Preface

Bimonoidal categories are categorical analogues of rings without additive inverses. They have been actively studied in category theory, homotopy theory, and algebraic K-theory since around 1970. There is an abundance of new applications and questions of bimonoidal categories in mathematics and other sciences. This work provides the first unified treatment of bimonoidal and higher ring-like categories, their connection with algebraic K-theory and homotopy theory, and applications to quantum groups and topological quantum computation. With ample background material, extensive coverage, detailed presentation of both well-known and new theorems, and a list of open questions, this work is a user-friendly resource for beginners and experts alike.

Bimonoidal and E_n-Monoidal Categories

A *bimonoidal category* C is a categorical analogue of a rig, which is a ring without additive inverses. In this categorification, the addition, multiplication, 0, and 1 of a rig are replaced by functors and objects in a bimonoidal category. Rig axioms are replaced by natural structure morphisms, along with suitable coherence axioms of their own.

More specifically, in place of the rig addition and multiplication, C has two monoidal structures

$$\big(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big) \quad \text{and} \quad \big(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}\big).$$

The first is symmetric monoidal, and called the *additive structure*. The second is plain monoidal, and called the *multiplicative structure*. As with plain monoidal categories, there are variants with braided or symmetric multiplicative structure, and a variety of intermediate multiplicative structures parametrized by E_n-operads.

In place of distributivity relations in a rig, a bimonoidal category has natural *distributivity monomorphisms* for objects A, B, and C:

$$A \otimes (B \oplus C) \xrightarrow{\delta^l_{A,B,C}} (A \otimes B) \oplus (A \otimes C)$$
$$(A \oplus B) \otimes C \xrightarrow{\delta^r_{A,B,C}} (A \otimes C) \oplus (B \otimes C).$$

These data are required to satisfy a finite list of axioms that (i) are checkable in practice and (ii) ensure that (symmetric/braided) bimonoidal categories have good coherence and other categorical properties. An important special case is a *tight* bimonoidal category, in which the distributivity monomorphisms δ^l and δ^r are *isomorphisms*.

A number of examples, arising in both algebraic and homotopical contexts, are discussed throughout the text. Here, we summarize three important ones. More examples are discussed in the next section about quantum science.

(1) The category of finite dimensional complex vector spaces, $\mathsf{Vect}^{\mathbb{C}}$, is a tight symmetric bimonoidal category with its additive and multiplicative structures given by the usual direct sum and tensor product of vector spaces. More generally, each distributive symmetric monoidal category is a tight symmetric bimonoidal category.
(2) The nonnegative integers and permutations form the objects and the morphisms of a tight symmetric bimonoidal category Σ, called the finite ordinal category.
(3) May's bipermutative categories, with the additional axiom $\xi^{\otimes}_{-,\mathbb{0}} = \mathrm{Id}$, are tight symmetric bimonoidal categories.

The definition and coherence theorems for symmetric bimonoidal categories are due to Laplaza [**Lap72a, Lap72b**]. These theorems and their plain bimonoidal analogues are discussed in detail in Part I.1. In addition to providing completely detailed proofs, we also correct some subtle and nontrivial inaccuracies in the original statements and proofs. See Sections I.3.11 and I.4.7 for related discussion. Just as applications of monoidal categories heavily depend on Mac Lane's coherence theorem, Laplaza's two coherence theorems for symmetric bimonoidal categories, as well as their plain and braided analogues, are crucial to their applications.

Part I.2 applies Laplaza's coherence theorems to prove a number of theorems about bimonoidal categories in the context of 2-dimensional categories. These include existence of a bi-initial object, confirming a conjecture of Baez [**Bae18**], and a symmetric monoidal bicategory of matrices, $\mathsf{Mat}^{\mathsf{C}}$, constructed from a tight symmetric bimonoidal category C. (Note the unfortunately subtle confluence of terminology that "symmetric monoidal *bicategory*" and "symmetric *bimonoidal* category" refer to wildly distinct algebraic structures.) In the case $\mathsf{C} = \mathsf{Vect}^{\mathbb{C}}$, $\mathsf{Mat}^{\mathsf{C}}$ is the symmetric monoidal bicategory of *coordinatized 2-vector spaces*, one version of the 2-vector spaces introduced by Kapranov and Voevodsky in [**KV94**].

Braided bimonoidal categories, along with their corresponding coherence and strictification theorems, are discussed in Part II.1. These structures are of interest for applications in quantum science, discussed below. The relevant coherence results are new, confirming a conjecture of Blass and Gurevich [**BG20a**].

Part II.2 introduces a similar but distinct categorification of rigs, called *E_n-monoidal categories*. These have *factorization morphisms* in place of distributivity monomorphisms, and are significant for the K-theoretic applications in Part 2. The E_n-monoidal structure is a generalization of n-fold monoidal structure due to Balteanu, Fiedorowicz, Schwänzel, and Vogt [**BFSV03**]. Special cases of E_n-monoidal categories include, or are closely related to, the *bipermutative categories* of May [**May77**], the *ring categories* of Elmendorf and Mandell [**EM06**], and the *braided bimonoidal categories* of Richter [**Ric10**].

Applications in Quantum Science

Due to the ubiquity of ring-like structures and categories, bimonoidal categories are increasingly applied in a variety of disciplines in mathematics and other formal sciences related to quantum algebra. To support readers with a variety of interests, this work includes summary and introduction to several such applications.

Quantum Groups. The first part of Chapter II.3 extends a well-known fact in quantum group theory. We observe that the category of left modules over a braided bialgebra, which is also known as a quasitriangular bialgebra in the literature, is a tight braided bimonoidal category.

Topological Quantum Computation. The second part of Chapter II.3 discusses applications of braided bimonoidal categories to topological quantum computation (TQC). We prove that the Fibonacci anyons and the Ising anyons, which are two of the most important models in TQC, are both tight braided bimonoidal categories.

Centers. Monoidal, braided monoidal, and symmetric monoidal categories are connected by the Drinfeld center and the symmetric center. Kassel [**Kas95**] and Majid [**Maj91**] explain the relationship between the Drinfeld center construction, due to Drinfeld in unpublished work, and modules over the Drinfeld double of a a finite dimensional Hopf algebra. (See Note II.1.7.2 for further explanation and context.) Bimonoidal and ring categorical analogues of these center constructions are discussed in Chapters II.4 and II.9.

