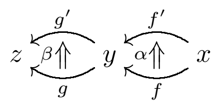
(4) If we have $\rho'\rho \cong \mathbb{1}_X, \rho\rho' \cong \mathbb{1}_Y$ for some 2-natural transformation $\rho' \colon Y \Rightarrow X$, then ρ is called a 2-*natural equivalence*, and this ρ' is called a *quasi-inverse* of ρ.

EXAMPLE 4.1.22. We fix a set $\mathbf{U}$ of 2-categories, (e.g., the whole of small 2-categories). Then we define a 2-category $\mathbf{C}$ as follows:

(1) Set $\mathbf{C}_0 := \mathbf{U}$.
(2) For each $x, y \in \mathbf{C}_0$ let $\mathbf{C}(x, y)$ be the 2-functor categories. Namely, it is the category, the object set $\mathbf{C}(x, y)_0$ of which is given by the whole of 2-functors $x \to y$, the morphisms of which are the 2-natural transformations and the compositions of which are the vertical compositions of 2-natural transformations.
(3) To determine a family $\circ := (\circ_{x,y,z} \colon \mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z))_{x,y,z\in\mathbf{C}_0}$ of functors with 2-variables take 2-functors and 2-natural transformations



$(x, y, z \in \mathbf{C}_0)$.

Then we define $g \circ f \colon x \to z$ to be the usual composite of 2-functors, and $\beta \circ \alpha$ to be the horizontal composite of 2-natural transformations.
(4) For each $x \in \mathbf{C}_0$ we set $u_x(*) := \mathbb{1}_x,\ u_x(\mathbb{1}_*) := \mathbb{1}_{\mathbf{1}_x}$.

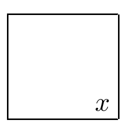
It is easily verified that the data above satisfy associativity and unitality. This 2-category $\mathbf{C}$ is denoted by 2-$\mathbf{Cat}(\mathbf{U})$. The 2-category 2-$\mathbf{Cat}(\mathbf{U})$ has modifications as "3-morphisms" and forms a "3-category".

The 2-category obtained from 2-$\mathbf{Cat}(\mathbf{U})$ by restricting the 2-morphisms to strict 2-natural transformations is denoted by 2-$\mathbf{Cat}^{\mathrm{st}}(\mathbf{U})$.

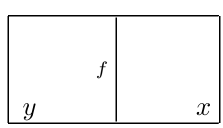
4.2. String diagrams

String diagrams make computations of 2-morphisms in a 2-category or natural transformations drastically easy in many cases. In this book we usually prove propositions on 2-morphisms or natural transformations using string diagrams. We let $\mathbf{C}$ be a 2-category and we explain how to draw them below. (In this book we compose 1-morphisms (resp. 2-morphisms) from right to left (resp. from bottom to top) according to [**45**][1]. We refer the reader to [**35**, §2] as a reference, but note that it composes 1-morphisms (resp. 2-morphisms) from left to right (resp. from bottom to top).)

(1) An object x of $\mathbf{C}$ is displayed as an area.

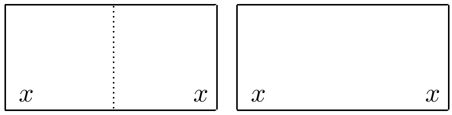


(2) 1-morphism $f \colon x \to y$ is displayed as a string (vertical line) (going from right to left):

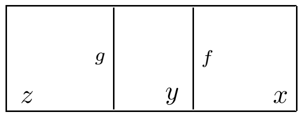


[1]The auther first learned how to draw string diagrams by these movies.

