CHAPTER 1

Constituents of the univariate antenna problem

There is no doubt that the study of the 1D bandlimited function of the form

$$t \mapsto \int_{\mathcal{F}_{\mathbb{Z}}} F(x)\, e^{-2\pi i t\cdot x}\, dx = F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t), \quad t \in \mathbb{R}, \quad F \in L^2(\mathcal{F}_{\mathbb{Z}}), \tag{1.1}$$

where $\mathcal{F}_{\mathbb{Z}} = [-\frac{1}{2}, \frac{1}{2})$ is the one-dimensional fundamental cell of the 1D unit lattice $\Lambda = \mathbb{Z}$, has significantly influenced the mathematical development during the last decades. In fact, the specific properties of the 1D Fourier integral (1.1) has led to a deepened mathematical discussion in two essential research areas initiated by engineering, namely

- regularization methods in inverse problems,
- sampling methods in signal analysis/processing.

As pointed out in the Preface, thirty years ago, the connections among the areas of regularization and sampling were rather tenuous. Researchers in one of the areas were often unfamiliar with the techniques and relevance of the other area. Today the situation, however, has changed, not least because of the 1D study of the bandlimited functions (1.1) as a bridge between two specific problem fields, namely

- antenna problem oriented approach to inverse problem theory,
- Shannon sampling oriented approach to signal analysis.

It turned out that the common thread among these problems led to key constituents in both inverse as well as sampling problem theory, so that it is time now to pass over to a description of the underlying mathematical ideas and concepts resulting canonically in a superordinate unifying framework:

Recovering an object (function, signal, picture) from partial or indirect information about the object.

As a matter of fact, a substantial amount of the machinery from functional analysis, Fourier analysis, theory of special functions, potential theory, lattice point theory, approximation theory, and numerical analysis has been brought to bear on the resolution and understanding of a recovery problem, and the interdisciplinary character of many recovery problems has emerged very clearly. The concept is to provide a new outlook within which scientific results can be better motivated and understood. Within this context, criteria can be given relative to which the scope and limitations of the various methods can be assessed. These aspects are important both in theory and practice.

In what follows, for purposes of motivation, our first interest is in showing that the univariate treatment of (1.1) within the antenna problem has provided deep and profound results in the theory of inverse problems. Simultaneously, the 1D Shannon sampling theory of the last century experienced far-reaching generalizations and innovative developments. Today, in a synopsis of both research fields, the concept of recovery becomes obvious and further developable, so that a common view has started to take a new prominent role within the traditional branches of mathematics, thereby leading to appropriate multivariate extensions and discoveries in other areas of mathematics and engineering.

1.1. Problem induced regularization

The following approach to a trend setting philosophy to treat ill-posed problems of mathematics is essentially influenced by M. Z. Nashed's understanding of the 1D antenna problem published in "IEEE Transactions on Antennas and Propagation, Vol. AP-29, No. 2, 220-231, 1981" under the title "Operator-Theoretic and Computational Approaches in Ill-Posed Problems with Applications to Antenna Theory" (cf. [**625**]).

In 1D antenna theory, the essential problem is the one in which the *far-field radiation pattern* $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ is known, either exactly or approximately, by means of physical measurements. The problem is to find the *aperture distribution* F producing this given pattern $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$. More explicitly, in a mathematical formalism, determine the aperture distribution $F \in L^2(\mathcal{F}_{\mathbb{Z}})$ from the far field $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ via the Fredholm integral equation of the first kind

$$AF(t) = \int_{\mathcal{F}_{\mathbb{Z}}} F(x)\, e^{-2\pi i t\cdot x}\, dx = F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t), \quad t \in \mathbb{R}. \tag{1.2}$$

In other words, the antenna problem (1.2) of determining the aperture distribution from far-field input requires particular solvability procedures which come from the theory of inverse problems (IPs).

Schematically, a *direct (forward) problem* can be formulated as follows:

object (e.g., aperture distribution) $\longrightarrow$ data information of the object (e.g., far-field).

The *inverse problem* is considered the "inverse" to the forward problem which relates the object (sub)information to the object:

data information of the object (e.g., far-field information) $\longrightarrow$ object (e.g., aperture distribution).

An object may be understood to be the systematic relationship of all data (sub)information, object parameters, and other auxiliary information.

