Preface

This book is devoted to a field at the intersection of fractal geometry and dynamical systems theory. Although certain parts of the theory have been around for a long time, in particular, the classical Cantor set and its relatives, as well as natural measures on them, in the last few decades it has seen an explosive growth. On the one hand, it was the insight of Mandelbrot who realized that diverse phenomena in many areas of Mathematics, Physics, Biology, and other Sciences, as well as in Nature, which have previously been viewed as "anomalous" or "weird" are actually widespread and have a common feature which he called "fractal". We will not attempt to give a definition of a fractal, but it has now become well established that introducing this notion was crucial and transformed the way we view many things in mathematics and far beyond. It has, of course, been tremendously aided by the advance of computers and the beautiful pictures which are easy to produce and bring the beauty of the subject to the general public. On the other hand, Dynamical Systems theory, in particular, of what is sometimes called the "chaotic" behavior, naturally lead to fractals, as many of the objects appearing there, such as attractors, repellers, basin boundaries, sets of parameters with distinguished properties, etc., are fractal.

There are many excellent books on fractals, among them we should mention the books by Rogers [**Rog70**], Falconer [**Fal85**, **Fal03**, **Fal97**], Barnsley [**Bar93**], Mattila [**Mat95**, **Mat15**], Edgar [**Edg98**], Bishop–Peres [**BP17**], and Fraser [**Fra20**]. Our goal is different, first of all, in that we focus almost exclusively on the class of *self-similar* and *self-affine* fractals. Roughly speaking, if we say that a "fractal" is a set where "zooming in" we see a picture that resembles the entire set, which is the general (imprecise) concept of self-similarity, then self-similar and self-affine sets are those for which this resemblance is precise and is given by a linear contracting similarity map or a general linear affine contracting map, respectively. These are the "model" and "test" cases, which have always been the first under investigation, and surprisingly, even simple-minded questions about them turn out to be very difficult. In the last decades, there have been many breakthroughs, and our aim is to give an introduction to some of them. Specific questions are usually about *"fractal" dimension*, the Hausdorff and box-counting dimensions being the most widely used, and on the topological nature of the fractals, such as: when do they have nonempty interior? In order to address these questions, it is unavoidable that we have to study *measures* on the fractal sets, so we are naturally led to self-similar and self-affine measures. We consistently use the framework of *iterated function systems*, first developed by Hutchinson, Diaconis, and Barnsley, and the intimately connected *symbolic spaces*, which provide the "coding", or "address" structure and facilitate the application of symbolic dynamics.

We concentrate on a few directions in the (relatively) recent developments. We try to choose the simplest special cases, where the proofs can be presented in a precise but less technical way, in order to introduce a range of important techniques for those interested to learn the subject in depth. It is unavoidable that many results that we rely on, are stated without proofs. Moreover, the subject has grown so much in recent years, that is impossible to give a comprehensive account in a single volume.

One of the key areas that we focus on is the investigation of self-similar sets with "overlaps". Such sets arise naturally, in particular, from orthogonal projections of nonoverlapping Cantor-like sets in higher dimensions. A closely related direction is the study of self-similar measures with overlaps, of which one of the best known is the family of so-called *infinite Bernoulli convolutions*, or simply Bernoulli convolutions. They are easy to define: for a parameter $0 < \lambda < 1$ consider ν_λ, the distribution of the random series $\sum_{n=0}^{\infty} \pm\lambda^n$, where the signs are chosen independently with probabilities $(1/2, 1/2)$. In the case $\lambda < 1/2$ we get the Cantor–Lebesgue measure on a homogeneous self-similar Cantor set of dimension less than 1, for $\lambda = 1/2$ this is a uniform measure on $[-2, 2]$, but the "overlapping" case $\lambda \in (1/2, 1)$ presents a major challenge. These measures and their close relatives come up in many areas, both pure and applied, and they have been studied since the 1930's by many authors, including Erdős, Salem, Kahane, and Garsia, by a variety of methods, with the help of harmonic analysis, number theory, and probability.