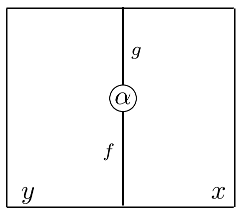
We draw $\mathbb{1}_x\colon x \to x$ by a dotted vertical line or nothing for all $x \in \mathbf{C}_0$:



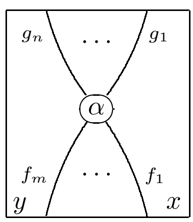
(3) The composite $g \circ f\colon x \to z$ of a 1-morphism $f\colon x \to y$ and $g\colon y \to z$ is displayed as follows (composed from right to left):



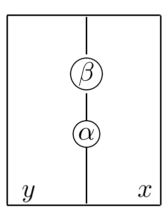
(4) A 2-morphism $y \overset{g}{\underset{f}{\overset{\alpha\Uparrow}{\leftleftarrows}}} x$ is displayed as a vertex drawn as a white circle with its name inside as follows (draw going from bottom to top):

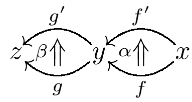


In particular, when $f = f_m \circ \cdots \circ f_1$, $g = g_n \circ \cdots \circ g_1$ are composites of 1-morphisms, a 2-morphism $\alpha\colon f_m \circ \cdots f_1 \Rightarrow g_n \circ \cdots g_1$ from a composite of 1-morphisms to a composite of 1-morphisms is displayed as follows:

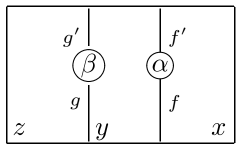


(5) For two 2-morphisms $y \overset{\beta\Uparrow}{\underset{\alpha\Uparrow}{\longleftarrow}} x$ the vertical composite $\beta \bullet \alpha$ is displayed as follows (composed from bottom to top):

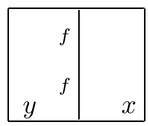


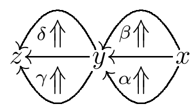
(6) For two 2-morphisms 

the horizontal composite $\beta \circ \alpha$ is displayed as follows:



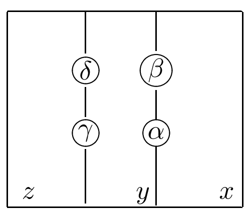
(7) We omit the identity 2-morphism $\mathbb{1}_f : f \Rightarrow f$:



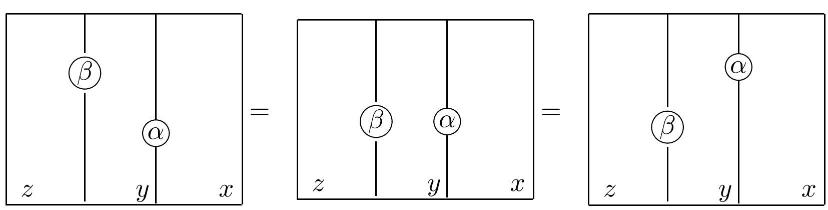
(8) For four 2-morphisms 

since we have the interchange law

$(\delta \bullet \gamma) \circ (\beta \bullet \alpha) = (\delta \circ \beta) \bullet (\gamma \circ \alpha)$, the following displays the common 2-morphism of both sides:



(9) By substituting the identity 2-morphisms for 2-morphisms above suitably we obtain the following *slide equations*:

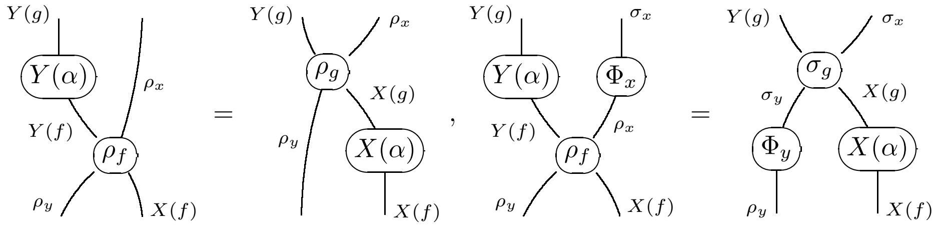


Thanks to computations (8) and (9), 2-morphisms become very simple. We freely make use of string diagrams below.

(10) We usually omit areas expressing objects of **C** and draw only lines and vertices.

We apply the drawing rules above also to the 2-category $\mathbf{Cat}(\mathscr{U})$ (Example 4.1.7) consisting of categories, functors and natural transformations.