In a mathematical abstraction of equations such as (1.2) involving an operator A we are usually confronted with the following situation: Given spaces X, Y equipped with the settings of "distance" (i.e., norm) and "angle" (i.e., inner product), i.e., X, Y are (ideally) assumed to be Hilbert spaces (especially, for the 1D antenna problem, we are led to the Hilbert spaces $X = L^2(\mathcal{F}_{\mathbb{Z}})$ and $Y = L^2(\mathbb{R})$). Consider a general mapping A from X to Y,

$$A : X \to Y. \tag{1.3}$$

The *direct problem* is as follows: Given $x \in X$, find $y = Ax \in Y$. The *inverse problem* is as follows: Given an observed output y, find an input x that produces it, i.e., $Ax = y \in Y$, or given a desired output z, find an input x that produces an output $y = Ax \in Y$ that is as "close" to z as possible, e.g., in the least squares sense:

$$\min_{x \in X} \|Ax - z\|^2. \tag{1.4}$$

A *well-posed (properly-posed) problem in the sense of Hadamard* is as follows: For $y \in Y$, the operator equation $X \ni x \mapsto Ax = y \in Y$ has one and only one solution $x \in X$, and the solution depends continuously on y. So, a mathematical problem is well-posed in the sense of Hadamard, if it satisfies the following properties:

(H1) *Existence: For given data, there exists a solution of the problem (in an appropriate sense).*

(H2) *Uniqueness: For given data, the solution is unique.*

(H3) *Stability: The solution depends continuously on the data.*

In accordance with this definition, *a problem is ill-posed in the sense of Hadamard*, if at least one of these three conditions is violated.

In this respect it should be noted that J. Hadamard (1865-1963) dismissed ill-posed problems as irrelevant to real world applications, but he was proven wrong four decades after his declaration. In fact, it turned out that Hadamard's classification (cf. [**390**, **391**]) had a tremendous influence on the development of practice-oriented mathematics.

Initially, the theory of inverse problems was influenced by two essential cornerstones, the *method of least squares* and the *concept of the pseudoinverse*: The actual reasoning for an adjustment theory with its core, the method of least squares, was already done by Gauss and Legendre around 1800. Gauss probably hit on the fundamental idea in the autumn 1794 reading a publication of Lambert concerning applications of mathematics (see A. Galle [**342**]), but Gauss published his method not before 1809 (cf. "Theoria motus corporum coelestium in sectionibus conicis solem ambientium"[**345**]). He closed this publication by giving an explanation of his method ("Determinatio orbitae observationibus quotcumque quam proximae satisfacientis"). Further works followed in the year 1810 ("Disquisitio de elementis ellipticis Palladis" [**346**]), in 1816 ("Bestimmung der Genauigkeit der Beobachtungen" 1880) and finally, in 1821 and 1823 in a systematic presentation ("Theoria combinationis observationum erroribus minimis obnoxiae, pars prior et pars posterior" [**350**]). Three years later these works were completed by the note "Supplementum theoriae combinationis observationum erroribus minimis obnoxiae" [**353**]. From then on, the concept of a pseudoinverse (or generalized inverse) $A^\dagger$ has rated considerable attention in mathematics as well as geosciences (a bibliography, for example, listing over 1700 references on the subject is due to M.Z. Nashed [**618**]).

Early interest in the first half of the last century in the subject of pseudoinverses was associated to a paper on matrices by R. Penrose [**684**]. However, this concept had been considered somewhat earlier. For example, E. H. Moore [**588**] presented a development of the notion (see also R. Baer [**57**], A. Bjerhammar [**89**], K. Friedrich

[**336**], E.H. Moore [**589**], R. Rado [**706**], and C.L. Siegel [**774**]). Moreover, in the setting of integral and differential operators the concept was considered even earlier by I. Fredholm [**247**] and W.A. Hurwitz [**452**], and by D. Hilbert [**433**] (see W.T. Reid [**716**] for a discussion of generalized inverses in classical analysis, and see also A. Ben-Israel and T.N.E. Greville [**77**], T.L. Boullion, P.L. Odell [**108**], M.Z. Nashed [**618**], W. Freeden, T. Sonar, B. Witte [**328**], and W. Freeden, B. Witte [**332**] for brief historical sketches on the subject). The relation between Gauss's ideas and the concept of the pseudoinverse was discussed by D.W. Robinson [**724**]. His paper attempts to show that although Gauss did not formalize the notion of a pseudoinverse, he provided the essential ingredients to produce one.

Today, the approach to the generalized inverse is well-described within the notational framework of Linear Algebra, more explicitly, within the reference structure of the linear space of $n \times m$ matrices $\mathbb{K}^{n\times m}$ (with $\mathbb{K} = \mathbb{R}$ for the set of real and $\mathbb{K} = \mathbb{C}$ for the set of complex numbers). Within this reference framework, the foundation is laid for the operator nomenclature of range, nullspace (kernel), adjoint, conventionally denoted by $\mathcal{R}(A)$, $\mathcal{N}(A)$, and A^*, respectively.

The *pseudoinverse (or generalized inverse)* $A^\dagger$ is the linear operator which assigns to each element of $\mathcal{D}(A^\dagger) = \mathcal{R}(A) + \mathcal{R}(A)^\perp$ the unique solution in $\mathcal{N}(A)^\perp$ of the *normal equation*

$$A^*Ax = A^*y \tag{1.5}$$

(the equation obtained by setting the first variation of $\|Ax - y\|^2$ equal to zero). It also follows that $A^\dagger$ can be characterized as the linear operator with the properties:

$$\mathcal{D}(A^\dagger) = \mathcal{R}(A) + \mathcal{R}(A)^\perp, \quad \mathcal{N}(A^\dagger) = \mathcal{R}(A)^\perp = \mathcal{N}(A^*) \tag{1.6}$$

and

$$\mathcal{R}(A^\dagger) = \mathcal{N}(A)^\perp. \tag{1.7}$$

Moreover,

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A. \tag{1.8}$$

The equivalence of these characterizations of $A^\dagger$ is established and generalized by M. Z. Nashed (see, e.g., [**627**]) to unbounded operators such as the integral operator A, given in (1.2).