In the 1990s and early 2000s, the method of "transversality" has been developed, which led to a first breakthrough in the understanding of the overlapping case, both for self-similar sets and measures. The authors of this book, with co-authors, have been among the main contributors in this development, and it constitutes a large and important part of the book. More recently, a further breakthrough occurred, starting with the work of Hochman, followed by Shmerkin and Varjú, with various co-authors. The details of this latest stage, which is currently very "hot" and is still being developed, remains beyond the scope of our book; however we do summarize the main results obtained as of this writing.

Another great challenge is computing the dimension of general self-affine sets, even in the plane when there are no overlaps. Here the difficulty, compared with the self-similar case, is "nonconformality". In the book we present first the results obtained using the transversality method, which generalize a classical theorem of Falconer. In the study of self-affine sets, dynamical systems and ergodic theory tools appear even more prominently. Here also there were recent breakthroughs; an important one was started by Feng and Hu, who adapted the approach of Ledrappier and Young from dynamical systems theory to the self-affine case. A part of this theory is presented in the book. In this area as well, the most recent developments, e.g., those by Käenmäki, Morris, Bárány, Hochman, and Rapaport, are only summarized.

Once again, we emphasize that in such an active and broad field it is impossible to be comprehensive; we focus on a few specific directions which we consider interesting and important. Among the many topics which are completely left out or only mentioned briefly, are graph-directed systems, random fractals, iterated function systems that are contracting on-average, iterated function systems with infinitely many maps (an excellent book by Mauldin and Urbański [**MU03**] is devoted to the latter), and there is only one chapter about nonlinear systems.

On the other hand, in a few places, for the sake of completeness, our presentation bears similarity to the existing books; notably, to Falconer [**Fal97**, Chapter 11] in parts of our chapter on multifractals, to Mattila's books: [**Mat95**, Chapter 9] in the proof of Marstrand's theorem and [**Mat15**, Chapters 8–9] in the treatment of Cantor measures with Pisot parameters and Bernoulli convolutions, as well as to Bishop–Peres [**BP17**, Chapter 9] in the discussion of projections of self-similar sets and the Bandt–Graf theorem.

