|_{\mathcal{N}(B)}) \cap \mathcal{R}((A+B)|_{\mathcal{P}})) \\ \beta(A+B) &= \dim(\mathcal{N}(A') \cap \mathcal{N}(B')) \\ &\quad + \dim(\overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) + \overline{(\mathcal{R}(A) \cap \mathcal{R}((A+B)|_{\mathcal{P}}))})). \end{aligned}$$

PROOF. With respect to the decompositions given below,  $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  have the following representations:

$$(5.12) \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix},$$

$$(5.13) \quad B = \begin{bmatrix} 0 & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix}.$$

So  $A+B$  is a Fredholm operator if and only if the operator matrix given by

$$\begin{bmatrix} A_1 & B_1 + A_2 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{P} \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{Q} \end{bmatrix}$$

is Fredholm.

Let  $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_1)} \oplus \mathcal{S}$  and  $\mathcal{P} = \mathcal{N}(B_2) \oplus \mathcal{W}$ . By Theorem 1.55 we have that  $A+B$  is a Fredholm operator if and only if the following three conditions are satisfied:

- (\*)  $A_1$  is left semi-Fredholm.
- (\*\*)  $B_2$  is right semi-Fredholm.
- (\*\*\*)  $P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)}$  is Fredholm.

Evidently, (\*) holds if and only if  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$  and  $\mathcal{R}(A|_{\mathcal{N}(B)})$  is closed.

Also, (\*\*) holds if and only if  $B'_2$  is left semi-Fredholm. Since  $\mathcal{N}(B'_2) = \mathcal{N}(B') \cap \mathcal{N}(A')$  it follows that (\*\*) holds if and only if the range of  $P_{\mathcal{Q}, \overline{\mathcal{R}(A)}} B$  is closed and  $\dim(\mathcal{N}(B') \cap \mathcal{N}(A')) < \infty$ .

Now, we consider condition (\*\*\*) taking into account that  $\mathcal{R}(A_1)$  is closed in both directions. The condition (\*\*\*) is satisfied if and only if

$$\dim(\mathcal{N}(P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)})) < \infty$$

and

$$\dim(\mathcal{S}/\mathcal{R}(P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)})) < \infty.$$

Since

$$\mathcal{N}(P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)}) = \{x \in \mathcal{P} \mid (A+B)x \in \mathcal{R}(A_1)\},$$

we have that

$$\begin{aligned} \dim(\mathcal{N}(P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)})) &= \dim(\mathcal{R}(A|_{\mathcal{N}(B)}) \cap \mathcal{R}((A+B)|_{\mathcal{P}})) \\ &\quad + \dim(\mathcal{P} \cap \mathcal{N}(A+B)). \end{aligned}$$

Also,

$$\dim \mathcal{S}/\mathcal{R}(P_{S, \overline{\mathcal{R}(A_1)} \oplus \mathcal{Q}}(A+B)|_{\mathcal{N}(B_2)}) < \infty,$$

if and only if there exists a finite-dimensional subspace  $\mathcal{M}$  such that  $\mathcal{S} = \mathcal{M} \oplus \overline{\mathcal{R}(P_{\mathcal{S}, \mathcal{R}(A_1)} \oplus \mathcal{Q}(A+B)|_{\mathcal{N}(B_2)})}$ . The last assertion is equivalent with the existence of a finite dimensional subspace  $\mathcal{N}$  such that

$$\overline{\mathcal{R}(A)} = \mathcal{N} \oplus (\mathcal{R}(A|_{\mathcal{N}(B)}) + \overline{\mathcal{R}(A)} \cap \mathcal{R}((A+B)|_{\mathcal{P}}));$$

i.e.,

$$\dim(\overline{\mathcal{R}(A)}/(\mathcal{R}(A|_{\mathcal{N}(B)}) + \overline{\mathcal{R}(A)} \cap \mathcal{R}((A+B)|_{\mathcal{P}}))) < \infty.$$

The rest of the proof follows by Remark 1.57.  $\square$

Notice that the condition  $\dim(\mathcal{N}(A') \cap \mathcal{N}(B')) < \infty$  from item (ii) of the previous theorem can be replaced by  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ .

In the special case when we take the orthogonal decompositions of spaces  $\mathcal{H}, \mathcal{K}$  in the proof of Theorem 5.42, we get the following result:

**THEOREM 5.43.** *Let  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  be given operators. Then  $A+B$  is Fredholm if and only if the conditions*

- (i)  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$ ,  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ ,
- (ii)  $AP_{\mathcal{N}(B)}$  and  $P_{\mathcal{R}(A)^\perp}B$  have closed ranges,
- (iii)  $P_{\mathcal{S}}^{\text{rst}}(A+B)|_{\mathcal{T}}$  is Fredholm

hold, where  $\mathcal{S} = \mathcal{N}(P_{\mathcal{N}(B)}A^*) \cap \overline{\mathcal{R}(A)}$  and  $\mathcal{T} = \mathcal{N}(P_{\mathcal{R}(A)^\perp}B) \cap \mathcal{N}(B)^\perp$ .

Furthermore, if (i)–(iii) hold, then

$$\begin{aligned} \text{ind}(A+B) &= \dim(\mathcal{N}(A) \cap \mathcal{N}(B)) - \dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) \\ &\quad + \text{ind}(P_{\mathcal{S}}^{\text{rst}}(A+B)|_{\mathcal{T}}). \end{aligned}$$

**PROOF.** Let  $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{N}(B)^\perp$  and  $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp$ . With respect to these decompositions  $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  have the following representations:

$$(5.14) \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{bmatrix},$$

$$(5.15) \quad B = \begin{bmatrix} 0 & B_1 \\ 0 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{bmatrix} \rightarrow \begin{bmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{bmatrix}.$$

Using the orthogonal decomposition  $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A_1)} \oplus (\mathcal{R}(A_1)^\perp \cap \overline{\mathcal{R}(A)})$  in the proof of Theorem 1.55 we get that the linear combination  $A+B$  is a Fredholm operator if and only if the following three conditions are satisfied:

- (i)  $A_1$  is left semi-Fredholm.
- (ii)  $B_2$  is right semi-Fredholm.
- (iii)  $P_{\overline{\mathcal{R}(A_1)^\perp \cap \overline{\mathcal{R}(A)}}}^{\text{rst}}(A_2+B_1)|_{\mathcal{N}(B_2)}$  is Fredholm.

Evidently, (i) holds if and only if  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$  and  $AP_{\mathcal{N}(B)}$  has closed range. Also, (ii) holds if and only if  $B_2^*$  is upper semi-Fredholm. Since

$$\mathcal{N}(B_2^*) = \mathcal{N}(B^*) \cap \mathcal{R}(A)^\perp = \mathcal{R}(B)^\perp \cap \mathcal{R}(A)^\perp,$$

it follows that (ii) holds if and only if the range of  $P_{\mathcal{R}(A)^\perp}B$  is closed and  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ . The third condition follows directly from the fact that  $\overline{\mathcal{R}(A_1)^\perp \cap \overline{\mathcal{R}(A)}} = \mathcal{N}(P_{\mathcal{N}(B)}A^*) \cap \overline{\mathcal{R}(A)}$  and  $\mathcal{N}(B_2) = \mathcal{N}(P_{\mathcal{R}(A)^\perp}B) \cap \mathcal{N}(B)^\perp$ .  $\square$

REMARK 5.44. If  $A_1$  and  $B_2$  from (5.1) and (5.2) are Fredholm, then  $\mathcal{N}(B_2)$  and  $\mathcal{S}$  defined in Theorem 5.42 are finite dimensional so the condition (\*\*\*) from the proof of Theorem 5.42 is satisfied. Hence, in this case taking  $\alpha A$  instead of  $A$  and  $\beta B$  instead of  $B$ , we get that for all constants  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ , the linear combination  $\alpha A + \beta B$  is a Fredholm operator.