As an evolution of Hadamard's classification, M.Z. Nashed's work [**618**], [**625**], [**627**] also shows the advantage of adopting a notion of well-posedness that is focused on infinite-dimensional problems (e.g., an inconsistent finite system of linear algebraic equations will not be ill-posed in Nashed's sense, while it is ill-posed in the sense of Hadamard). Indeed, it follows immediately from the open mapping theorem in functional analysis (see, e.g., [**39**]) that the following statements are equivalent:

(N1) *the inverse problem is well-posed,*

(N2) *the range $\mathcal{R}(A)$ of the operator A is closed,*

(N3) *the pseudoinverse $A^\dagger$ of the operator A is bounded.*

Thus, we are led to call an IP *well-posed in the sense of Nashed*, if $\mathcal{R}(A)$ is closed. If $\mathcal{R}(A)$ is not closed (such as (1.2) in the L^2-nomenclature), IP is called *ill-posed in the sense of Nashed.*

Regularization strategies. The rationale in most methods for the resolution of an inverse problem IP such as (1.2) is to construct a "solution" that is acceptable physically as a meaningful approximation and is sufficiently stable from the computational standpoint. The main dilemma of modeling inverse problems is that most of them are ill-posed. The characteristic of such problems is that the closer the mathematical model describes the ill-posed problem (IPP), the worse is the "condition" of the associated computational problem (i.e., the more sensitive to errors). Therefore, the indispensable problem is to bring additional information about the desired solution, compromises, or new outlooks as aids to the resolution of ill-posed problems (IPPs). It is conventional to use the phrase *"regularization of an ill-posed problem"* to refer to various approaches to circumvent the lack of continuous dependence as well as to bring about existence and uniqueness if necessary. Roughly speaking, this entails an analysis of an IPP via an analysis of an associated well-posed problem, i.e., a family (usually a sequence or a net) of well-posed problems (WPPs), yielding meaningful answers to the IPP.

Nowadays, from the standpoint of mathematical and numerical analysis, one can roughly group "regularization methods" into several (canonically overlapping) categories:

- *Regularization methods in function spaces* is one category. This includes Tikhonov-type regularization (see, e.g., [**618**], [**821**]), the method of quasi-reversibility, the use for certain function spaces such as scale spaces in multiresolutions, the method of generalized inverses (pseudoinverses), e.g., in reproducing kernel Hilbert spaces (cf. [**640**], [**641**]), and multiscale wavelet regularization (see, e.g., [**266**], [**289**, **292**], [**313**]).
- *Resolution of ill-posed problems by "control of dimensionality"* is another category (see, e.g., [**629**] and the literature therein). This includes projection methods and moment-discretization schemes. The success of these methods hinges on the possibility of obtaining an approximate solution while keeping the dimensionality of the finite dimensional problem within the "range of numerical stability". It also depends on deriving error estimates for the approximate solutions that is crucial to the control of the dimensionality.
- A third category is formed by *iterative methods* (see, e.g., [**487**], [**488**]) which can be applied either to the problem in function spaces or to a discrete version of it. The crucial ingredient in iterative methods is to stop the iteration before instability creeps into the process. Thus iterative methods have to be modified or accelerated so as to provide a desirable accuracy by the time a stopping rule is applied.
- A fourth category constitutes *filter methods.* Filter methods refer to procedures where, for example, values producing highly oscillatory solutions are eliminated. Various "lowpass" filters can, of course, be used. They are also crucial for the determination of a stopping rule. *Mollifiers* (see, e.g., [**218**], [**215**], [**296**], [**291**], [**270**]) are known as smooth filter functions with special properties to create sequences of smooth functions approximating a nonsmooth or singular function. Thus, we compromise by changing the problem into a more well-posed one, namely that of trying to determine a mollified version of the solution. Once more, the heuristic motivation

often is that the trouble comes from high frequency components of the data and of the solution, which are damped out by mollification.