EXAMPLE 4.2.1. Axioms of 2-natural transformations or modifications are displayed by string diagrams as follows:



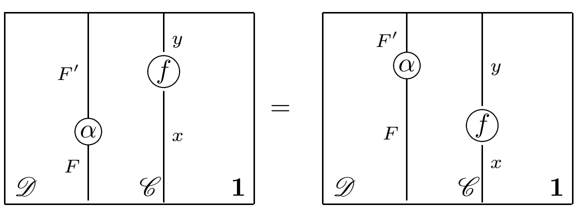
EXAMPLE 4.2.2. Let $\mathscr{C}$ be a category. Then by the following way we can use string diagrams for objects and morphisms of a category (cf. [**35**, §2.1]). Recall that we let $\mathbf{1}$ be a discrete category with a unique object $*$. Then each object x of $\mathscr{C}$ is identified with a functor $\mathbf{1} \to \mathscr{C}$ ($* \mapsto x, \mathbb{1}_* \mapsto \mathbb{1}_x$). We denote this functor also by x. Thus $x(*) := x, x(\mathbb{1}_*) := \mathbb{1}_x$. Then we can regard each morphism $f\colon x \to y$ of $\mathscr{C}$ as a natural transformation

$$\mathbf{1} \underset{y}{\overset{x}{\Downarrow f}} \mathscr{C}\ ,\ (f_*\colon x(*) \to y(*)) := (f\colon x \to y).$$

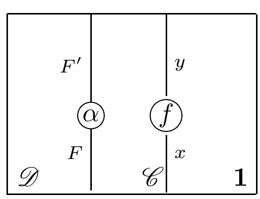
Here, consider two functors $F, F'\colon \mathscr{C} \to \mathscr{D}$ between categories and a family $\alpha := (\alpha_x\colon F(x) \to F'(x))_{x\in\mathscr{C}_0}$ of morphisms in $\mathscr{D}$. Then α is a natural transformation if and only if for each morphism $f\colon x \to y$ of $\mathscr{C}$, the diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & F'(x) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F'(f)} \\ F(y) & \xrightarrow[\alpha_y]{} & F'(y) \end{array}$$

is commutative, i.e., we have $F'(f) \circ \alpha_x = \alpha_y \circ F(f)$. This is expressed by the following slide equation:



(Note that since $F \circ x = F(x), \alpha \circ x = \alpha_x$, and so on, the equality above is written as $(F' \circ f) \bullet (\alpha \circ x) = (\alpha \circ y) \bullet (F \circ f)$.) Therefore we may display the common morphism of $\mathscr{D}$ by the following diagram:



This will be used later in the proof of Theorem 4.4.7 that gives a meaning of adjoint functors.

4.3. Lax functors, colax functors, and pseudofunctors

Definition of 2-functors are sometimes too strong to discuss general theory of 2-categories. We define weaker notions using 2-morphisms in 2-categories below. These are called "lax functors" (functors with relaxed conditions). There is a general definition of lax functors from a 2-category to a 2-category (Definition 8.2.1). However, since at present we do not need this general definition, we instead give a definition of those from a category to a 2-category. First of all we start from colax functors, which are obtained by substituting 2-morphisms for equalities in the definition of functors.

Definition 4.3.1. Let I be a category and $\mathbf{C}$ a 2-category. A sequence of the following data satisfying the axioms below is called a *colax functor* from I to $\mathbf{C}$, and is denoted by $X\colon I \to \mathbf{C}$.

Data:

- A map $X\colon I_0 \to \mathbf{C}_0$,
- A family of maps $({}_jX_i\colon I(i,j) \to \mathbf{C}(X(i), X(j)))_{i,j\in I_0}$, (where ${}_jX_i(a)$ is abbreviated as $X(a)$ for all $a \in I(i,j)$),
- A family $(X_i\colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)})_{i\in I_0}$ of 2-morphisms of $\mathbf{C}$,
- A family $(X_{b,a}\colon X(ba) \Rightarrow X(b)X(a))_{(b,a)\in \mathrm{com}(I_1)}$ of 2-morphisms of $\mathbf{C}$, where $\mathrm{com}(I_1) := \{(b,a) \in I_1 \times I_1 \mid \mathrm{dom}(b) = \mathrm{cod}(a)\}$.

