

# Contents

This is a three-volume work; each volume consists of two parts. The preface and backmatter are identical across all three volumes. The table of contents for each volume is followed by a copy of the tables of contents for the other two volumes.

*Reference Convention.* In each of Volumes II and III, the numbering of parts, chapters, and pages restarts at 1. Page numbers are prefixed with a volume number so that, for example, page  $n$  of Volume II is printed II. $n$ . References (chapters, definitions, theorems, equations, etc.) to a different volume are preceded by I, II, or III according to the volume. For example,  $a.b.c$  in Volume I is denoted by I. $a.b.c$  in Volumes II and III, and Chapter  $N$  in Volume III is denoted by Chapter III. $N$  in Volumes I and II.

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