Reversible Programming. Section I.2.6 is a brief illustration that symmetric bimonoidal categories naturally arise in reversible programming of finite types. We observe that there is a symmetric bimonoidal groupoid whose objects are syntax of finite types. Note I.2.7.5 directs the reader to further applications in the sheet diagrams of [**CDH∞**] and the work of [**Hin13**] on quantum circuits.

Applications in Algebraic K-Theory

Uses of additive and multiplicative categorical structure in homotopy theory are among the earliest and most well-developed applications of the material from Volumes I and II of this work. Volume III focuses on those applications, beginning with the work of Segal [**Seg74**] that assembles structured ring spectra from permutative categories. Under the Segal K-theory functor K^{Se}, the symmetric monoidal structure of a permutative category C results in the additive structure in the cohomology theory represented by the spectrum $\mathsf{K}^{\mathsf{Se}}(\mathsf{C})$.

The computational importance of multiplicative structure in cohomology motivates significant interest in multiplicative structure for the representing spectra, leading to the highly-structured spectra that are presaged in work of Adams [**Ada95**] and realized in the S-modules of Elmendorf-Kriz-Mandell-May [**EKMM97**], the symmetric spectra of Hovey-Shipley-Smith [**HSS00**], and the orthogonal spectra of Mandell-May-Schwede-Shipley [**MMSS01**], among other equivalent models. Thus, there is corresponding interest in bimonoidal structures for the input categories.

One difficulty, however, is that Segal K-theory does *not* preserve the multiplicative aspect of such structures. As a resolution, the work of Elmendorf and Mandell [**EM06, EM09**] introduces an alternative construction that (a) is suitably equivalent to Segal's construction and (b) preserves multiplicative structure. This is known as Elmendorf-Mandell K-theory, K^{EM}.

The formalism in which these statements can be made precise is that of multicategories, where multilinear functors between permutative categories encode the relevant multiplicative structure. In these terms, the essential difference between the Segal and Elmendorf-Mandell constructions is that the latter is *multifunctorial*, while the former is merely functorial.

Part 1 develops the necessary supporting theory of enriched monoidal categories and multicategories, but is also of independent interest. Part 2 contains the applications to algebraic K-theory, including a review of the relevant background in homotopy theory and detailed treatments of both the Segal and Elmendorf-Mandell constructions.

The second half of Part 2 applies the Elmendorf-Mandell K-theory multifunctor K^{EM} to the E_n-monoidal categories in Part II.2 to produce structured ring spectra. We prove in detail that K^{EM} sends

- small ring categories to strict ring symmetric spectra,
- small bipermutative categories to E_∞-symmetric spectra,
- small braided ring categories to E_2-symmetric spectra, and
- small E_n-monoidal categories to E_n-symmetric spectra for $2 \leq n < \infty$.

The strict ring and E_∞ cases are from [**EM06, EM09**]. The $1 < n < \infty$ cases are new results.

Audience and Features

This work is aimed at graduate students and advanced researchers with an interest in category theory, homotopy theory, algebraic K-theory, and their applications. Below are some features that make this work a unique and user-friendly resource.

Unified Presentation: The literature on bimonoidal categories, higher ring-like categories, enriched monoidal categories, multicategories, and their connection with algebraic K-theory, homotopy theory, and the sciences is scattered across many journal articles over several decades, with varying definitions, notations, and terminology. This work presents these topics in a unified manner, with both well-known and new theorems.

Background Material: To make this work self-contained and to bring the reader quickly up to speed, there is extensive background material on

- basic category theory (Chapter I.1),
- 2-dimensional categories (Chapter I.6),
- braided structures (Chapter II.1),
- abelian categories (Section II.2.3),
- braided, also known as quasitriangular, bialgebras (Section II.3.1),
- enriched monoidal categories (Chapters 1, 2, and 3),
- pointed objects and pointed diagram categories (Chapter 4),
- enriched multicategories (Chapters 5 and 6), and
- homotopy theory (Chapter 7).

These chapters and sections form a substantial portion of this work.

Open Questions: Appendix A discusses open questions related to the topics of this work. The reader is encouraged to take advantage of these open questions and use them as a springboard to read the main text.

Detailed Discussion: This work contains many highly detailed and carefully structured proofs for both known and new theorems. For each major result, our discussion has much more detail than one would normally find in the literature. Our detailed discussion has several purposes.

Exercises with Solutions. Our detailed presentation makes the material accessible to a diverse audience, including those who are new to bimonoidal and higher ring-like categories and algebraic K-theory. Students are encouraged to regard the numerous detailed proofs as exercises with full solutions. Each result whose proof has many different parts has been carefully structured to make it easy for the reader to jump forward and backward.

Axioms. Symmetric bimonoidal categories are defined by 24 axioms, and the list of axioms for (braided) bimonoidal categories is similarly substantial. Our detailed discussion helps the reader see exactly where these axioms are used and why they are needed.

Laplaza's Theorems. The Coherence Theorems I.3.9.1 and I.4.4.3 for symmetric bimonoidal categories are central results in this subject that have been cited and used numerous times in the literature. Their original proofs given by Laplaza in [**Lap72a, Lap72b**] were written in outline form, with much detail and some cases in the proofs completely omitted. Moreover, Laplaza's original proofs and statements of these theorems have several subtle and nontrivial inaccuracies that have never been made explicit before and are not easy to spot. For both archival and educational purposes, we present fully detailed proofs of these theorems and correct the inaccuracies. Sections I.3.11 and I.4.7 have more related discussion.

K-Theory Multifunctors. The K-theory multifunctors in Chapters 9 and 10, due to Elmendorf-Mandell [**EM06, EM09**], are fundamental constructions for multiplicative structure of algebraic K-theory spectra. They are essential for our development of E_n-monoidal symmetric spectra from corresponding structure on small permutative categories. We use the theory of enriched monoidal categories and enriched multicategories from Part 1 to give complete explanations of the constructions and their properties. This treatment corrects an inaccuracy in the statement of [**EM09**, Theorem 1.3] and some other statements about expanding the domain of the K-theory multifunctor. The basic issue has to do with monoidal units and, to the authors' knowledge, has not been previously explained. See Note 10.8.2 for further discussion.

