Preface for Volumes II and III

The two new volumes.

In the thirty or more years since *Algebras, Lattices, Varieties Volume I* appeared, the authors—and our readers—have seen the growth of many branches of our field of general algebra. Our first volume included five chapters—the first four laid the foundation for our enterprise, while the fifth provided a serious look at direct products and, particularly, at some circumstances under which a unique direct factorization result (like the familiar one for finite Abelian groups) might hold.

For presentation in the second and third volumes, following the reissuing of Volume I by the American Mathematical Society in their Chelsea series in 2018, we have selected six threads of development (three per volume), and have written six new chapters corresponding to these themes. (The three-volume set thus has eleven chapters, numbered consecutively.)

The six new chapters.

We begin Volume II with Chapter 6, *Classification of Varieties*, which describes the many classes (or properties) of varieties that arise from the existence (or not) of various term operations (as typified in the theory of Mal'tsev conditions). Chapter 7 *Equational Logic*, presents the study of equational theories through a detailed analysis of equational proofs—especially undecidability results and the investigation of finite axiomatizability of equational theories—and the small-scale analysis of subdirect representation. Chapter 8 *Rudiments of Model Theory*, provides tools and motivations from model theory, including logical compactness, reduced products, Jónsson's Lemma, and Baker's Finite Basis Theorem. (As noted in the Introduction to Volume 1, general algebra and model theory have become very distinct fields, but the more basic parts of model theory continue to play a strong role in our understanding of general algebra.)

In Chapter 9, *Finite Algebras and their Clones*, we examine, for a set A, the collection $\mathbf{Clo}\,A$ of all operations $A^n \longrightarrow A$ (with n ranging over ω, or sometimes over $\omega\smallsetminus\{0\}$). By a *clone of operations on A*— or simply *a clone*—we mean a subset C of $\mathbf{Clo}\,A$ that contains the coordinate projections and is closed under forming compositions. For example if C is a clone and $f \in C$, then $g(x_0,x_1,x_2,x_3) = f(f(x_0,x_1), f(x_2,x_3))$ must also belong to C. The *clone of an algebra* $\mathbf{A} = \langle A, F_t\rangle_{t\in T}$ is the smallest clone of operations on A that contains all the basic operations F_t. In many contexts, the interesting properties of $\mathbf{A}$ depend only on this clone. Landmark theorems in this chapter include: for A a two-element set, a description of all the clones on A (Emil L. Post, 1941); and for A an arbitrary finite set, a description of all the maximal proper subclones of $\mathbf{Clo}\,A$ (Ivo Rosenberg, 1970).

In Chapter 10, *Abstract Clone Theory*, the composition of operations in **Clo** A is regarded as forming an algebraic system in its own right. The family of clones $C \subseteq \mathbf{Clo}\,A$ (with A ranging over all sets) can be axiomatized in a way that makes no mention of the underlying sets A. (So abstract clones are related to clones of operations in the same way that abstract groups are related to permutation groups.) Abstract clones have also been known as *algebraic theories* (F. William Lawvere, 1963). Abstract clones support the study of varieties in the following way: we may attach to each variety $\mathcal{V}$ (of any signature) the clone of operations of the free algebra $\mathbf{F}_{\mathcal{V}}(\omega)$, which we may also designate as $\mathbf{C}(\mathcal{V})$. Now the correspondence $\mathcal{V} \longleftrightarrow \mathbf{C}(\mathcal{V})$ makes term-equivalence classes of varieties correspond to isomorphism classes of clones. And it makes interpretations of a variety $\mathcal{V}$ in a variety $\mathcal{W}$ correspond bijectively to clone homomorphisms $\mathbf{C}(\mathcal{V}) \longrightarrow \mathbf{C}(\mathcal{W})$. In this way many varietal properties can be seen as arising in tandem with algebraic properties at the clone level. Thus in Chapter 10 we reprise the subject of Mal'tsev conditions from Chapter 6. Varieties can be ordered by taking $\mathcal{V} \leq_{\text{int}} \mathcal{W}$ iff there is an interpretation of $\mathcal{V}$ in $\mathcal{W}$. The resulting ordered class is a lattice $\mathbf{L}_{\text{int}}$, and each Mal'tsev condition defines a filter on $\mathbf{L}_{\text{int}}$.