- The root of the *Backus-Gilbert method (BG method)* was geophysical (cf. [**54**], [**55**], [**56**]). The characterization involved in the model is known as moment problem in the mathemetical literature. The BG method can be thought of as resulting from discretizing an integral equation of the first kind. Where other regularization methods, such as the frequently used Tikhonov regularization method, seek to impose smoothness constraints on the solution, the BG method instead realizes stability constraints (cf. [**867**]). As a consequence, the solution is varying as little as possible if the input data were resampled multiple times. The common feature between the mollifier and the BG method is that an approximate inverse is determined independently from the right side of the equation.
- The usual determination of parameters by inverse analysis assumes that the parameters are deterministic, that is that they are unknown but constants. The statistical approach to estimating parameters assumes that the parameters are random and that some knowledge of their statistical properties is known via a prior probability density distribution. Originally, *statistical regularization* is the name given to a form of estimation which resembles the usual Tikhonov-Phillips regularization method in which the regularization parameter is related to the variance of the prior. In modern nomenclature, the designation *stochastic regularization* is used to reduce the sensitivity to the degree of uncertainty in the parameters implying an estimation of the data, for example, a mean square estimation of a parameter which has a Gaussian distribution when the input data is also Gaussian distributed.
- Linear models for nonlinear relationships usually describe the reality in a very limited way. Thus it is required to work with nonlinear concepts. Because of the diversity of nonlinear phenomena, there is no closed inverse theory. As a consequence, nonlinear inverse problems are mainly based on the specific properties of the individual case.

Each method for resolution involves a critical "parameter" whose "optimal value" is crucial to the amenability and numerical implementation of the method. In *Tikhonov-type regularization* it is the regularization parameter, or more generally the choice of the regularization operator. In *multiscale methods* it is the scale parameter to determine the scale space in which multiresolution is realizable relative to the data width and noise. In *projection* and other discrete methods, it is the optimal dimension of the approximating subspaces. *Mollifier regularization* of integral operators (cf. [**270**], [**100**]) uses a kernel function that convolved with a particular function yields a filtered function, which is "close" to the function but "smoother". Usually, a mollifier is not only one kernel function but a sequence, or usually a one-parameter (i.e., scale dependent) family of kernels (also known as an approximate identity), so that the kernels of a mollifier have to tend to the Dirac kernel. As a consequence, mollifier kernels become more and more space-localized, and this make them extremely attractive for numerical purposes. In addition, filtering (convolution) with mollifier wavelets constituting the difference of two subsequent mollifier kernels provides band-pass information which, with increasing scales, deliver better and better details up to a certain scale. In discretization methods it is the choice of the mesh size beyond which a further refinement will lead to instability. In *iterative methods* it is the moment at which one should terminate the iteration (i.e., it is the number of iterates). In *filter-truncation methods* it is the number of terms to be included, etc. Our work intends to delineate suitable principles that quantify the choice of the parameter, the type of estimates, and a priori information that are needed to arrive at an "optimal" value for this parameter.

Expansion theorems play a fundamental role for an ill-posed *inverse problem*. Two problems are particularly significant: The determination of an (object) function from the inner product with a given set of functions (i.e., the *moment problem* (MP)), and the recovery of a function from its values approaches on a subset of its domain (i.e., the *reconstruction from samples* from sampling expansions). Moreover, new directions in expansion approaches are of major interest, such as in reproducing kernel Hilbert spaces, translation/rotation-invariant spaces, Sobolev spaces, etc. As a consequence, our textbook is concerned with a variety of function spaces in which functions admit representative recovery expansions.

Dilemmas and methodologies of the (re)solution of ill-posed problems and their numerical implementations are examined in this framework with particular reference to the problem of finding minimum weighted-norm least-squares solutions of first kind integral equations (and more generally of linear operator equations with nonclosed range). A common problem in all these methods is delineated: Each method reduces the problem of resolution to a "nonstandard" minimization problem involving an unknown critical "parameter" whose "optimal" value is crucial to the numerical realization and amenability of the method. The "nonstandardness" results from the fact that one does not have explicitly, or a priori, the function to be minimized: It has to built up using additional information, convergence rate estimates, and robustness conditions, etc. Several results are discussed that represent and complement recent advances in numerical analysis and regularization of inverse and ill-posed problems. An emphasis is placed on the role of constraints, function space methods, the role of generalized inverses, and reproducing kernels in the regularization and stable computational resolution of these problems.

The thrust of the work is devoted to the interdisciplinary character of operator-theoretic and numerical methods for ill-posed problems. In fact, our purpose is to provide a new outlook within which technical results can be better motivated and understood. Within this framework, criteria can be given relative to which the scope and limitations of the various methods can be assessed. This is important both in theory and practice since there is no cure-all method for ill-posed problems. Therefore it is imperative to be able to clarify why a certain method works in some context as well as when not to use that method. The work discusses at length the intuitive principles that underlie the various methods and establishes some results within this framework. The development in the first part of the work mostly based on a functional analytic background is fairly general in scope and theory, and the applicability covers a wide range.