Reading Guides: In addition to a detailed introduction, almost every chapter has a brief *Reading Guide* that provides an alternative to reading that chapter linearly. Our presentation in the main text follows a straightly logical order and has a lot of detail. By following the reading guide, it is possible to first obtain a bird's-eye view of that chapter before digesting all the detail. The end of this Preface also includes several thematic reading guides for salient topics that span multiple chapters.

Motivation and Explanation: Main definitions and results are often preceded by discussion that motivates the upcoming definitions and proofs. Whenever useful, definitions and results are followed by a detailed explanation that interprets and unpacks the various components. In the text, these are clearly marked as *Motivation* and *Explanation*, respectively. Examples include Motivation I.2.1.1, Explanation I.2.4.7, and Section I.4.1.

Organization: There are extensive cross-references throughout the text. In addition to a detailed index, there are lists of main facts and notations, each organized by chapters. While the text follows a strictly logical order, it is not necessary to read the chapters in a linear order. The reader can jump straight to a section and use the extensive cross-references to fill in the necessary definitions and facts.

Part and Chapter Summaries

Part I.1: Symmetric Bimonoidal Categories

This part studies symmetric bimonoidal categories and bimonoidal categories (Chapter I.2). It presents highly detailed proofs of Laplaza's Coherence Theorems for symmetric bimonoidal categories (Chapters I.3 and I.4), May's Strictification Theorem for tight symmetric bimonoidal categories (Chapter I.5), and their non-symmetric analogues for bimonoidal categories. The only prerequisite for this part is some basic knowledge of category theory, which is summarized in Chapter I.1.

Part I.2: Bicategorical Aspects of Symmetric Bimonoidal Categories

Applying Laplaza's Coherence Theorems, this part proves several new theorems on the connection between symmetric bimonoidal categories and bicategories. All the necessary definitions of 2-dimensional category theory are summarized in Chapter I.6. The first main result is a confirmation of Baez's Conjecture (Chapter I.7) that proves the existence of a bi-initial object in a 2-category of symmetric bimonoidal categories. Chapter I.8 proves that a matrix construction $\mathsf{Mat}^{\mathsf{c}}$ sends each tight symmetric bimonoidal category to a symmetric monoidal bicategory.

Part II.1: Braided Bimonoidal Categories

Starting with a preliminary chapter on the braid groups and braided monoidal categories, this part is a detailed study of braided bimonoidal categories (Chapter II.2), which are strictly more general than Richter's [**Ric10**] and the BD categories of Blass-Gurevich [**BG20a**]. This part discusses applications to quantum groups and topological quantum computation (Chapter II.3), bimonoidal centers (Chapter II.4), coherence and strictification of braided bimonoidal categories (Chapters II.5 and II.6), and the braided versions of Baez's Conjecture and the matrix construction (Chapters II.7 and II.8). Our coherence and strictification theorems confirm the Blass-Gurevich Conjecture. The main theorems in Parts I.1 and I.2 are used in this part.

Part II.2: E_n-Monoidal Categories

This part studies a closely related variant of bimonoidal categories, called ring categories, and their bipermutative, braided, and higher analogues, called E_n-monoidal categories. Ring and bipermutative categories are due to Elmendorf-Mandell [**EM06, EM09**]. An E_n-monoidal category combines n ring categories with a common additive structure and an n-fold monoidal category as in [**BFSV03**]. The categories in this part are applied in Part 2 to obtain E_n-symmetric spectra via algebraic K-theory. This part is independent of the earlier parts, except for some definitions and statements of theorems.

Part 1: Enriched Monoidal Categories and Multicategories

To prepare for Part 2, this part lays the groundwork on enriched monoidal categories (Chapters 1, 2, and 3), smash products (Chapter 4), and multicategories (Chapters 5 and 6). In addition to their roles in the Segal K-theory functor and the Elmendorf-Mandell K-theory multifunctor, the detailed discussion of enriched monoidal categories—including change of enrichment, coherence, self-enrichment, and the Enriched Yoneda Lemma—and multicategories is also of independent interest. These chapters assume only a basic knowledge of monoidal categories, as summarized in Section 1.1.

Part 2: Algebraic K-Theory

This part studies the interconnection between E_n-monoidal categories (Part II.2), homotopy theory (Chapter 7), and algebraic K-theory. The first half discusses in detail the Segal K-theory functor (Chapter 8) and the Elmendorf-Mandell K-theory multifunctor (Chapters 9 and 10) from small permutative categories to symmetric spectra. The second half (Chapters 11, 12, and 13) applies the K-theory multifunctor to small ring, bipermutative, braided ring, and E_n-monoidal categories to obtain, respectively, strict ring, E_∞-, E_2-, and E_n-symmetric spectra. These structured ring spectra are fundamental objects in homotopy theory. Our discussion shows how they arise from E_n-monoidal categories via algebraic K-theory.

In the main text, each chapter starts with a detailed introduction. A summary of each chapter follows.

Part I.1: Symmetric Bimonoidal Categories

Chapter I.1: Basic Category Theory

To make this book self-contained, this chapter reviews the basics of category theory, starting from the definitions of categories, functors, and natural transformations. Then it discusses adjunctions, equivalences of categories, (co)limits, (co)ends, and Kan extensions. The remaining sections review (symmetric) monoidal categories, (symmetric) monoidal functors, monoidal natural transformations, and their coherence theorems.

Chapter I.2: Symmetric Bimonoidal Categories

This chapter introduces symmetric bimonoidal categories and bimonoidal categories. Then we prove Laplaza's Theorem I.2.2.13 that says that half of the 24 symmetric bimonoidal category axioms are formal consequences of the other 12 axioms. The weaker bimonoidal analogue is Proposition I.2.2.14. The remaining sections discuss examples of symmetric bimonoidal categories, including distributive symmetric monoidal categories, the finite ordinal category Σ, a variant Σ', and left and right bipermutative categories. The finite ordinal category Σ is an important part of (i) the distortion category $\mathcal{D}$ (Chapter I.4) used in Laplaza's Second Coherence Theorem I.4.4.3, (ii) Baez's Conjecture (Chapter I.7), and (iii) the braided version of Baez's Conjecture (Chapter II.7). Section I.2.6 contains an application of symmetric bimonoidal categories to reversible programming of finite types.

