Preface

The primary driver for writing this book is the desire to provide an on-ramp to the new, developing, and mathematically diverse research related to the inverse eigenvalue problem for graphs (IEP-G) and the related area of zero forcing, propagation, and throttling. Inverse problems play a central role in mathematics and naturally arise in applications. In many instances the inverse problem reduces to a question about the existence of a matrix with a prescribed set of eigenvalues and prescribed structure. The IEP-G studies such questions.

Due to the lack of effective tools for the IEP-G, much early research focused on ancillary problems (subquestions) that could lead to progress on the IEP-G. The most important of these is the study of the maximum possible multiplicity of an eigenvalue among matrices described by G, or equivalently maximum possible nullity or minimum possible rank. Since the birth of the minimum rank problem in a 1997 paper by Peter Nylen, the IEP-G and related questions have proven to be intriguing, but difficult problems. The 2006 American Institute of Mathematics Workshop 'Spectra of families of matrices described by graphs, digraphs, and sign patterns' and the small research group that grew out of it, 'Minimum rank of symmetric matrices described by a graph,' catalyzed two new research areas, which we call Strong Properties of Matrices and Zero-Forcing of Graphs, related to the IEP-G. Since then there has been a rapid, expansive development of the areas, which has resulted in many deep results for the IEP-G. Because of the growth of these emerging areas, we believe that there is a need for a book like this which provides the essential concepts, techniques and results in a unified way, and which suggests topics for future research.

A pleasing aspect of these new topics is their interplay with other areas of mathematics and applications. The topic of Strong Properties of Matrices is closely related to the Implicit Function Theorem and the transversal intersection of manifolds. It provides algebraic conditions on a matrix with a certain spectral property and graph that guarantee the existence of a matrix with the same spectral property for each graph in a related family of graphs. Additionally, strong properties provide interesting combinatorial matrix theoretic and graph minor problems.

Zero forcing originated independently in both the IEP-G setting and in the study of control of quantum systems (where it is often referred to as graph infection). Zero forcing is a game on a graph in which vertices are initially filled or unfilled, and at each stage certain vertices change from unfilled to filled according to some rule. The goal of the game is to fill all vertices. Zero forcing gives a graph-theoretic method to bound the multiplicity of an eigenvalue of a symmetric matrix with a given zero-nonzero structure. Zero forcing is closely related to the power domination process that provides a model for placing monitoring units in an electric network, and has connections to graph searching, including the well-studied Cops and Robbers game in graph theory. Additionally, variants of zero forcing have arisen in various applications. Related graph parameters, such as the number of rounds needed to fill all vertices (propagation time) and minimizing a combination of the number of initially filled vertices and number of rounds needed (called throttling) are active areas of study.

The book is partitioned into four parts. Part 1 comprises Chapters 1 and 2 and provides a gentle introduction to the IEP-G, zero forcing, and some ancillary problems of the IEP-G. Chapter 1 also provides motivation and describes applications and connections with other areas of mathematics.

Part 2 focuses on various strong properties. Chapter 3 describes how the Implicit Function Theorem can be used to show that under certain conditions a solution to an inverse problem implies a solution to all 'nearby' inverse problems. Each strong property can be phrased as a matrix theoretic criterion. Chapter 4 shows how the strong properties can be used and establishes key consequences of the various strong properties. Chapter 5 provides a unified and technical theoretical derivation of the strong properties. This is done for both for completeness and in case readers need to develop other strong properties. However, we note that researchers may take the strong properties for granted and immediately use them in research.

Part 3 contains additional discussion of subproblems of the IEP-G and applies strong properties to these problems. In particular, Chapter 6 discusses ordered multiplicity lists for spectra of matrices described by G, Chapter 7 discusses rigid linkages, which provide additional information about eigenvalue multiplicities, and Chapter 8 discusses $q(G)$, the minimum number of distinct eigenvalues among matrices with off-diagonal nonzero pattern described by the edges of G.

Part 4 comprises Chapters 9–11 and focuses on zero forcing and its variants, related problems, and applications. Chapter 9 discusses zero forcing, its variants, connections to various ancillary problems of the IEP-G, and other graph coloring/searching parameters, such as power domination and Cops and Robbers. Chapter 10 discusses 'time' to complete coloring all the vertices, called propagation time or capture time. Chapter 11 discusses minimizing some combination of number of initially filled vertices and time.

Some graph and linear algebra terminology and notation is introduced throughout the book. Since there is more ambiguity in graph terminology (starting with the definition of a graph), a more complete summary is provided in Appendix A. Background material from linear algebra can be found in any standard linear algebra reference, such as *Handbook of Linear Algebra* [**178**], Horn and Johnson's *Matrix Analysis* [**196**], or Zhang's *Matrix Theory* [**271**].

Partial drafts of this book has been used as background material for two research communities, the American Mathematical Society Mathematics Research Community *Finding Needles in Haystacks: Approaches to Inverse Problems using Combinatorics and Linear Algebra* [**12**], and the American Institute of Mathematics Research Community *Inverse eigenvalue problems for graphs and zero forcing* [**9**]. We thank the Haystacks MRC and IEPG-ZF ARC participants, especially Ben Small, for their feedback. We also thank the reviewers for their suggestions. All of these comments have helped to improve the book.

A few places in the book refer to websites that contain some applications or videos that may be of interest to the reader. These urls were active at the time of publication, but may change in the future, and readers will need to search for and find these or similar sites.