Univariate antenna problem in L^2-nomenclature. The particular genealogical role of Nashed's 1D antenna note [**625**] in the recovery context is as follows: The operator A as given by (1.2) is a linear mapping from the space $L^2(\mathcal{F}_{\mathbb{Z}})$ to the range

$$\mathcal{R}(A) = \left\{ \mathbb{R} \ni x \mapsto F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(x) = \int_{\mathcal{F}_{\mathbb{Z}}} F(t)\, e^{-2\pi i t \cdot x}\, dt : F \in L^2(\mathcal{F}_{\mathbb{Z}}) \right\}. \tag{1.9}$$

Central for the understanding of the regularization as well as sampling constituents in our recovery context below is the fact that the space $\mathcal{R}(A)$ is nonclosed in $L^2(\mathbb{R})$. To show this we give a specific Fourier theoretic and a general functional analytic based verification:

- *Fourier theoretic verification.* For every sufficiently small $\varepsilon > 0$, the characteristic function $\chi_{[-\varepsilon,\varepsilon]}$ given by

$$\chi_{[-\varepsilon,\varepsilon]}(x) = \left\{ \begin{array}{ccc} 0 & , & x > \varepsilon, \\ 1 & , & x \leq \varepsilon, \end{array} \right. \tag{1.10}$$

  is in the class $L^2(\mathcal{F}_{\mathbb{Z}})$, and $\frac{1}{2\varepsilon}(\chi_{[-\varepsilon,\varepsilon]})^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ explicitly expressed by

$$\frac{1}{2\varepsilon}(\chi_{[-\varepsilon,\varepsilon]})^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(x) = \frac{1}{2\varepsilon} \int_{\mathcal{F}_{\mathbb{Z}}} \chi_{[-\varepsilon,\varepsilon]}\, e^{-2\pi i t \cdot x}\, dt = \frac{\sin(2\pi\varepsilon x)}{2\pi\varepsilon x} \tag{1.11}$$

  represents a function in $\mathcal{R}(A)$. The limit function of (1.11) with $\varepsilon \to 0$ is equal to 1 and, therefore, not in $L^2(\mathbb{R})$. In other words, the range of the linear operator $A : L^2(\mathcal{F}_{\mathbb{Z}}) \to L^2(\mathbb{R})$, as defined by (1.2) mapping the space $L^2(\mathcal{F}_{\mathbb{Z}})$ to the range $\mathcal{R}(A)$ is nonclosed in $L^2(\mathbb{R})$.
- *Functional analytic verification.* It is a well-known result in functional analysis (see, e.g., [**435**]) that $A : L^2(\mathcal{F}_{\mathbb{Z}}) \to L^2(\mathbb{R})$, as defined by the integral equation (1.2), is a compact linear operator mapping the space $L^2(\mathcal{F}_{\mathbb{Z}})$ to the infinite-dimensional range $\mathcal{R}(A)$, so that A is nonclosed understood in the framework of $L^2(\mathbb{R})$. The nonclosedness indeed is a consequence of the open mapping theorem, and it is valid not only for the operator in (1.2), but also for all compact operators with infinite-dimensional range.

In order to overcome the nonclosedness of the range of the linear operator $A : L^2(\mathcal{F}_{\mathbb{Z}}) \to L^2(\mathbb{R})$, as defined by (1.2), a way out (see [**625**]) is the observation that the range $\mathcal{R}(A)$ can be endowed with a new topology that would make this space a reproducing kernel Hilbert space (RKHS) with useful properties, that means, we are able to shrink the image space $L^2(\mathbb{R})$ to the range $\mathcal{R}(A)$ and understand the latter space as the so-called Paley-Wiener space $B_{\mathcal{F}_{\mathbb{Z}}}$ with an associated topology to guarantee that the range is closed. As a consequence, in analyzing the 1D antenna problem, we are led to a remarkable transition, namely from an ill-posed problem in the sense of Hadamard (in the nomenclature of $L^2(\mathbb{R})$) to a well-posed problem in the Paley-Wiener reproducing kernel Hilbert space $B_{\mathcal{F}_{\mathbb{Z}}}$ in a canonically defined topology (see (1.14)):

Paley-Wiener reproducing kernel Hilbert space. The *Paley-Wiener space* $B_{\mathcal{F}_{\mathbb{Z}}}$ defined by (1.12)

$$B_{\mathcal{F}_{\mathbb{Z}}} = \left\{ \mathbb{R} \ni y \mapsto F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(y) = \int_{\mathcal{F}_{\mathbb{Z}}} F(t)\, e^{-2\pi i t \cdot y}\, dt : F \in L^2(\mathcal{F}_{\mathbb{Z}}) \right\} \subset C^{(0)}(\mathbb{R}) \tag{1.12}$$

is a reproducing kernel Hilbert space (RKHS) with the uniquely determined reproducing kernel (RK)

$$K_{\mathcal{F}_{\mathbb{Z}}}(x-y) = \int_{\mathcal{F}_{\mathbb{Z}}} e^{2\pi i t\cdot(x-y)}\, dt = \frac{\sin(\pi(x-y))}{\pi(x-y)} = \mathrm{sinc}\,(\pi(x-y))\,, \tag{1.13}$$

so that

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(y) = \int_{\mathbb{R}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(x) \left(\int_{\mathcal{F}_{\mathbb{Z}}} e^{2\pi i a\cdot(x-y)}\, da \right) dx = \int_{\mathbb{R}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(x)\; K_{\mathcal{F}_{\mathbb{Z}}}(x-y)\, dx \tag{1.14}$$

for all $y \in \mathbb{R}$.