Chapter I.3: Coherence of Symmetric Bimonoidal Categories

This chapter proves Laplaza's First Coherence Theorem I.3.9.1 for symmetric bimonoidal categories that satisfy a monomorphism assumption. This assumption is automatically satisfied if tightness—that is, the invertibility of the distributivity morphisms δ^l and δ^r—is assumed, but the general form of this theorem only requires that the distributivity morphisms be natural monomorphisms. The analogue of this coherence theorem for bimonoidal categories is Theorem I.3.10.7. Section I.3.11 discusses the main differences between this chapter and Laplaza's original work in [**Lap72a**].

Chapter I.4: Coherence of Symmetric Bimonoidal Categories II

This chapter proves Laplaza's Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories that satisfy the same monomorphism assumption as in Theorem I.3.9.1. The analogue of this coherence theorem for bimonoidal categories is Theorem I.4.5.8. Section I.4.7 discusses the main differences between this chapter and Laplaza's original work in [**Lap72b**]. Both Coherence Theorems I.3.9.1 and I.4.4.3 say that some formal diagrams in certain symmetric bimonoidal categories commute. The first theorem has an assumption called *regularity* on the common domain of the two paths involved, which is analogous to Mac Lane's Coherence Theorem I.1.3.3 for monoidal categories. The second theorem has an assumption about the two paths themselves, which is reminiscent of the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories. In Chapter II.5, we observe that the second, but not the first, theorem has a braided analogue.

Chapter I.5: Strictification of Tight Symmetric Bimonoidal Categories

This chapter proves May's Strictification Theorem I.5.4.6 of *tight* symmetric bimonoidal categories to right bipermutative categories. The latter are tight symmetric bimonoidal categories whose additive structures and multiplicative structures are both permutative categories, and whose structure morphisms $\lambda^\bullet$, $\rho^\bullet$, δ^r, and $\xi^{\otimes}_{-,\mathbb{0}}$ are identities. Unlike the Coherence Theorems I.3.9.1 and I.4.4.3, the strictification theorem requires the tightness assumption. Our detailed proofs show exactly where the invertibility of δ^l and δ^r is used. Theorem I.5.4.7 is another version of the strictification theorem involving *left* bipermutative categories, in which δ^l, instead of δ^r, is the identity. Theorems I.5.5.11 and I.5.5.12 are the corresponding strictification results for tight bimonoidal categories. Section I.5.6 briefly discusses the history of related strictification theorems and claims. The proofs in this chapter are repurposed in Chapter II.6 to prove the strictification form of the Blass-Gurevich conjecture for braided bimonoidal categories.

Part I.2: Bicategorical Aspects of Symmetric Bimonoidal Categories

Chapter I.6: Definitions from Bicategory Theory

Without assuming any knowledge of 2-dimensional categories, in this chapter we review the basics of 2-/bicategories, pasting diagrams, lax functors, lax transformations, modifications, and adjunctions in bicategories. Then it reviews multiplicative structures, including monoidal bicategories, their braided, sylleptic, and symmetric analogues, the Gray tensor product for 2-categories, (permutative) Gray monoids, and permutative 2-categories. Most of these topics are discussed in detail in the book [**JY21**].

Chapter I.7: Baez's Conjecture

This chapter proves Baez's Conjecture (Theorems I.7.8.1 and I.7.8.3). Section I.7.1 defines a 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ with *flat* small symmetric bimonoidal categories as objects. Flatness (Definition I.3.9.9) is much weaker than tightness, and it guarantees that the Coherence Theorems I.3.9.1 and I.4.4.3 are applicable. The first version of Baez's Conjecture (Theorem I.7.8.1) says that the finite ordinal category Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. Another version is Theorem I.7.8.3, which says that the variant Σ' of Σ is also such a lax bicolimit. We emphasize that our proof of Baez's Conjecture does *not* use the Strictification Theorems I.5.4.6 and I.5.4.7. This allows us to use flat small symmetric bimonoidal categories in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$, instead of the smaller class of tight ones. Section I.7.9 discusses the relationship between our version of Baez's Conjecture and the more restricted version in [**CDH∞, Elg21**] for rig categories, which are multiplicatively nonsymmetric and tight.

Chapter I.8: Symmetric Monoidal Bicategorification

This chapter proves Theorem I.8.15.4. It says that, for each tight symmetric bimonoidal category C, a matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory, with no strict structure morphisms in general. Therefore, the construction $\mathsf{Mat}^{\mathsf{C}}$ is a direct connection between tight symmetric bimonoidal categories and symmetric monoidal bicategories. The objects in $\mathsf{Mat}^{\mathsf{C}}$ are nonnegative integers. Its 1-/2-cells are matrices whose entries are objects/morphisms in C. The horizontal composition in the bicategory $\mathsf{Mat}^{\mathsf{C}}$ uses the usual matrix product. The monoidal composition in its monoidal bicategory structure uses the matrix tensor product, which is also known as the Kronecker product. The category of coordinatized 2-vector spaces, which is $\mathsf{Mat}^{\mathsf{C}}$ with $\mathsf{C} = \mathsf{Vect}^{\mathbb{C}}$, is such a symmetric monoidal bicategory. This chapter uses the Coherence Theorems I.3.9.1 and I.3.10.7 and the graph theoretic machinery in Chapter I.3, but neither the Coherence Theorem I.4.4.3 nor the Strictification Theorems I.5.4.6 and I.5.4.7.

Part II.1: Braided Bimonoidal Categories

Chapter II.1: Preliminaries on Braided Structures

To prepare for the rest of Part II.1, this chapter discusses the braid groups and braided monoidal categories. First it defines the braid groups and proves some useful properties for sum braids and block braids. Then it reviews braided monoidal categories and proves some basic properties, including two manifestations of the third Reidemeister move. Next it proves in detail that the Drinfeld center of a monoidal category is a braided monoidal category and that the symmetric center of a braided monoidal category is a symmetric monoidal category. Then it recalls the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories.

