# Preface

It is well known that if two independent identically distributed random variables are Gaussian, then their sum and difference are also independent. It turns out that only Gaussian random variables have such property. This statement, known as the Kac–Bernstein theorem, is a typical example of a so-called characterization theorem. Characterization theorems in mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions of these random variables.

The first characterization theorems in mathematical statistics are associated with the names of such classics of mathematics of the 20th century as G. Pólya, M. Kac, S.N. Bernstein, Yu.V. Linnik. The proofs of these theorems required the development of powerful methods of complex analysis. By now, the corresponding theory on the real line has basically been constructed. The problem of extending the classical characterization theorems to various algebraic structures has been actively studied in recent decades. Characterization theorems were studied on Lie groups, quantum groups, symmetric spaces, and Banach spaces. Many papers were especially devoted to the study of characterization theorems on locally compact Abelian groups.

The purpose of the book is to present the results obtained in this direction in the last 15 years. We have also included some important results obtained earlier. On the whole, the book gives a comprehensive and self-contained overview of the current state of the theory of characterization problems on locally compact Abelian groups.

When proving characterization theorems in mathematical statistics on groups, two related problems arise. The first problem is the same as in the classical case of the real line: the description of possible distributions of random variables through some properties of functions of these random variables. As a rule, this problem is solved either in a certain class of groups or for a particular group. From this point of view, the real line is just a very important but a particular locally compact Abelian group. The second problem is to find out when the distributions arising in this description, depending on the properties of a group, are such known distributions as either Gaussian distributions or idempotent distributions or their convolutions.

We shall briefly describe the main contents of the book.

Chapter I is of an auxiliary nature. In order to make the presentation independent, we present in Chapter I well-known facts related to abstract harmonic analysis and results on probability distributions on topological Abelian groups. A separate section is devoted to Gaussian distributions on locally compact Abelian groups.

In Chapter II we study the distributions of independent random variables $\xi_1$ and $\xi_2$ with values in a locally compact Abelian group $X$ and having independent sum and difference. If, in addition, $\xi_1$ and $\xi_2$ are identically distributed with distribution $\mu$, then $\mu$ is called a Gaussian distribution in the sense of Bernstein. First, for various classes of locally compact Abelian groups we solve the main problems on Gaussian distributions in the sense of Bernstein. For groups $X$ whose connected component of the zero contains a finite number of elements of order 2 we prove a decomposition theorem for such distributions: every Gaussian distribution in the sense of Bernstein is a convolution of a Gaussian distribution, an idempotent distribution, and a signed measure supported in a subgroup generated by elements of order 2. Based on this result, we give a complete description of locally compact Abelian groups on which every Gaussian distribution in the sense of Bernstein is represented as a convolution of a Gaussian distribution and an idempotent distribution. These are groups whose connected component of the zero contains no more than one element of order 2. We describe supports of Gaussian distributions in the sense of Bernstein and also prove a zero-one law for them. Next, we study the distributions of independent random variables with independent sum and difference. In so doing, we do not assume that the random variables are identically distributed.

The Kac–Bernstein theorem is a special case of a more general statement known as the Skitovich–Darmois theorem. In this theorem the Gaussian distribution on the real line is characterized by the independence of two linear forms $L_1$ and $L_2$ of independent random variables $\xi_j$ with distributions $\mu_j$. In Chapter III we study analogues of the Skitovich–Darmois theorem for various classes of locally compact Abelian groups $X$. In so doing, coefficients of $L_1$ and $L_2$ are topological automorphisms of $X$. First, we assume that the characteristic functions of $\xi_j$ do not vanish. We prove that if a group $X$ contains no subgroups topologically isomorphic to the circle group $\mathbb{T}$, then the independence of $L_1$ and $L_2$ implies that all distributions $\mu_j$ are Gaussian. We also prove that for an arbitrary locally compact Abelian group $X$ the description of distributions which are characterized by the independence of $L_1$ and $L_2$ is reduced to the case when independent random variables take values in a group of the form $\mathbb{R}^{\mathfrak{n}} \times \mathbb{T}^{\mathfrak{m}}$, where $\mathbb{T}$ is the circle group. For two independent random variables we prove that if a group $X$ has no subgroups topologically isomorphic to the group $\mathbb{T}^2$, then the independence of $L_1$ and $L_2$ implies that $\mu_j$ are either Gaussian distributions or convolutions of Gaussian distributions and signed measures supported in a subgroup of $X$ generated by an element of $X$ of order 2. In the classes of discrete and compact totally disconnected Abelian groups we prove the series of theorems where shifts of idempotent distributions are characterized. In so doing, the characteristic functions of independent random variables can vanish. To conclude this chapter, we prove that in contrast to the case of two linear forms of two independent random variables, if we consider three linear forms of three independent random variables with values in the cylinder $\mathbb{R} \times \mathbb{T}$, then only Gaussian distributions are characterised by the independence of these linear forms.