Univariate antenna problem in $B_{\mathcal{F}_\Lambda}$-nomenclature. Under the assumption that $\mathcal{R}(A)$ is the subspace $B_{\mathcal{F}_{\mathbb{Z}}}$ of $L^2(\mathbb{R})$, the operator A defined by the integral equation (1.2) becomes *onto and*, hence, by the open mapping theorem of functional analysis, the inverse operator $A^{-1} : B_{\mathcal{F}_{\mathbb{Z}}} \to L^2(\mathcal{F}_{\mathbb{Z}})$ is *bounded.* Therefore, methods of constructive approximation in reproducing kernel Hilbert spaces (for example, associated spline methods as proposed in [**298**]) become efficiently applicable.

1.2. Problem induced sampling

The essential objective of sampling is to deal with the reconstruction of a bandlimited function, usually given in the form (1.1), by a sum in terms of the function itself at a discrete set of points. Sampling is a keystone of signal processing. The most common form of sampling today is lattice point sampling of a bandlimited signal $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ over the fundamental cell $\mathcal{F}_{\mathbb{Z}} = [-\frac{1}{2}, \frac{1}{2})$ of the unit lattice $\mathbb{Z}$ for a square-integrable function F on $\mathcal{F}_{\mathbb{Z}}$.

Classical Fourier analytic methodology. Fourier analysis offers different ways to the *1D Shannon sampling theorem* (based, e.g., on tools already known to L.J. Mordell [**590**], [**591**], [**592**], and H.G. Hardy [**412**], [**413**], [**414**]). Continuing the nomenclature used in the discussion of (1.1) we are able to explain the sampling theorem as follows:

Let $F^{\wedge}_{\mathcal{F}_\Lambda}$ be the Fourier transform for some $F \in L^2([-\frac{1}{2}, \frac{1}{2}))$ extended $\mathbb{Z}$-periodically to the real line $\mathbb{R}$. We expand the extension in a series of the form

$$F(x) = \sum_{h\in\mathbb{Z}} c_h\; \Phi_h(x), \tag{1.15}$$

where the system $\{\Phi_h\}_{h\in\mathbb{Z}}$ of one-dimensional "periodic polynomials" is given by

$$\Phi_h(x) = e(h\cdot x) = e^{2\pi i h\cdot x}, \quad h \in \mathbb{Z}, \tag{1.16}$$

and $\sum_{h\in\mathbb{Z}} \ldots$ means that the sum is extended over all h of the lattice $\mathbb{Z}$. It is not hard to see that

$$\int_{\mathcal{F}_{\mathbb{Z}}} \Phi_h(x)\, \overline{\Phi_{h'}(x)}\, dx = \int_{\mathcal{F}_{\mathbb{Z}}} e^{2\pi i h\cdot x}\; e^{-2\pi i h'\cdot x}\, dx = \delta_{h,h'} = \begin{cases} 1, & h = h', \\ 0, & h \neq h'. \end{cases} \tag{1.17}$$

Hence, the system $\{\Phi_h\}_{h\in\mathbb{Z}}$ is orthonormal with respect to the L^2-inner product ($\delta_{h,h'}$ is the Kronecker symbol). As a consequence, the coefficients c_h are given in the form

$$c_h = \int_{\mathcal{F}_{\mathbb{Z}}} F(x)\, \overline{\Phi_h(x)}\, dx = \int_{\mathcal{F}_{\mathbb{Z}}} F(x)\; e^{-2\pi i h\cdot x}\, dx, \tag{1.18}$$

i.e., $c_h = F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h)$. Substituting (1.15) into (1.18) we obtain the identity

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t) = \int_{\mathcal{F}_{\mathbb{Z}}} \sum_{h\in\mathbb{Z}} c_h \; e^{2\pi i h\cdot x} \; e^{-2\pi i x\cdot t} \; dx, \quad t \in \mathbb{R}, \tag{1.19}$$

such that by observing (1.2) and interchanging the order of integration and summation

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t) = \sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h) K_{\mathcal{F}_{\mathbb{Z}}}(h-t) = \sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h) \int_{\mathcal{F}_{\mathbb{Z}}} e^{2\pi i x\cdot(h-t)} \; dx. \tag{1.20}$$

This identity yields the celebrated classical expansion of a bandlimited signal for the "base band" $\mathcal{F}_{\mathbb{Z}} = [-\frac{1}{2}, \frac{1}{2})$:

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t) = \sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h) \; \frac{\sin(\pi(h-t))}{\pi(h-t)}. \tag{1.21}$$

Hence, documented in the lattice point nomenclature as used in this work, Shannon's sampling result (1.21) can be rephrased as follows:

Any function $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ *bandlimited to the fundamental cell* $\mathcal{F}_{\mathbb{Z}} = \left[-\frac{1}{2}, \frac{1}{2}\right)$ *of the unit lattice* $\mathbb{Z}$, *i.e., the Fourier transform* (1.1) *may be reconstructed from the sequence of samples*

$$\{F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h)\}_{h\in\mathbb{Z}} \tag{1.22}$$

via the so-called cardinal series

$$\sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h) \; \int_{\mathcal{F}_{\mathbb{Z}}} e^{2\pi i x\cdot(h-t)} \; dx, \quad t \in \mathbb{R}, \tag{1.23}$$

where the series is absolutely and uniformly convergent on any compact set of the real line $\mathbb{R}$.