Chapter II.2: Braided Bimonoidal Categories

This chapter defines braided bimonoidal categories. They are defined using 12 of the 24 Laplaza axioms of a symmetric bimonoidal category, together with two additional axioms that are variants of the only two Laplaza axioms involving the braiding $\xi^{\otimes}$. In a symmetric bimonoidal category, each of these two variant axioms is equivalent to the original Laplaza axiom. This is reminiscent of the fact that a braided monoidal category has two hexagon axioms, which are equivalent to each other in a symmetric monoidal category. A *tight* braided bimonoidal category—that is, one with invertible distributivity morphisms δ^l and δ^r—is equivalent to a BD category in the sense of Blass and Gurevich [**BG20a**]. The first main observation in this chapter is Theorem II.2.2.1, which says that each braided bimonoidal category satisfies all 24 Laplaza axioms. Therefore, a symmetric bimonoidal category is precisely a braided bimonoidal category whose braiding satisfies the symmetry axiom. The second main result in this chapter says that an abelian category with a compatible (symmetric/braided) monoidal structure is a tight (symmetric/braided) bimonoidal category. The additive structure comes from the abelian structure, and the multiplicative structure comes from the monoidal structure. The braided case of this result is due to Blass and Gurevich [**BG20a**].

Chapter II.3: Applications to Quantum Groups and Topological Quantum Computation

This chapter shows that braided bimonoidal categories arise naturally in quantum groups and topological quantum computation (TQC). The first main observation is Theorem II.3.2.19. It says that for a (symmetric/braided) bialgebra A, the category $\mathsf{Mod}(A)$ of left A-modules, equipped with the usual direct sum and tensor product, is a tight (symmetric/braided) bimonoidal category. This is an extension of the important fact in quantum group theory that, for a braided bialgebra A, $\mathsf{Mod}(A)$ is a braided monoidal category. Next we prove in detail that Fibonacci anyons and Ising anyons, which are two central models in TQC, are both tight braided bimonoidal categories. In each case, the additive structure comes from an abelian category structure, and the multiplicative structure comes from the fusion rules of anyons.

Chapter II.4: Bimonoidal Centers

This chapter generalizes the Drinfeld center of a monoidal category and the symmetric center of a braided monoidal category (Sections II.1.4 and II.1.5) to the bimonoidal setting. Generalizing the Drinfeld center, Theorem II.4.4.3 says that, for each tight bimonoidal category C, the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided bimonoidal category. Tightness is required for this theorem because the invertibility of δ^l and δ^r is used in the construction of $\overline{\mathsf{C}}^{\mathsf{bi}}$. The proof of this theorem is another good illustration of the axioms of a braided bimonoidal category, since we will use all 24 Laplaza axioms and the two variant axioms in the braided case. Generalizing the symmetric center, Theorem II.4.5.3 says that, for each braided bimonoidal category C, the bimonoidal symmetric center $\mathsf{C}^{\mathsf{sym}}$ is a symmetric bimonoidal category.

Chapter II.5: Coherence of Braided Bimonoidal Categories

This chapter proves the Coherence Theorem II.5.4.4 for braided bimonoidal categories that satisfy a monomorphism assumption. As in the symmetric case (Theorems I.3.9.1 and I.4.4.3), the monomorphism assumption in Theorem II.5.4.4 is automatically satisfied if tightness is assumed. This theorem is the braided analogue of Laplaza's Second Coherence Theorem I.4.4.3. It uses a braided version $\mathcal{D}^{\mathsf{br}}$ of the distortion category that involves the symmetric groups and the braid groups to keep track of, respectively, the additive symmetry $\xi^{\oplus}$ and the braiding $\xi^{\otimes}$. Reminiscent of the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories, Theorem II.5.4.4 says that, if two paths have the same image in the braided distortion category $\mathcal{D}^{\mathsf{br}}$, then they have the same value in the braided bimonoidal category in question. This condition of having the same image in $\mathcal{D}^{\mathsf{br}}$ is very much checkable in practice. In fact, the proofs of the main results in Chapters II.6, II.7, and II.8 all use Theorem II.5.4.4 and involve checking this condition many times. In [**BG20a**], Blass and Gurevich conjectured the existence of a coherence theorem for their BD categories, which are equivalent to our tight braided bimonoidal categories. Theorem II.5.4.4 confirms the Blass-Gurevich Conjecture in the form of commutative formal diagrams.

Chapter II.6: Strictification of Tight Braided Bimonoidal Categories

This chapter proves two Strictification Theorems II.6.3.6 and II.6.3.7 for tight braided bimonoidal categories. As in the symmetric case (Theorems I.5.4.6 and I.5.4.7), strictification requires the tightness assumption because the construction of the strictification uses the invertibility of the distributivity morphisms δ^l and δ^r. A *right permbraided category* is a tight braided bimonoidal category with both the additive and the multiplicative structures strict monoidal, and with identities for the structure morphisms $\lambda^\bullet$, $\rho^\bullet$, δ^r, $\xi^\otimes_{-,\mathbb{0}}$, and $\xi^\otimes_{\mathbb{0},-}$. Theorem II.6.3.6 says that each tight braided bimonoidal category is adjoint equivalent to a right permbraided category via strong braided bimonoidal functors. Theorem II.6.3.7 is the analogue that strictifies each tight braided bimonoidal category to a *left* permbraided category, in which δ^l, instead of δ^r, is the identity. Theorems II.6.3.6 and II.6.3.7 are two further positive answers to the Blass-Gurevich Conjecture [**BG20a**] in the form of strictification.

Chapter II.7: The Braided Baez Conjecture

This chapter proves the braided version of Baez's Conjecture. Section II.7.1 defines the 2-category $\mathsf{Bi}_\mathsf{r}^{\mathsf{fbr}}$ with *flat* small braided bimonoidal categories as objects. As in the symmetric case, flatness (Definition II.5.4.5) is much weaker than tightness, and it guarantees that the Braided Bimonoidal Coherence Theorem II.5.4.4 is applicable. The first version of the Braided Baez Conjecture (Theorem II.7.3.4) says that the finite ordinal category Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_\mathsf{r}^{\mathsf{fbr}}$. Another version is Theorem II.7.3.6, which says that the variant Σ' of Σ is also such a lax bicolimit. Also like the symmetric case, the proofs of the Braided Baez Conjecture do *not* use the Strictification Theorems II.6.3.6 and II.6.3.7. This allows us to use flat small braided bimonoidal categories in the 2-category $\mathsf{Bi}_\mathsf{r}^{\mathsf{fbr}}$, instead of the smaller class of tight ones. The reader may wonder why the finite ordinal category Σ and its variant Σ' are bi-initial in both the symmetric case (Theorems I.7.8.1 and I.7.8.3) and the braided case. This is analogous to the fact that the ring of integers is initial in both the category of rings and the category of commutative rings.