Axioms:

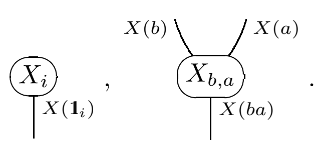
(1) The following are commutative for all morphisms $a\colon i \to j$ of I:

$$\begin{array}{ccc} X(a\mathbb{1}_i) & \overset{X_{a,\mathbb{1}_i}}{\Longrightarrow} & X(a)X(\mathbb{1}_i) \\ & \searrow\!\!\!= & \Big\Downarrow {\scriptstyle X(a)X_i} \\ & & X(a)\mathbb{1}_{X(i)} \end{array} \qquad \begin{array}{ccc} X(\mathbb{1}_j a) & \overset{X_{\mathbb{1}_j,a}}{\Longrightarrow} & X(\mathbb{1}_j)X(a) \\ & \searrow\!\!\!= & \Big\Downarrow {\scriptstyle X_jX(a)} \\ & & \mathbb{1}_{X(j)}X(a) \end{array}.$$

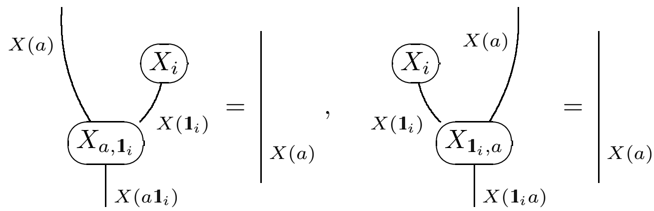
(2) The following are commutative for all paths $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I:

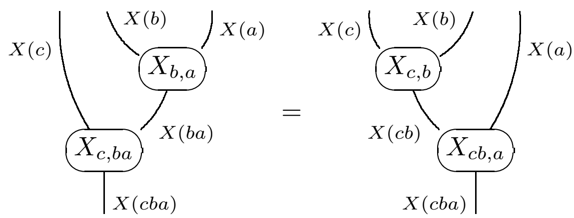
$$\begin{array}{ccc} X(cba) & \overset{X_{c,ba}}{\Longrightarrow} & X(c)X(ba) \\ {\scriptstyle X_{cb,a}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle X(c)X_{b,a}} \\ X(cb)X(a) & \underset{X_{c,b}X(a)}{\Longrightarrow} & X(c)X(b)X(a). \end{array}$$

Remark 4.3.2. In the form of string diagrams the axioms of colax functors $X\colon I \to \mathbf{C}$ is expressed as follows: First, 2-morphisms $X_i, X_{b,a}$ are displayed as follows, respectively:



Using these the axioms (a), (b) are displayed as follows by string diagrams, respectively:





EXAMPLE 4.3.3. A colax functor $X\colon \mathbf{1} \to \mathbf{Cat}$ from $\mathbf{1}$ to the 2-category $\mathbf{Cat}$ of all small categories is nothing but a comonad $T := X(\mathbb{1}_*)$ over a small category $\mathscr{C} := X(*)$.

DEFINITION 4.3.4. Let $\mathbf{C}$ be a 2-category.

- A colax functor $I \to \mathbf{C}^{\mathrm{co}}$ is called a *lax functor* $I \to \mathbf{C}$.
- A colax functor $X\colon I \to \mathbf{C}$ is called a *pseudofunctor* if all $X_i, X_{b,a}$ are 2-isomorphisms.
- A lax functor $X\colon I \to \mathbf{C}$ is called a *lax pseudofunctor* if all $X_i, X_{b,a}$ are 2-isomorphisms. Clearly, this notion is equivalent to that of a pseudofunctor defined above. This terminology is used to distinguish these equivalent notions. To contrast, the latter should be called a *colax pseudofunctor*, but we call it simply a pseudofunctor in this book.
- A colax functor $X\colon I \to \mathbf{C}$ with all $X_i, X_{b,a}$ identity 2-morphisms is nothing but a 2-functor $I \to \mathbf{C}$.