Also if one of the operators  $A_1$  and  $B_2$  is Fredholm and the other one is not, at least one of the conditions (\*)–(\*\*\*) from the proof of Theorem 5.42 is not satisfied. Indeed, if conditions (\*) and (\*\*) are satisfied, one of spaces  $\mathcal{N}(B_2)$  and  $\mathcal{S}$  is finite dimensional and the other one is infinite dimensional, so the condition (\*\*\*) is not satisfied and  $A + B$  is not a Fredholm operator. Moreover, in this case the linear combination  $\alpha A + \beta B$  is not a Fredholm operator for any constants  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . Thus we have the following result:

THEOREM 5.45. *Let  $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  be given operators and let  $\mathcal{H} = \mathcal{N}(B) \oplus \mathcal{P}$  and  $\mathcal{K} = \overline{\mathcal{R}(A)} \oplus \mathcal{Q}$ . Let  $A_1 = P_{\overline{\mathcal{R}(A)}, \mathcal{Q}}^{\text{rst}} A|_{\mathcal{N}(B)}$  and  $B_2 = P_{\mathcal{Q}, \overline{\mathcal{R}(A)}}^{\text{rst}} B|_{\mathcal{P}}$ .*

- (i) *If the operators  $A_1$  and  $B_2$  are Fredholm, then for all constants  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  the linear combination  $\alpha A + \beta B$  is a Fredholm operator.*
- (ii) *If one of operators  $A_1$  and  $B_2$  is Fredholm and the other one is not, then for all constants  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  the linear combination  $\alpha A + \beta B$  is not a Fredholm operator.*

REMARK 5.46. If we take orthogonal decompositions of spaces  $\mathcal{H}$  and  $\mathcal{K}$ , as in Theorem 5.43, we have that  $\mathcal{S} = \mathcal{N}(A_1^*) = \{x \in \mathcal{N}(A^*)^\perp \mid A^*x \in \overline{\mathcal{R}(B^*)}\}$  and  $\mathcal{T} = \mathcal{N}(B_2) = \{x \in \mathcal{N}(B)^\perp \mid Bx \in \overline{\mathcal{R}(A)}\}$ . It can be checked that  $\dim(\mathcal{N}(A_1^*)) = \dim(\mathcal{R}(A^*) \cap \overline{\mathcal{R}(B^*)})$  and  $\dim(\mathcal{N}(B_2)) = \dim(\overline{\mathcal{R}(A)} \cap \mathcal{R}(B))$ . So, we are ready to present some particular cases when the linear combination  $\alpha A + \beta B$  is independent of the choice of scalars  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ :

THEOREM 5.47. *Let  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and  $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  and let  $m = \dim(\mathcal{R}(A^*) \cap \overline{\mathcal{R}(B^*)})$  and  $n = \dim(\overline{\mathcal{R}(A)} \cap \mathcal{R}(B))$ . If one of the conditions*

- (i)  $\max\{m, n\} < \infty$ ,
- (ii)  $\min\{m, n\} < \max\{m, n\} = \infty$

*holds, then Fredholmnness of the linear combination  $\alpha A + \beta B$  is independent of the choice of scalars  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . Furthermore, we have in the case (i) that  $A + B$  is Fredholm if and only if  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$ ,  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ , and  $AP_{\mathcal{N}(B)}$ ,  $P_{\overline{\mathcal{R}(A)}}B$  have closed ranges, while in the case (ii) we have that  $A + B$  is not Fredholm.*

PROOF. The proof follows by Theorem 5.43 and Remark 5.46. □

From a previous theorem, we have that if  $m, n$  are both infinite or if one is finite and the other is infinite, then Fredholmnness of the linear combination  $\alpha A + \beta B$  is independent of the choice of scalars  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . What happens in the case when both  $m, n$  are infinite? The next example gives an answer to this question:

EXAMPLE 5.48. Let  $\mathcal{H}$  be infinite-dimensional Hilbert space and let  $A = I_{\mathcal{H}}$  and  $B = -2I_{\mathcal{H}}$ . Evidently  $\alpha A + \beta B$  is a Fredholm operator except in the case when  $\alpha = 2\beta$  which means that Fredholmnness of the linear combination  $\alpha A + \beta B$  depends on the choice of  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . On the other side if  $A \in \mathcal{B}(H)$  is non-Fredholm and  $B = A$ , we get that  $\alpha A + \beta B$ , is a non-Fredholm operator for any  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$



which means that Fredholmness of the linear combination  $\alpha A + \beta B$  is independent of the choice of scalars  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ .

From Theorem 5.43, we can deduce the following well-known result whose proof will be given presently since it is quite different from all others that can be found in the literature:

**COROLLARY 5.49.** Let  $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  be an operator of finite rank and let  $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ . Then  $A + B$  is Fredholm if and only if  $B$  is Fredholm.

**PROOF.** Since  $\dim(\mathcal{R}(A)) < \infty$  from Theorem 5.47, we get that  $A + B$  is Fredholm if and only if the following conditions hold:

- (i)  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$ ,  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ .
- (ii)  $P_{\mathcal{R}(A)^\perp} B$  has closed range.

These conditions are satisfied if and only if  $B$  is Fredholm. Indeed, if  $B$  is Fredholm, the condition (i) is obviously satisfied. Also  $P_{\mathcal{R}(A)^\perp} B$  is a left semi-Fredholm operator as the product of two such operators, so its range is closed. Conversely, if conditions (i) and (ii) are satisfied, then from the closedness of  $\mathcal{R}(P_{\mathcal{R}(A)^\perp} B)$  we have that  $\mathcal{R}(B^* P_{\mathcal{R}(A)^\perp})$  is closed. Now, from  $\mathcal{R}(B^*) = \mathcal{R}(B^* P_{\mathcal{R}(A)^\perp}) + \mathcal{R}(B^* P_{\mathcal{R}(A)})$ , we get that  $\mathcal{R}(B)$  is closed. Using the fact that in any vector space  $\mathcal{X}$  if  $\mathcal{X}_1, \mathcal{X}_2$ , and  $\mathcal{M}$  are subspaces of  $\mathcal{X}$  such that  $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ ,  $\dim \mathcal{X}_2 < \infty$ , and  $\dim(\mathcal{M} \cap \mathcal{X}_1) < \infty$ , then  $\dim \mathcal{M} < \infty$ , from  $\dim(\mathcal{N}(A) \cap \mathcal{N}(B)) < \infty$ ,  $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(A^*)$ , and  $\dim(\mathcal{R}(A^*)) < \infty$  it follows that  $\dim(\mathcal{N}(B)) < \infty$ . In the same manner from  $\dim(\mathcal{R}(A)^\perp \cap \mathcal{R}(B)^\perp) < \infty$ ,  $\mathcal{K} = \mathcal{R}(A)^\perp \oplus \mathcal{R}(A)$ , and  $\dim(\mathcal{R}(A)) < \infty$ , we get that  $\dim(\mathcal{R}(B)^\perp) < \infty$ . So  $B$  is a Fredholm operator.  $\square$

The Fredholmness of a difference, sum, and in general a linear combination of idempotents and orthogonal projectors has been considered in several papers (see [67, 78, 79, 117–119, 167, 169]). There are results in the literature pertaining to the independence of the linear combination  $c_1 P_1 + c_2 P_2$  being left semi-Fredholm, right semi-Fredholm, and Fredholm from the constants  $c_1, c_2$  in the case when  $P_1$  and  $P_2$  are orthogonal projectors or idempotents [78, 119]. Also, in the case when  $P_1$  and  $P_2$  are orthogonal projectors some necessary and sufficient conditions for  $P_1 + P_2$  being Fredholm are known [117], which combined with the result given above furnish necessary and sufficient conditions for  $c_1 P_1 + c_2 P_2$  being Fredholm as well as those under which the difference of two Hilbert space orthogonal projectors is a Fredholm operator. Here we present the results from [50] that give necessary and sufficient conditions for  $c_1 P_1 + c_2 P_2$  to be Fredholm in the case when  $P_1$  and  $P_2$  are idempotents. We have the following two results whose proofs are given below as one:

**THEOREM 5.50.** Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be idempotents and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ ,  $\alpha + \beta \neq 0$ . Then  $\alpha P + \beta Q$  is Fredholm if and only if the following conditions hold:

- (i)  $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$  and  $\dim(\mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp) < \infty$ .
- (ii)  $P(I - Q)$  and  $(I - P)Q$  have closed ranges.
- (iii)  $\dim(\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q)) < \infty$  and  $\dim(\mathcal{R}(P)/(\mathcal{R}(P(I - Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)))) < \infty$ .

Furthermore, if (i)–(iii) hold, then

$$\begin{aligned}\alpha(\alpha P + \beta Q) &= \dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q)), \\ \beta(\alpha P + \beta Q) &= \dim(\mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp) \\ &\quad + \dim(\mathcal{R}(P)/(\mathcal{R}(P(I - Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q)))).\end{aligned}$$

**THEOREM 5.51.** *Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be idempotents. Then  $P - Q$  is Fredholm if and only if the following conditions hold:*

- (i)  $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$  and  $\dim(\mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp) < \infty$ .
- (ii)  $(I - P)Q$  has closed range.
- (iii)  $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$  and  $\dim(\mathcal{R}(P)/\mathcal{R}(P(I - Q))) < \infty$ .

Furthermore, if (i)–(iii) hold, then

$$\begin{aligned}\alpha(P - Q) &= \dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q)), \\ \beta(P - Q) &= \dim(\mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp) + \dim(\mathcal{R}(P)/\mathcal{R}(P(I - Q))).\end{aligned}$$

**PROOF.** Let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ . By Theorem 5.42, we have that  $\alpha P + \beta Q$  is a Fredholm operator if and only if:

- (i)  $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$  and  $\dim(\mathcal{N}(P') \cap \mathcal{N}(Q')) < \infty$ .
- (ii)  $P(I - Q)$  and  $(I - P)Q$  have closed ranges.
- (iii)  $\dim(\mathcal{R}(Q) \cap \mathcal{N}(\alpha P + \beta Q)) < \infty$ ,  $\dim(\mathcal{R}(P|_{\mathcal{N}(Q)}) \cap \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{R}(Q)})) < \infty$ , and  $\dim(\mathcal{R}(P) \setminus (\mathcal{R}(P|_{\mathcal{N}(Q)}) + (\mathcal{R}(P) \cap \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{R}(Q)})))) < \infty$ .

Notice that

$$\mathcal{N}(P') \cap \mathcal{N}(Q') = \mathcal{R}(P)^\circ \cap \mathcal{R}(Q)^\circ = (\mathcal{R}(P) + \mathcal{R}(Q))^\circ$$

which implies that

$$\dim(\mathcal{N}(P') \cap \mathcal{N}(Q')) = \dim(\mathcal{R}(P) + \mathcal{R}(Q))^\circ = \dim(\overline{\mathcal{H}/(\mathcal{R}(P) + \mathcal{R}(Q))}).$$

Hence, we have that  $\dim(\mathcal{N}(P') \cap \mathcal{N}(Q')) < \infty$  if and only if  $\dim(\mathcal{R}(P)^\perp \cap \mathcal{R}(Q)^\perp) < \infty$ .