Chapter II.8: Monoidal Bicategorification

The main Theorem II.8.4.7 in this chapter says that, for each tight braided bimonoidal category C, the matrix construction Mat^c is a monoidal bicategory. While most of the definitions for Mat^c are the same as in the symmetric case in Chapter I.8, there are two subtleties. First, in the current braided case, the lax functoriality constraint $\boxtimes^2$ of the monoidal composition $\boxtimes$ in Mat^c has two additional conditions about the braided distortions of the two paths involved; see (II.8.2.15) and (II.8.2.16). These conditions about the braided distortions are necessary because a braid is not determined by its underlying permutation, and the braided distortion category $\mathcal{D}^{\mathsf{br}}$ involves the braid groups. The second subtle point is that, even if C is a tight braided bimonoidal category, the monoidal bicategory Mat^c does *not* seem to have any reasonable braided monoidal bicategory structure in general. We will explain this point in more detail near the end of Section II.8.4. The difficulty once again comes from the fact that the braided distortion category $\mathcal{D}^{\mathsf{br}}$ involves the braid groups, and a braid with an identity underlying permutation is usually not the identity braid.

Part II.2: E_n-Monoidal Categories

Chapter II.9: Ring, Bipermutative, and Braided Ring Categories

This chapter discusses ring and bipermutative categories in the sense of Elmendorf-Mandell and the braided version. The main difference between these categorical notions and their bimonoidal counterparts in Parts I.1 and II.1 is that ring categories have generally non-invertible *factorization morphisms*

$$(A \otimes C) \oplus (B \otimes C) \xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C$$
$$(A \otimes B) \oplus (A \otimes C) \xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C)$$

that go in the opposite direction as the distributivity morphisms δ^r and δ^l. Ring categories with invertible factorization morphisms are special cases of tight bimonoidal categories, so the latter's strictification theorems in Chapter I.5 also apply to such ring categories. The bipermutative and braided analogues are also true. Similar to the endomorphism rig of a commutative monoid, each small permutative category C yields an endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$. Similar to the reduction of Laplaza's axioms in symmetric bimonoidal categories in Section I.2.2 and the braided version in Theorem II.2.2.1, about half of the ring category axioms are redundant in a bipermutative category and a braided ring category. This is an extension of an observation in [**EM06**, Fig. 1]. Moreover, the Drinfeld center and the symmetric center have natural analogues for these ring-like categories. As we will discuss in Chapters 11 and 12, the Elmendorf-Mandell K-theory multifunctor sends small ring, braided ring, and bipermutative categories to, respectively, strict ring, E_2-, and E_∞-symmetric spectra. The strict ring and E_∞ cases are due to Elmendorf-Mandell [**EM06, EM09**], and the E_2 case is new.

Chapter II.10: Iterated and E_n-Monoidal Categories

Keeping in mind that the ring-like categories in Chapter II.9 correspond to E_n-symmetric spectra for $n \in \{1, 2, \infty\}$ via algebriac K-theory, this chapter discusses the categorical structure for the general E_n cases. An n-fold monoidal category in the sense of [**BFSV03**] has n monoidal structures $\otimes_1, \ldots, \otimes_n$ that are strictly associative and unital and interact via the exchange natural transformations

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$

for $1 \le i < j \le n$. Monoids in the monoidal category of small n-fold monoidal categories are precisely small $(n+1)$-fold monoidal categories. We introduce the notion of an *E_n-monoidal category* as a permutative category $(\mathsf{C}, \oplus)$ equipped with an n-fold monoidal structure $\{\otimes_i, \eta^{i,j}\}$ and factorization morphisms $\{\partial^{l,i}, \partial^{r,i}\}$ for each monoidal structure $\otimes_i$, such that (i) each $(\oplus, \otimes_i, \partial^{l,i}, \partial^{r,i})$ is a ring category and (ii) several axioms relating $\eta^{i,j}$, $\partial^{l,i}$, and $\partial^{r,i}$ hold. Ring categories are E_1-monoidal categories. Braided ring categories and bipermutative categories are special cases of, respectively, E_2- and E_n-monoidal categories for $n \ge 2$. Moreover, each small category generates a free E_n-monoidal category. In Chapter 13, we will show that the Elmendorf-Mandell K-theory of a small E_n-monoidal category is an E_n-symmetric spectrum for $n \ge 1$.

Part 1: Enriched Monoidal Categories and Multicategories

Chapter 1: Enriched Monoidal Categories

This chapter gives the basic definitions and properties for enriched monoidal categories, including plain, braided, and symmetric variants. Definition 1.4.25 describes 2-categories of each, with 1- and 2-cells given by appropriately monoidal enriched functors and natural transformations, respectively. For our applications to K-theory in Part 2, the enriching category V is symmetric monoidal closed.

However, our treatment in this chapter addresses the more general case that V is merely monoidal, with additional assumptions about braided or symmetric monoidal structure stated as necessary.

Section 1.5 discusses the important special case V = Cat, the category of small categories with its Cartesian product. Explanation 1.5.3 describes how the monoidal V-categories in this case are strict versions of monoidal bicategories. The braided and symmetric cases are similarly compared.

Chapter 2: Change of Enrichment

This chapter describes change of enriching category induced by a symmetric monoidal functor, showing that monoidal structures are preserved. Sections 2.1 through 2.4 give a thorough treatment of 2-functoriality results. As an application, Corollary 2.4.17 shows that taking underlying categories gives a 2-functor from small monoidal V-categories, V-functors, and V-natural transformations to ordinary monoidal categories, functors, and natural transformations. Similar statements hold for the braided and symmetric cases.