4.4. Adjoints and equivalences

We can consider adjoints for 1-morphisms in 2-categories ("inner" adjoints) and adjoints for 2-functors between 2-categories ("outer" adjoints). We start from inner adjoints.

4.4.a. Adjoints and equivalences in a 2-category.

DEFINITION 4.4.1. Let $\mathbf{C}$ be a 2-category.

(1) A quadruple $(f, g, \eta, \varepsilon)$ of the following data is called a *pre-adjoint system* in $\mathbf{C}$.

Data:

- 1-morphisms of $\mathbf{C}$ $x \underset{g}{\overset{f}{\rightleftarrows}} y$,
- 2-morphisms of $\mathbf{C}$ $\eta\colon \mathbb{1}_x \Rightarrow g \circ f$ and $\varepsilon\colon f \circ g \Rightarrow \mathbb{1}_y$.

APPENDIX A

Set theory for the foundation of category theory

In this book, we adopt ZFC (Zermelo-Fraenkel set-theory (ZF) with the axiom of choice (C)) as axioms of set theory, and we do not assume the existence of urelements. For any sets x and y, the pair (x, y) is defined as the Kuratowski's pair $(x, y) := \{\{x\}, \{x, y\}\}$, and the set of all maps from x to y is denoted by $\mathrm{Map}(x, y)$ or by y^x. For each set x, the power set (resp. the cardinality) of x is denoted by $\mathscr{P}x$ (resp. $|x|$), and we set $\omega := |\mathbb{N}|$. In the following, the class of all sets (resp. ordinal numbers) is denoted by SET (resp. ORD).

A.1. Universes

Before defining universes, we will introduce tools to control them in terms of ordinal numbers.

Definition A.1.1. A map $\mathsf{V}\colon \mathsf{ORD} \to \mathsf{SET}$ is defined as follows: For each $\alpha \in \mathsf{ORD}$,

$$\mathsf{V}_\alpha := \begin{cases} \emptyset & (\alpha = 0); \\ \mathscr{P}\mathsf{V}_\beta & (\alpha = \beta + 1, \exists \beta \in \mathsf{ORD}); \\ \bigcup_{\beta<\alpha} \mathsf{V}_\beta & (\alpha \text{ is a limit ordinal}). \end{cases}$$

This map is called the *von Neumann hierarchy*.

An element of V_ω is called a *hereditarily finite set*. Since $|\mathsf{V}_0| = 0$, and it holds that $|\mathsf{V}_{n+1}| = 2^{|\mathsf{V}_n|}$ for all $n \in \mathbb{N}$, we have $|A| < \omega$ for all $A \in \mathsf{V}_\omega$ and $|\mathsf{V}_\omega| = \omega$.

The following is well known (see e.g., [**39**, Lemma 7.5.14, 7.5.18] for a proof):

Lemma A.1.2. *The following hold.*

(1) *The map* V *is monotone, i.e., for any* $\alpha, \beta \in \mathsf{ORD}$*, we have "*$\alpha \leq \beta \Rightarrow \mathsf{V}_\alpha \subseteq \mathsf{V}_\beta$*".*
(2) *It holds that* $\mathsf{SET} = \bigcup_{\alpha \in \mathsf{ORD}} \mathsf{V}_\alpha$.

Definition A.1.3. A *Grothendieck universe* (or a *universe* for short) is a set $\mathfrak{U}$ that satisfies the following conditions:

(1) $x \in \mathfrak{U}, y \in x \Rightarrow y \in \mathfrak{U}$;
(2) $\emptyset \in \mathfrak{U}$;
(3) $x, y \in \mathfrak{U} \Rightarrow \{x, y\} \in \mathfrak{U}$;
(4) $I \in \mathfrak{U}, x_i \in \mathfrak{U}\ (i \in I) \Rightarrow \bigcup_{i \in I} x_i \in \mathfrak{U}$;
(5) $x \in \mathfrak{U} \Rightarrow \mathscr{P}x \in \mathfrak{U}$.