A few words about the contents and organization of the book. The long introductory chapter is aimed at a newcomer to the subject; there we present many examples, introduce the basic notions, and outline the main directions of what follows. A more experienced reader can skip it after a quick perusal and only return to it as necessary. The second chapter contains a few results from geometric measure theory that we need, among them results about densities of Radon measures, the Marstrand projection theorem, whose proof by Kaufman has been a "precursor" of the transversality method, and slicing theorems, with proofs or partial proofs, as well as additional background material on dimension of measures. In Chapter 3 we begin an in-depth investigation of self-similar sets and measures. We show that some results hold without any assumptions of nonoverlapping, such as equality of the Hausdorff and box-counting dimensions and lower semicontinuity in parameter dependence. We also prove "Laws of Pure Types" for self-similar measures and show the existence of L^q dimensions and the entropy dimensions for them. In Chapter 4 we turn to various conditions of being overlapping or nonoverlapping, the classical one being the Open Set Condition. The main part of this chapter is devoted to a large class of systems for which weaker forms of separation holds (namely, some of the "cylinders" are allowed to coincide). This turns out to be remarkably subtle, and we present a few featured special systems in which the methods can be understood more clearly. Some of these more specialized sections may be skipped on first reading. A short Chapter 5 gives a glimpse into the Multifractal Analysis for self-similar measures, which is mainly beyond the scope of the book. Chapter 6 contains the key parts of the transversality method for self-similar sets and measures and includes a discussion of more recent developments. In Chapter 7 we mainly focus on the questions of positivity of the Hausdorff and packing measure of self-similar sets in their dimension. This includes connections with the weak separation property and the Assouad dimension. The chapter also contains sections on important examples, such as projections of planar self-similar sets (without rotations) and the $\{0, 1, 3\}$-problem. Chapter 8 contains a mixture of old and more recent results, with a focus on Number Theory connections, often with the help of Fourier Analysis. It includes classical results of Erdős and Garsia on Bernoulli convolutions with a modern exposition, connections with β-expansions, and a discussion of recent breakthrough results on dimensions of Bernoulli convolutions. Chapter 9 presents some elements of Ergodic Theory, mostly without proof, and it opens the informal second part of the book devoted to self-affine sets and measures. It plays a role similar to Chapter 2 on elements of geometric measure theory. Ergodic Theory is used throughout the book, including the earlier chapters, in particular, the Ergodic and Shannon–McMillan–Breiman Theorem. In order to help a reader unfamiliar with the basics, some of the notions and results of Ergodic Theory are presented already in Chapter 1. In Chapters 10–13 the use of Ergodic Theory becomes more sophisticated, which includes Rokhlin's Theorem on disintegration of measures, Multiplicative Ergodic Theorem of Oseledeč, and Thermodynamic Formalism. Chapters 10–13 constitute the second "main core" of the book; they are devoted to selected results on self-affine sets and measures. In the first part of Chapter 10 we present the self-affine transversality condition and its consequences, following Jordan, Pollicott, and Simon. After that we introduce the sub-additive thermodynamic formalism for self-affine IFS's, including a presentation of Käenmäki's results and the notion of domination, with proofs in special cases. Chapter 11 is devoted to the Feng–Hu theory, which may be thought of as the adaptation of the Ledrappier–Young theory to the context of diagonally-self-affine IFS. We first present it the simplest nontrivial case, in order to motivate the notion of projection entropy. As an application, we consider a large variety of self-affine families, starting with the classical Bedford–McMullen carpets. In Chapter 12 we present a selection of results on dimension conservation and exact dimensionality for self-affine measures, highlighting the role of the Furstenberg measure and the Ledrappier–Young formula, extended to the general planar case. Chapter 13 studies projections of self-affine sets and measures, using the method of entropy averages introduced by Hochman and Shmerkin. In most of the results in Chapters 10–13 we restrict ourselves to the planar case, in order to make the presentation more accessible. Chapter 14 is somewhat special, in that it treats nonlinear conformal IFS on the line. We demonstrate how the transversality methods presented in Chapter 6 may be adapted to a nonlinear setting. Finally, several appendices supply the needed background material from diverse fields.

Parts of this book have been used by the authors in graduate and advanced undergraduate courses over the years: at the Budapest University of Technology and Economics (B.B. & K.S.), at the University of Washington and the Bar-Ilan University (B.S.), as well as lecture series and mini-courses at various schools and workshops. In particular, we think that Chapter 1 (Sections 1.1–1.6, 1.8–1.12); Chapter 2 (Sections 2.1–2.4); Chapter 3 (Sections 3.1–3.4); Chapter 4 (Section 4.1); Chapter 5 (Sections 5.1 and 5.3); Chapter 6 (Sections 6.1 and 6.2), and Chapter 8 (Section 8.1), are suitable material for an advanced undergraduate course. Much of the rest can be used for a graduate course, according to the Lecturer's taste.

In spite of our best efforts, errors (hopefully, minor) undoubtedly remain. We would appreciate receiving email about errors found by the readers. Corrections will be posted on the authors' personal websites.

Acknowledgment. The authors would like to express their gratitude to Simon Baker, Christoph Bandt, Jonathan M. Fraser, Thomas Jordan, Antti Käenmäki, Vilma Orgoványi, Yuval Peres, Mark Pollicott, Dániel Rudolf Prokaj, Feliks Przytycki, Ariel Rapaport, Alex Rutar, Adam Śpiewak, Mariusz Urbański, and the anonymous referees for their careful proofreading, useful comments and valuable suggestions. We are grateful to Pablo Shmerkin for his kind permission to use some excerpts of his unpublished Lecture Notes.