Now the theorems follow from the following equalities:

$$\begin{aligned}\mathcal{R}(Q) \cap \mathcal{N}(\alpha P + \beta Q) &= \begin{cases} \{0\} & \text{if } \alpha + \beta \neq 0, \\ \mathcal{R}(P) \cap \mathcal{R}(Q) & \text{if } \alpha + \beta = 0, \end{cases} \\ \mathcal{R}(P|_{\mathcal{N}(Q)}) \cap \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{R}(Q)}) &= \begin{cases} \mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q) & \text{if } \alpha + \beta \neq 0, \\ \{0\} & \text{if } \alpha + \beta = 0, \end{cases} \\ \mathcal{R}(P) \cap \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{R}(Q)}) &= \begin{cases} \mathcal{R}(P) \cap \mathcal{R}(Q) & \text{if } \alpha + \beta \neq 0, \\ \{0\} & \text{if } \alpha + \beta = 0. \end{cases}\end{aligned}$$

In the case  $\alpha + \beta = 0$ , the condition

$$\dim(\mathcal{R}(P)/(\mathcal{R}(P|_{\mathcal{N}(Q)}) + (\mathcal{R}(P) \cap \mathcal{R}((\alpha P + \beta Q)|_{\mathcal{R}(Q)})))) < \infty$$

is equivalent to the condition  $\dim(\mathcal{R}(P)/\mathcal{R}(P(I - Q))) < \infty$ , which implies that  $\mathcal{R}(P(I - Q))$  is closed.  $\square$

In the case when  $P, Q \in \mathcal{B}(\mathcal{H})$  are orthogonal projectors, the well-known result below follows from our Theorems 5.50 and 5.51:

**THEOREM 5.52.** *Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be orthogonal projectors on a Hilbert space  $\mathcal{H}$ .*

- (1) *If  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ ,  $\alpha + \beta \neq 0$ , then the following are equivalent:*
  - (i)  *$\alpha P + \beta Q$  is Fredholm.*
  - (ii)  *$\mathcal{R}(P) + \mathcal{R}(Q)$  is closed and  $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$ .*
- (2)  *$P - Q$  is Fredholm if and only if  $P + Q$  is Fredholm and  $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$ .*

Furthermore, if  $\alpha P + \beta Q$  is Fredholm, then  $\text{ind}(\alpha P + \beta Q) = 0$ .

**PROOF.** It is evident that condition (i) from Theorem 5.50 is equivalent with  $\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) < \infty$  while using Lemma 2.4 of [117] we have that the condition (ii) from Theorem 5.50 is equivalent with the fact that  $\mathcal{R}(P) + \mathcal{R}(Q)$  is closed. Now, the proof follows having in mind that

$$\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q) = \{0\}$$

and from the fact that  $\mathcal{R}(P) + \mathcal{R}(Q)$  is closed implies

$$\mathcal{R}(P)/(\mathcal{R}(P(I - Q)) + (\mathcal{R}(P) \cap \mathcal{R}(Q))) = \{0\}. \quad \square$$

Let us mention that using the result from [119] which says that for any two idempotents  $P, Q$  and  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  such that  $\alpha + \beta \neq 0$ ,

$$\alpha(\alpha P + \beta Q) = \dim(\mathcal{N}((I - P)Q) \cap \mathcal{N}(P)),$$

and our Theorem 5.50, we get the following result:

**THEOREM 5.53.** *Let  $P, Q \in \mathcal{B}(\mathcal{H})$  be idempotents and let  $\mathcal{R}(P(I - Q))$  be closed. Then*

$$\dim(\mathcal{N}(P) \cap \mathcal{N}(Q)) + \dim(\mathcal{R}(P(I - Q)) \cap \mathcal{R}(Q)) = \dim(\mathcal{N}((I - P)Q) \cap \mathcal{N}(P)).$$

Notice that if we remove the condition that  $\mathcal{R}(P(I - Q))$  is closed, the equality from the previous theorem is still valid.