Obviously, the Fourier transform $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ (considered as a one-dimensional signal) is an infinitely repeated replication of the samples at points h of the lattice $\mathbb{Z}$.

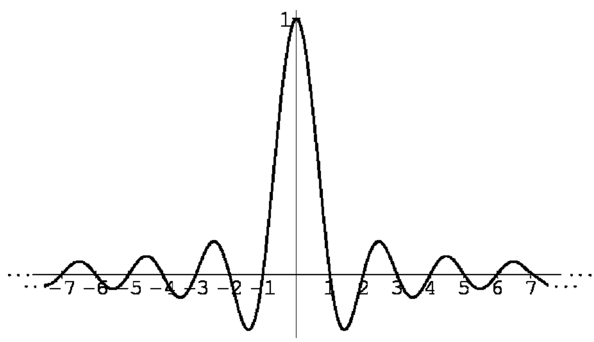

FIGURE 1. Illustration of the function $t \mapsto \operatorname{sinc}(\pi t) = \frac{\sin(\pi t)}{\pi t}$, $t \in \mathbb{R}$.

The *interpolatory background of bandlimited sampling* can be seen briefly as follows: A mathematical way to interpolate samples $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h')$, $h' \in \mathbb{Z}$, may be realized by use of sinc-functions (cf. Figure 1), since the sequence

$$\{\operatorname{sinc}(\pi\,(h-\cdot))\}_{h\in\mathbb{Z}} \tag{1.24}$$

shows the discrete orthogonality property:

$$\operatorname{sinc}(\pi(h-h')) = \delta_{h,h'} = \begin{cases} 1, & h = h', \\ 0, & h \neq h'. \end{cases} \tag{1.25}$$

As a consequence, $F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}$ given by the so-called *cardinal series*

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t) = \sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h)\ \operatorname{sinc}(\pi\,(h-t)), \quad t \in \mathbb{R}, \tag{1.26}$$

satisfies the interpolatory property

$$F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(t)|_{t=h'} = \sum_{h\in\mathbb{Z}} F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h)\ \operatorname{sinc}(\pi\,(h-t))|_{t=h'} = F^{\wedge}_{\mathcal{F}_{\mathbb{Z}}}(h'), \quad h' \in \mathbb{Z}. \tag{1.27}$$

It should be noted that the classical proof of the Shannon sampling theorem is indeed rigorous. The interchange of integration and summation from (1.19) to (1.20) can be easily justified. However, the proof is very revealing (as already pointed out by M.Z. Nashed [**629**]): We perform this interchange and a theorem pops up, but what is its essential mathematical core?

Sampling strategies. In fact, many questions arise when thinking about generalizations and modifications of the 1D Shannon sampling context (see also some aspects in P.L. Butzer [**123**, **128**], M. Unser [**827**], M. Unser, A. Aldroubi [**828**], W. Freeden, M.Z. Nashed [**298**]):

- *Engineering reflected aspects.* Engineering uses the 1D Shannon sampling theorem to express a continuous (time-dependent) signal in a discrete context, but what are the essential ingredients how to do so, seen from a mathematical point of view? Is there a deeper interrelationship between Shannon sampling and a periodically reflected Dirac distributional expression (Dirac impulse) leading to the specification of lattice reflected Green functions (lattice functions) and resulting Euler and Poisson summation formulas?
- *Functional analytic manifestations.* Are there functional analytic strategies to transfer reproducing kernel Hilbert/Banach space structures to sampling and/or to induce structures in reproducing kernel Hilbert/Banach spaces from sampling concepts? In this respect, it is of tremendous importance that both continuous and discrete variants of reproducing kernel Hilbert spaces arise from given positive definite kernels, i.e., a corresponding pre-Hilbert form and subsequently a Hilbert space completion. What are the specific characteristics of sampling procedures for continuous and discrete variants? While the better known and classical sampling algorithms (Shannon and others) are based on interpolation, are there theories to go beyond this? For example, is it possible to make precise the notion of "nonuniform point sampling" in a discrete reproducing kernel Hilbert/Banach space context?
- *Geometric number theoretic manifestations.* Are there straightforward multivariate counterparts of the univariate Shannon sampling theorem, not only by Parzen's iteration of 1D framework, but also on general qD "base bands" independent of the choice of the general qD lattice? In other words, is a specifically multivariate Shannon sampling variant valid for a

class of bandlimited sample functions related not only to the base band determined by the fundamental cell of the underlying lattice, but also to "finite bandwidth bands" such as, e.g., certain "potato-like" (i.e., regular) regions $\mathcal{G}$ in Euclidean space $\mathbb{R}^q$? How can we describe geometrically resulting procedures of over- and undersampling in sampling related to "base band" regions $\mathcal{G}$ by explicitly determining the aliasing error? How can we come to the explicit knowledge about the fortunate situation that there is no aliasing error between the cardinal series and the (Fourier transformed) functional value in bandlimited sampling?