Example A.1.4. The set V_ω of all hereditarily finite sets is a universe. However, since each element of V_ω is a finite set, we have $\mathbb{N} \notin \mathsf{V}_\omega$.

REMARK A.1.5. Let $\mathfrak{U}$ be a universe. Then the following are immediate from conditions in Definition A.1.3:

(1) $x \in \mathfrak{U} \Rightarrow x \subseteq \mathfrak{U}$;
(2) $x \subseteq y, y \in \mathfrak{U} \Rightarrow x \in \mathfrak{U}$;
(3) $x, y \in \mathfrak{U} \Rightarrow (x, y) \in \mathfrak{U}$;
(4) $x, y \in \mathfrak{U} \Rightarrow x \cup y, x \times y \in \mathfrak{U}$;
(5) $x, y \in \mathfrak{U} \Rightarrow \mathrm{Map}(x, y) \in \mathfrak{U}$;
(6) $I \in \mathfrak{U}, x_i \in \mathfrak{U}\ (i \in I) \Rightarrow \prod_{i\in I} x_i, \bigsqcup_{i\in I} x_i \in \mathfrak{U}$;
(7) $\emptyset \neq I \in \mathfrak{U}, x_i \in \mathfrak{U}\ (i \in I) \Rightarrow \bigcap_{i\in I} x_i, \in \mathfrak{U}$;
(8) $x \in \mathfrak{U} \Rightarrow x \cup \{x\} \in \mathfrak{U}$;
This implies that $\mathbb{N} \subseteq \mathfrak{U}$. (Note, however, that it is not possible to prove that $\mathbb{N} \in \mathfrak{U}$ by the example above.) Therefore in particular, all finite unions, finite direct products, and finite disjoint unions of elements of $\mathfrak{U}$ belong to $\mathfrak{U}$.
(9) $x \in \mathfrak{U} \Rightarrow \bigcup x \in \mathfrak{U}$.
(10) $x \in \mathfrak{U}, y \subseteq \mathfrak{U}, f\colon x \to y$ is a surjection $\Rightarrow y \in \mathfrak{U}$.
(11) $\mathfrak{U} \notin \mathfrak{U}$.

PROOF. We omit the proofs since they are all straightforward, but show only (11). Assume contrarily that $\mathfrak{U} \in \mathfrak{U}$ holds. Then by Definition A.1.3 (5), we have $\mathscr{P}\mathfrak{U} \in \mathfrak{U}$. By (1) above, $\mathscr{P}\mathfrak{U} \subseteq \mathfrak{U}$. This shows that $|\mathscr{P}\mathfrak{U}| \leq |\mathfrak{U}|$. However, as long as $\mathfrak{U}$ is a set, Cantor's diagonal argument says that $|\mathfrak{U}| < |\mathscr{P}\mathfrak{U}|$, a contradiction. □

Universes are closely related to the following inaccessible cardinals[1].

DEFINITION A.1.6 (cf. [**44**]). A cardinal α is called an *inaccessible cardinal* if the following two conditions are satisfied:

(1) For any set I and any family $(B_i)_{i\in I}$ of sets, it holds that

$$|I| < \alpha, |B_i| < \alpha\ (i \in I) \Rightarrow \left|\bigcup_{i\in I} B_i\right| < \alpha;$$

(2) For any cardinals β, γ, we have

$$\beta, \gamma < \alpha \Rightarrow \beta^\gamma < \alpha.$$

We set the class of all universes (resp. the class of all inaccessible cardinals) to be Univ (resp. Ina). Then there exists a bijection between them (see [**47**] for a proof) as shown in the following theorem.