A partial reverse of Corollary 2.4.17 is given in Theorem 2.5.1. The theorem shows that, for given V-enriched data, various enriched monoidal axioms are satisfied if and only if the corresponding monoidal axioms for the underlying data are satisfied. This provides a mechanism to lift ordinary monoidal structures to enriched monoidal structures.

Sections 2.5 and 2.6 apply Theorem 2.5.1 to lift coherence and strictification results for ordinary monoidal, braided, and symmetric monoidal categories to their enriched counterparts. The Enriched Monoidal Coherence Theorem 2.5.6 and Enriched Epstein's Coherence Theorem 2.5.8 play a significant role in subsequent chapters.

Chapter 3: Self-Enrichment and Enriched Yoneda

This chapter restricts to the case that V is a symmetric monoidal closed category. Theorem 3.3.2 shows, via Theorem 2.5.1, that the canonical enrichment of V over itself is symmetric monoidal as a V-category. The next several sections develop the theory of V-enriched co/ends followed by the V-Yoneda Lemma (Theorem 3.6.9) and an equivalent form called the V-Yoneda Density Theorem 3.7.8. These are applied to develop the Day convolution and internal hom for enriched diagram categories (Theorem 3.7.22). The remainder of the chapter discusses additional theory of enriched diagram categories and tensor/cotensor structures that will be important for the development of enriched K-theory functors in Part 2.

Chapter 4: Pointed Objects, Smash Products, and Pointed Homs

This chapter gives the definitions and properties of smash products and pointed homs. These will be used throughout Part 2, and the smash product of pointed multicategories, developed in Chapter 5, will be particularly important.

Section 4.3 uses the Day convolution and internal hom to develop symmetric monoidal closed structure for pointed diagram categories. The results are summarized in Theorem 4.3.37. Applications of this material appear in Chapters 8, 9, and 10, where the Segal and Elmendorf-Mandell K-theory constructions are given via certain pointed diagram categories.

Chapter 5: Multicategories

This chapter gives relevant background on multicategories, multifunctors, and multinatural transformations. Theorem 5.5.14 shows that the category of small multicategories is complete and cocomplete. The Boardman-Vogt tensor product of multicategories, and the associated smash product for pointed multicategories, are developed in Section 5.6. The corresponding internal hom and its pointed variant are developed in Section 5.7.

Chapter 6: Enriched Multicategories

This chapter develops basic definitions and properties for enriched multicategories. One of our important applications, developed in Section 6.3, is the enriched multicategory associated to an enriched symmetric monoidal category. Our first use of this is in Section 6.4 where we describe the Cat-enriched multicategory structure on $\mathsf{Multicat}$, the category of small multicategories. It is induced by showing that the tensor product makes $\mathsf{Multicat}$ symmetric monoidal as a Cat-enriched category (Theorem 6.4.3). The pointed variant, with the smash product of small pointed multicategories, is given in Theorem 6.4.4 and will be essential for Part 2.

Sections 6.5 and 6.6 cover our second important application of enriched multicategories. The category $\mathsf{PermCat}^{\mathsf{su}}$, consisting of small permutative categories and strictly unital symmetric monoidal functors, has a Cat-enriched multicategory structure given by multilinear functors and multilinear transformations (Definitions 6.5.4 and 6.5.11). Propositions 6.5.10 and 6.5.13 show that this Cat-enriched multicategory structure is induced from that of small pointed multicategories and their smash product. Section 6.6 gives a second, direct proof of the Cat-enriched multicategory axioms.

Part 2: Algebraic K-Theory

Chapter 7: Homotopy Theory Background

This chapter gives relevant background from homotopy theory. Sections 7.1 and 7.2 introduce simplicial sets and simplicial homotopy, along with the nerve and geometric realization functors. The category of symmetric spectra, with its symmetric monoidal closed structure, is presented in Sections 7.3 through 7.6. Then, Sections 7.7 and 7.8 give a short review of Quillen model categories and a number of key examples.

Chapter 8: Segal K-Theory of Permutative Categories

This chapter presents the K-theory functor K^{Se} due to Segal [**Seg74**]. Its inputs are small permutative categories and its outputs are symmetric spectra. Section 8.3 describes the key construction as given by Segal. Sections 8.4 and 8.5 describe an equivalent construction that compares more readily with the K-theory multifunctor of Elmendorf-Mandell, K^{EM}.

Chapter 9: Categories of $\mathcal{G}_*$-Objects

This chapter is the first of two that replace the Segal K-theory functor with a simplicially-enriched multifunctor due to Elmendorf-Mandell [**EM06, EM09**]. This chapter focuses on the replacement of Γ-categories and Γ-simplicial sets with pointed diagrams out of a larger indexing category $\mathcal{G}$. The construction of symmetric spectra from such diagram categories is given in Section 9.3 and is denoted $\mathsf{K}^{\mathcal{G}}$. Sections 9.2 and 9.4 use the material from Part 1 to explain that the new diagram categories and the new functor $\mathsf{K}^{\mathcal{G}}$ are symmetric monoidal, in the enriched sense of Chapter 1, over the category of pointed simplicial sets.

Chapter 10: Elmendorf-Mandell K-Theory of Permutative Categories

This chapter is the second of two that replace the Segal K-theory functor with a simplicially enriched multifunctor due to Elmendorf-Mandell [**EM06, EM09**]. This chapter focuses on the construction of $\mathcal{G}_*$-categories from small permutative categories, replacing Segal's construction of Γ-categories from the same. Additional material from Part 1 is used throughout the chapter to explain that the multi/categories and multi/functors are enriched either in the symmetric monoidal sense of Chapter 1 or in the multicategorical and multifunctorial sense of Chapter 6. Section 10.6 contains the proof that the Segal and Elmendorf-Mandell K-theory symmetric spectra associated to a small permutative category C are level equivalent (Theorem 10.6.10). Because K^{EM} is an enriched multifunctor, it preserves operad actions. We state this result as Theorem 10.3.33 and apply it in Chapters 11, 12, and 13.