Chapter IV is devoted to analogues of the well-known Heyde theorem, where the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another. We study Heyde's theorem for various classes of locally compact Abelian groups $X$. Coefficients of linear forms, as in the Skitovich–Darmois theorem, are topological automorphisms of $X$. Particular attention is paid to the case of two independent random variables. If we consider the linear forms $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha\xi_2$, where $\alpha$ is a topological automorphism of $X$, then the kernel

Ker$(I + \alpha)$ plays an important role in the description of distributions $\mu_j$. In addition, it is important whether a group $X$ contains an element of order 2. This is the fundamental difference between the group situation and the case of the real line. First, we assume that the characteristic functions of independent random variables do not vanish and $X$ contains no elements of order 2. We prove a series of theorems characterizing the Gaussian distribution. Particular attention is paid to Heyde's theorem on $\boldsymbol{a}$-adic solenoids. Then we study Heyde's theorem for discrete Abelian groups and compact totally disconnected Abelian groups. In this case the characteristic functions of independent random variables can vanish, but $X$ still contains no elements of order 2. For these classes of groups we prove some theorems, where shifts of idempotent distributions are characterized. In the final section of the chapter we study Heyde's theorem for some groups containing an element of order 2.

Chapter V is devoted to characterization theorems for probability distributions in the case when independent random variables $\xi_1$ and $\xi_2$ take values in the additive group of the field of $p$-adic numbers $\Omega_p$. We study analogues of the Skitovich–Darmois and Heyde theorems for the group $\Omega_p$. We also prove for the field $\Omega_p$, a theorem, where shifts of idempotent distributions are characterized through the independence of sum and difference squared of $\xi_1$ and $\xi_2$. We also prove a similar theorem for discrete fields.

According to the well-known Rao theorem, if we have two linear forms $L_1$ and $L_2$ of three independent random variables $\xi_j$ with nonvanishing characteristic functions, then under certain conditions on the coefficients of the forms, the distribution of the random vector $(L_1, L_2)$ determines the distributions of $\xi_j$ up to a shift. C.R. Rao also proved that if four independent random variables $\xi_j$ are considered, then the distribution of the random vector $(L_1, L_2)$ determines the distributions of $\xi_j$ up to a convolution with a Gaussian distribution. In Chapter VI we prove for an arbitrary locally compact Abelian group $X$ an analogue of the first Rao theorem and for $\boldsymbol{a}$-adic solenoids an analogue of the second Rao theorem. In so doing, coefficients of the linear forms are continuous endomorphisms of $X$. We also prove for $\boldsymbol{a}$-adic solenoids an analogue of the well-known generalized Pólya theorem, where the Gaussian distribution on the real line is characterized by the property of equidistribution of a monomial and a linear form of independent identically distributed random variables.

Comments are given at the end of each chapter.

Characterization problems studied in the book are reduced to solving various functional equations. These equations are considered either on the character group of the original group or on the group itself. In most cases, solutions are sought in the class of characteristic functions (Fourier transforms) of the corresponding distributions.

We hope that the book will be useful to everyone who is interested in abstract harmonic analysis, probability distributions on groups or functional equations on groups.

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