From Theorem 5.50 we easily obtain some known results:

**THEOREM 5.54** ([78]). *Let  $P$  and  $Q$  be idempotents on a Hilbert space  $\mathcal{H}$ , and let  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  such that  $\alpha + \beta \neq 0$ . Then  $\alpha P + \beta Q$  is Fredholm if and only if  $P + Q$  is Fredholm and  $\text{ind}(\alpha P + \beta Q) = \text{ind}(P + Q)$ .*

**COROLLARY 5.55** ([78]). *Let  $P$  and  $Q$  be idempotents on  $\mathcal{H}$ . If  $P - Q$  is Fredholm, then so is  $P + Q$ . Also, if  $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) < \infty$ , then  $P - Q$  is Fredholm if and only if  $P + Q$  is.*

The next example shows that the formula given in Corollary 4 from [78] stating that  $\text{ind}(P - Q) = \text{ind}(P + Q) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q))$ , is not true in the case when  $P, Q \in \mathcal{B}(\mathcal{H})$  are orthogonal projectors:

**EXAMPLE 5.56.** Let  $P, Q \in \mathcal{B}(l_2)$  be defined by

$$Px = (x_1, 0, x_3, 0, x_5, \dots),$$

$$Qx = (x_1, x_2, 0, x_4, 0, x_6, \dots),$$

for any  $x = (x_n)_{n=1}^\infty \in l_2$ . It is easy to see that  $P, Q$  are orthogonal projectors and that  $\text{ind}(P + Q) = \text{ind}(P - Q) = 0$  and  $\dim(\mathcal{R}(P) \cap \mathcal{R}(Q)) = 1$ . Hence,  $\text{ind}(P - Q) \neq \text{ind}(P + Q) + \dim(\mathcal{R}(P) \cap \mathcal{R}(Q))$ .

Unlike the case of orthogonal projectors and idempotents, for some other classes of operators Fredholmness of linear combinations of operators in general depends on the choice of scalars  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  such that  $\alpha + \beta \neq 0$ . The following examples illustrate this fact for the classes of  $k$ -potent, nilpotent, and partial isometry operators.

EXAMPLE 5.57 (*k-potent operators*). Let  $A, B \in \mathcal{B}(l_2)$  be defined by

$$Ax = \left( \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, \dots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k + \frac{1}{2}x_{k+1}, \dots \right),$$

$$Bx = \left( \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2, \dots, \frac{1}{2}x_k + \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k - \frac{1}{2}x_{k+1}, \dots \right),$$

for any  $x = (x_n)_{n=1}^\infty \in l_2$ . It is easy to see that  $A^7 = A$  and  $B^3 = B$ , so  $A$  and  $B$  are  $k$ -potent operators. Also, for  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ , if  $\alpha \neq \pm\beta$ , we have that  $\alpha A + \beta B$  is invertible, so it is Fredholm. On the other hand,  $A+B$  and  $A-B$  are not Fredholm since  $\alpha(A+B) = \alpha(A-B) = \infty$ .

EXAMPLE 5.58 (*Nilpotent operators*). Let  $A, B \in \mathcal{B}(l_2)$  be defined by

$$Ax = (x_2 + x_3, x_3, 0, x_5 + x_6, x_6, 0, \dots, x_{3k-1} + x_{3k}, x_{3k}, 0, \dots),$$

$$Bx = (0, 2x_3, x_1, 0, 2x_6, x_4, \dots, 0, 2x_{3k}, x_{3k-2}, \dots),$$

for any  $x = (x_n)_{n=1}^\infty \in l_2$ . Then  $A^3 = B^3 = 0$ . It is easy to see that  $A+B$  is invertible, so it is Fredholm, and that  $2A-B$  is not Fredholm since  $\alpha(2A-B) = \beta(2A-B) = \infty$ .

EXAMPLE 5.59 (*Partial isometry*). Let  $A, B \in \mathcal{B}(l_2)$  be defined by

$$Ax = \left( \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, -\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2, \dots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, -\frac{\sqrt{3}}{2}x_k - \frac{1}{2}x_{k+1}, \dots \right),$$

$$Bx = \left( \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}x_2, \frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, \dots, \frac{1}{2}x_k - \frac{\sqrt{3}}{2}x_{k+1}, \frac{\sqrt{3}}{2}x_k + \frac{1}{2}x_{k+1}, \dots \right),$$

for any  $x = (x_n)_{n=1}^\infty \in l_2$ . It is easy to check that  $A+B$  is not a Fredholm operator while  $A+2B$  is Fredholm.