- *Fourier theoretic manifestations.* Are there certain conditions at infinity involving Fourier transforms to enable multivariate nonbandlimited Shannon-type sampling, i.e., how can we concretize the asymptotic properties of functions at infinity reflecting infinite bandwidth sample methodologies and explicit representations of the aliasing error?
- *Potential theoretic manifestations.* Harmonic kernel structures are known to be involved in thin plate and cardinal spline concepts. Are there metaharmonic reproducing kernel structures to allow Shannon-type sampling procedures and Paley-Wiener related framework to deduce, for example, Paley-Wiener and Slepian splines for inverse sampling application in multivariate antenna theory?
- *Special functions summability based manifestations.* Are there bandlimited/nonbandlimited methodologies to decorrelate a certain signal (signature) in an adequate Shannon-type wavelet context provided by special function originated features such as multiscale Gauss-Weierstrass (averaging) sampling?

Most of these questions will be studied in this book by choosing standard as well as nonstandard approaches to sampling.

1.3. Recovery as common thread

Since the middle of the last century, the rudiments of sampling are covered in almost every engineering textbook on signal analysis, but sampling theory could be found only in rare exceptions in the mathematical literature. The connections among the areas (inverse problems, sampling in signal processing, and image analysis) were rather tenuous. Not least the results for the antenna problem have shown, that our understanding today of inverse problems and sampling in signal analysis is very clear (cf. [**629**]):

- *Inverse problems* deal with determining for a given input-output system an input that produces an observed output, or of determining an input that produces a desired output (or comes as close to it as possible), often in the presence of noise. Most inverse problems are ill-posed and demand regularization. In particular, *Moment problems* deal with the recovery of a function or signal from its moments, and the construction of efficient stable algorithms for determining or approximating the function.
- *Signal analysis/processing* deals with digital representations of signals and their analog reconstructions from digital representations. Sampling

methodologies such as expansions, filters, constituents of reproducing kernel spaces, various function space specifications, and techniques of functional and Fourier analysis, computational and (meta)harmonic analysis play pivotal roles in this area. In particular, *Image analysis/processing* deals with image refinement and recovery, and include all facets of imaging.

The common thread among all these disciplines canonically is the *problem of recovering an object (function, signal, picture) from partial or indirect information about the object.*

1.4. Organization of the work

Reflecting the 1D framework described in the Introduction for the antenna problem we are led to organize the parts of the book as follows:

- Part I is devoted to the study of the Fourier transform over the one-dimensional fundamental cell of the 1D unit lattice. It is documented that the Fourier integral bandlimited with respect to a fundamental cell of the 1D unit lattice has led to a deepened mathematical discussion in two essential research areas both initiated by engineering, namely regularization methods in inverse problems and sampling methods in signal analysis/processing.
- Part II presents some essential tools, that serve as auxiliary material for the first part of the book concerned with inverse problems and their regularization. It contains selected key items of classical functional and Fourier analysis.
- Part III is concerned with inverse problems and regularization strategies. Resolution methodologies involving generalized inverses and singular value decompositions are discussed for matrix equations. A broad collection of reconstruction and regularization methods for compact operator equations are outlined mostly with the aid of functional analytic means.
- Part IV illustrates some regularization methodologies in three-dimensional Euclidean space, applied to three significant inverse potential problems. The presented examples are of great interest in today's geoengineering, namely (geodetic) satellite gravitation gradiometry as well as inverse (exploration) gravimetry and magnetometry.
- Part V presents some essential tools, that serve as preparatory material for the second part of the book concerned with sampling. It contains selected key items of geometric number theory, metaharmonic special function theory such as Bessel and Kelvin functions, and Gauss-Weierstrass as well as Abel-Poisson variants in the Fourier inversion.
- Part VI deals with sampling problems and solution strategies. As prototype for various developments, the 1D Shannon sampling theorem is deduced in a twofold way, on the one hand by classical Fourier analytic concepts, on the other hand by lattice point theoretically based Euler/Poisson summation formulas. A palette of multivariate sampling methodologies using harmonic and metaharmonic tools and structures is given resulting in both finite and infinite bandwidth sampling.

- Part VII explains two important examples of sampling methodology: The inversion of the gravimetry problem in geoexploration is documented by certain expressions, which reproduce exponentials. The solvability of the multivariate antenna problem is analyzed within the framework of Paley-Wiener spline theory in multi-dimensional Euclidean spaces.
- Part VIII explains the idea of recovery as an interconnecting whole of inverse and sampling theory in essential publication organs and its manifestations in today's literature.