THEOREM A.1.7. *The following hold.*

(1) *If α is an inaccessible cardinal, then V_α is a universe.*
(2) *If $\mathfrak{U}$ is a universe, then $|\mathfrak{U}|$ is an inaccessible cardinal.*
(3) *Therefore monotone maps $\mathsf{V}\colon \mathsf{ORD} \to \mathsf{SET}$ and $|\text{-}|\colon \mathsf{SET} \to \mathsf{ORD}$ induce their restrictions*

$$\mathsf{V}\colon \mathsf{Ina} \to \mathsf{Univ}, \quad |\text{-}|\colon \mathsf{Univ} \to \mathsf{Ina},$$

respectively. These are inverse maps to each other. Therefore, Univ *and* Ina *are isomorphic as ordered classes. In particular, since* $\mathsf{Ina} \subseteq \mathsf{ORD}$*, both of them are well-ordered.*

[1] Here we deal with strongly inaccessible cardinals.

Remark A.1.8. Since $\mathbb{N} \in \mathsf{V}_{\omega+1}$, all universes but V_ω has $\mathbb{N}$ as their element.

In addition to the ZFC, we assume the following axiom.

Axiom of universe. Every set belongs to a universe.

The axiomatic system that adds this axiom to ZFC is called ZFCU. By Theorem A.1.7, Axiom of universe is equivalent to the following axiom under ZFC:

Axiom of inaccessible cardinals. Every cardinal is smaller than an inaccessible cardinal.

Axiom of inaccessible cardinal is known to be independent of the ZFC. Therefore, Axiom of universe is also independent of the ZFC.

Definition A.1.9. A universe having $\mathbb{N}$ as its element is called an *infinite universe*. By the axiom above, an infinite universe exists, and by Remark A.1.8, all universes except for V_ω are infinite universes.

Throughout this book, we fix an infinite universe $\mathfrak{U}$.

Remark A.1.10. Since $\mathbb{N} \in \mathfrak{U}$, we have $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in \mathfrak{U}$. Therefore, all structures constructed from these using operations in set theory also belong to $\mathfrak{U}$.

Definition A.1.11. Let A be a set.

(1) A is called a $\mathfrak{U}$-*small* set if $A \in \mathfrak{U}$.
(2) A is called a $\mathfrak{U}$-*class* if $A \subseteq \mathfrak{U}$.
(3) A $\mathfrak{U}$-class that is not a $\mathfrak{U}$-small set is called a *proper* $\mathfrak{U}$-*class*.
(4) A is called an *essentially* $\mathfrak{U}$-*small* set if there exists a $\mathfrak{U}$-small set B such that there exists a bijection $f\colon B \to A$.

In the following, we omit "$\mathfrak{U}$-" if there seems to be no confusion[2].

Remark A.1.12.

(1) By Remark A.1.5 (1), small sets are classes.
(2) Since $\mathfrak{U} \notin \mathfrak{U}$, $\mathfrak{U}$ is a proper class.
(3) By Remark A.1.5 (10), essentially small classes are small sets.
(4) A class A is a proper class if and only if A is not an element of any classes. Indeed, assume that A is a proper class. If A is an element of some class C, then since $C \subseteq \mathfrak{U}$, we have $A \in \mathfrak{U}$, a contradiction. Since $\mathfrak{U}$ is a class, the converse is obvious. This fact shows, in particular, that *a set is not a class if it has a proper class as its element* .

Example A.1.13 (essentially small set that is neither small nor a class). Let C be a proper class (e.g., $C = \mathfrak{U}$). Then the set $\{C\}$ has only one element, and it is essentially small. However, since it has a proper class as its element, it is not a class (therefore not a small set, either) by Remark A.1.12 (4).

Thus we have $\{C\} \in \mathscr{P}^2\mathfrak{U} \setminus \mathscr{P}\mathfrak{U}$, and hence $\mathscr{P}\mathfrak{U} \subsetneq \mathscr{P}^2\mathfrak{U}$. Therefore, for each $k \in \mathbb{N}$, we also have $\mathscr{P}^k\mathfrak{U} \subsetneq \mathscr{P}^{k+1}\mathfrak{U}$.

[2] If "$\mathfrak{U}$-" is omitted, then there is a risk of confusion between a class that is not a set and a $\mathfrak{U}$-class. In such a case, we carefully distinguish them.