Our final chapter (11) is *The Commutator*. The commutator is a binary operation $\langle \alpha, \beta \rangle \longmapsto [\alpha, \beta] \in \mathbf{Con\,A}$, defined on **Con A**. Its theory was begun in the 1970's and 1980's, marking a new era in universal algebra. In that period important structural results were proved for the algebras in any congruence modular variety: results from group theory, ring theory and other classical algebraic systems were extended to this general framework. Chapter 11 revisits congruence-modular, congruence-distributive and Abelian varieties, as well as discussing nilpotence, solvability, finite axiomatizability, direct representability and residual smallness. In particular, the chapter represents a further study of congruence modularity. The chapter concludes with a proof of McKenzie's 1996 Finite Basis Theorem: *Every congruence-modular variety of finite signature that has a finite residual bound is finitely based.* (See Theorem 11.119 on page 367 of Volume III.) The commutator is defined in every variety and, in the last several years, it has been used to obtain substantial results for varieties that are not necessarily congruence modular. For example, Theorem 11.57 characterizes when an Abelian algebra is affine (polynomially equivalent to a module). Most of the results involving the commutator in nonmodular varieties are in §11.5 and §11.6.

Each of the six themes already appeared in Volume I. Chapter 6 really began, for example, in §4.7 and §4.9, with permutability and with distributivity of congruence lattices. The free-algebra material in §4.11 is a good basis for our Chapter 7. Birkhoff's ***HSP***-theorem (§4.11) was the earliest example of a preservation theorem, a theme that is developed more extensively in Chapter 8. The clones of term operations and of polynomial operations, which dominate Chapter 9, were set out in the first section of Chapter 4. Chapter 10 builds both on the category-theoretic axiomatization of clones introduced in Chapter 3 (pages 136–137), and on the interpretability notions of §4.12. Chapter 11 expands very thoroughly on the commutator material that was presented in §4.13. It is our hope that this continuity of subject matter will facilitate the readers' entry into these latest volumes.

Motifs.

There are leitmotifs that run throughout these two volumes. One example is the application of available *Mal'tsev conditions* to obtain insights about varieties, equational theories, and clones. Chapter 6 is largely devoted to laying the foundations of this motif. It re-emerges in later chapters to provide key hypotheses for decidability and finite axiomatizability results, for Jónsson's Lemma, and for many other results. Another leitmotif is the question, for a given (finite) algebra **A**, of *whether the equational theory* $\Theta\,\mathbf{A}$ *of the algebra* **A** *is finitely axiomatizable.* While this question does not occur in Chapter 6, in retrospect it is only because the concepts introduced in that chapter seem to be almost designed to provide methods for use in later chapters. The finite axiomatizability question is one of the central concerns of Chapter 7. Then in Chapter 8 we find Baker's Finite Basis Theorem according to which $\Theta\,\mathbf{A}$ is finitely axiomatizable provided **A** is of finite signature and the variety generated by **A** is congruence distributive. Some further results on finite axiomatizability occur in Chapters 8 and 11.

Yet a third motif is the consideration of the *subdirectly irreducible algebras in a variety*. When the subdirectly irreducible algebras in a variety can be completely understood, this leads to a deep understanding of the whole variety. This happens, for instance, when the signature is finite and there is a finite upper bound of the cardinalities of the subdirectly irreducible algebras. In Chapter 7 having such a finite residual bound is a key hypothesis for finite axiomatizability results, but it is also used to show that there is no computable procedure for determining, of a finite algebra of finite signature, whether the variety it generates has a finite residual bound. In Chapter 8 one great power of Jónsson's Lemma lies in its ability to describe all the subdirectly irreducible algebras in a congruence distributive variety.

We have written the six new chapters to be independent expositions of their respective topics. There are, nevertheless, important links between the chapters, especially where an earlier chapter supports a result in a later chapter. We mention here just a few examples of such dependency. Chapter 6 is largely devoted to a detailed development of particular Mal'tsev conditions. In Chapter 10 a general theory of Mal'tsev conditions is laid out in the categorical context of abstract clone theory. Then in Chapter 11 we show that certain properties connected with the commutator, such as having a difference term or a weak difference term, are definable by a Mal'tsev condition. Chapter 6 shows congruence meet semidistributivity is definable by a Mal'tsev condition, while Chapter 11 shows this property is equivalent to congruence neutrality, that is, the commutator equals the meet. The theory of directly representable varieties, introduced in Chapter 8, is expanded in Chapter 9 and in Chapter 11, using the properties of the commutator developed in Chapter 11. On a few occasions we have found it necessary to have forward references. For instance the Compactness Theorem is applied in Chapter 7, but not proved until Chapter 8; the characterization of congruence join semidistributivity given in Chapter 6, requires substantial results from Chapter 11 in order to complete its proof.

The examples of such dependency found in the exercises are too numerous to list here. A final example in the main text is the theory of Boolean powers, found here in Chapter 8, which is used in Chapter 9 to demonstrate that varieties generated by primal algebras are all categorically equivalent.

Our selection of topics has also been attentive to our (rather personal) observations, over time, of the ideas and methods that have been found useful and productive in our own, and in others', further investigations. Here we might cite the *recursive solution* (or lack thereof) of the word problem for *presentations* of an algebra **A** in a variety $\mathcal{V}$; the *decidability* (or not) of the set Θ**A** of equations valid in **A**; also *finite axiomatizability* of Θ**A**—especially for **A** a finite lattice with operators, and for **A** belonging to a variety with distributive congruence lattices; the strength of $\mathcal{V}$ (or of **A**) as measured by the family of Mal'tsev conditions that it satisfies (or not); the utility of the notion of a *system of definitions* of one variety $\mathcal{V}$ in a second variety $\mathcal{W}$; the *lattice of subvarieties* of a given variety; the *linearity* (*or not*) of various Mal'tsev conditions; the basic properties of *Taylor terms*.

Omissions and Inclusions.

The resulting conception of these two volumes has not, in fact, diverged far from what we envisioned thirty years ago. Meanwhile, of course, during this period the field has grown significantly, producing new themes (and many individual theorems) that we cannot accommodate in these volumes. Notable omissions are constraint satisfaction problems, tame congruence theory, natural duality theory (except for a departure point found in our section on Boolean powers, Chapter 8), the general theory of quasivarieties, topological algebra, substructural logics, higher-dimension commutators, the theory of free lattices, and others. Also missing are several significant theorems, like the Oates-Powell Theorem according to which the equational theory of each finite group is finitely axiomatizable, whose proofs use concepts and methods from parts of mathematics that could not be included in this volume, or whose proofs, like Jaroslav Ježek's profound study of subsets of lattices of equational theories definable in first-order logic, were so intricate and extensive that the space they would require was beyond what was available to us.

Nevertheless we have ventured to include a selection of results that were not available to us in 1986. Here we mention a few: congruence-semidistributive varieties (§6.4); the minimal idempotent variety of Olšák (§6.8); shift-automorphism algebras (§7.3); Willard's Finite Basis Theorem, etc. (§7.4); McKenzie's resolution of Tarski's Finite Basis Problem (§7.11); Nation's counterexample to the Finite Height Conjecture (§8.4). There are also a few results here that have not appeared in the literature—notably Don Pigozzi's representation of an algebraic lattice with an additional new top element as a principal filter in a lattice of equational theories.

Useful Background.

The best preparation a reader can bring to the study of the new volumes of *Algebras, Lattices, Varieties* is a command of the material in Volume I. Indeed, in the present volume we make reference to Volume I, usually giving the page number, for background material, the statements of particular definitions, theorems, and examples—although we have made an effort to restate most of these within the later volumes. But readers acquainted with *Universal Algebra* by Clifford Bergman (2012) should be comfortable continuing their study of our field in the present two volumes. The short volume *Allgemeine Algebra* by Thomas Ihringer (2003) or the older expositions *A Course in Universal Algebra* by S. Burris and H. P. Sankappanavar (1981) or the second edition of *Universal Algebra* by G. Grätzer (1979) would be useful background.

We have provided most sections in this volume with exercises, against which diligent readers might test their understanding of the material.

The beautiful edifice that we strive to portray in these volumes is the product of numerous mathematicians, who have worked over the last century to uncover and understand the fundamental structures of general algebra. During the course of writing we have gone again and again to the literature, coming away with a deep appreciation of this work. It is our hope that our readers will find here some of the beauty, joy, and power that we have found in this domain of mathematics.

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