Chapter 11: K-Theory of Ring and Bipermutative Categories

This is the first of three chapters that contain algebraic K-theory applications of the ring-like categories in Part II.2. The main K-theory results in this chapter, Corollaries 11.3.16 and 11.6.12, are from [**EM06, EM09**], and they are the E_1 and the E_∞ cases. These results state that the Elmendorf-Mandell K-theory multifunctor K^{EM} sends (i) small ring categories to strict ring symmetric spectra and (ii) small bipermutative categories to E_∞-symmetric spectra. They are obtained by combining the multifunctor K^{EM} and the fact that the associative operad and the Barratt-Eccles operad parametrize, respectively, ring and bipermutative category structures on small permutative categories. Since the associative operad has monoids as algebras and the Barratt-Eccles operad is an E_∞-operad, the K-theory results follow.

Chapter 12: K-Theory of Braided Ring Categories

This chapter contains the E_2 analogues of the results in Chapter 11. The first part of this chapter discusses the braid operad Br, which generalizes the Barratt-Eccles operad. This is a categorical E_2-operad (Theorem 12.2.4) whose algebras in Cat are small braided strict monoidal categories (Proposition 12.3.22). The main categorical input is Theorem 12.4.5, which says that Br parametrizes braided ring category structures, as in Chapter II.9, on small permutative categories. Applying the K-theory multifunctor K^{EM}, it follows that K^{EM} sends small braided ring categories to E_2-symmetric spectra (Corollary 12.5.3). The K-theory result, Corollary 12.5.3, and the main categorical input, Theorem 12.4.5, are new results.

Chapter 13: K-Theory of E_n-Monoidal Categories

This chapter contains the general E_n analogues for $n \geq 1$ of the categorical and K-theory results in Chapters 11 and 12. The first part of this chapter discusses the n-fold monoidal category operad Mon^n. This is a categorical E_n-operad (Theorem 13.2.1) whose algebras in Cat are small n-fold monoidal categories (Proposition 13.3.18) as in Chapter II.10. The main categorical input is Theorem 13.4.12, which says that Mon^n parametrizes E_n-monoidal category structures on small permutative categories. Applying the K-theory multifunctor K^{EM}, it follows that K^{EM} sends small E_n-monoidal categories to E_n-symmetric spectra for $n \geq 1$ (Corollary 13.5.2). As in Chapter 12, the K-theory result, Corollary 13.5.2, and the main categorical input, Theorem 13.4.12, are new results.

Appendix A: Open Questions

This chapter discusses open questions related to the topics of this work. We encourage the reader to read these open questions at any time and use them as additional motivation for the main text.

Reading Guides

Supplementing the chapter introductions and individual reading guides therein, the following guides describe themes that span multiple chapters. For especially broad topics, we include a selection of general references for background or further reading. The Notes section at the end of each chapter provides additional references relevant to the content of that chapter.

Category Theory. For a refresher of basic category theory, including braided and symmetric monoidal categories, read Chapters I.1 and II.1. For bicategories and 2-categories, read Chapter I.6. For abelian categories, read Section II.2.3. For enriched category theory, read Section 1.2 and Chapter 3. Bimonoidal categories are built upon monoidal categories. Thus, a thorough understanding of the definitions and coherence of monoidal categories is necessary to understand bimonoidal categories and their coherence.

References for basic category theory include [**Awo10, BK00, Gra18, Lei14, Rie16, Rom17, Sim11**]. References for more advanced category theory include [**Bor94a, Bor94b, Bor94c, ML98, Mit65, Sch72**]. References for Abelian categories include [**EGNO15, Fre03, Mit65**]. References for enriched categories include [**Bor94b, Cru09, For04, Kel05**]. References for ends and coends include [**Day70, DK69, Lor21**]. For further reference on 2-dimensional categories, we highly recommend [**JY21**].

Symmetric Bimonoidal Categories. To review the axioms of a symmetric bimonoidal category, read Sections I.2.1 and I.2.2. For Laplaza's Coherence Theorems for symmetric bimonoidal categories, read Theorems I.3.9.1 and I.4.4.3. For strictification theorems, read Theorems I.5.4.6, I.5.4.7, I.5.5.11, and I.5.5.12. For their bimonoidal analogues, read Theorems I.3.10.7 and I.4.5.8. For Baez's Conjecture, read Definition I.7.1.8 and Theorem I.7.8.1. The introductions of Chapters I.3, I.4, I.5, and I.7 have more detailed description and reading suggestions. This material on (symmetric) bimonoidal categories is used extensively in Volume II in the discussion of braided bimonoidal categories.

Braided Bimonoidal Categories. To review the axioms of a braided bimonoidal category, read Definition II.2.1.29 and Theorem II.2.2.1. For their coherence and strictification (i.e., the Blass-Gurevich Conjecture), read Theorems II.5.4.4, II.6.3.6, and II.6.3.7. For the braided version of Baez's Conjecture, read Definition II.7.1.5 and Theorem II.7.3.4. The introductions of these chapters have more detailed discussion and reading guides. This material on braided bimonoidal categories is not heavily used in Volume III in our discussion of algebraic K-theory. Instead, these braided structures provide a great illustration of the coherence theory in Volume I and have many applications in the sciences, some of which are discussed in this work.

Applications to Quantum Groups, TQC, and Programming. For applications to quantum group theory, read Sections II.3.1 and II.3.2. For applications to topological quantum computation, read Theorem II.2.4.22 and Sections II.3.3 through II.3.6. For applications to reversible programming, read Section I.2.6. Our treatment is self-contained and assumes no prior knowledge of these topics. These applications are not used in Volume III. Instead, they illustrate the much larger scientific context where categorical structures discussed in this text are applied.

Enriched Monoidal Categories. To review enriched monoidal categories, read Sections 1.3 through 1.5. For their coherence theory, read Sections 2.5 and 2.6. For change of enrichment, read Proposition 2.1.2, Theorems 2.2.7, 2.3.7, and 2.4.10, and Corollary 2.4.17. For symmetric monoidal closed structures in the pointed context, read Theorems 4.1.8, 4.2.3, 4.3.19, and 4.3.37. Much of this material is known to experts, but it is not easily accessible in the literature in a unified format. Our discussion is self-contained and highly detailed, so the reader can thoroughly learn these topics just from our text. This material is necessary to understand the intricate multicategorical properties of Elmendorf-Mandell K-theory discussed in later chapters.

Multicategories and Operads. To review enriched multicategories, read Section 6.1. The Boardman-Vogt tensor product of multicategories and its pointed variant are discussed in Sections 5.6 and 6.4. For the passage from enriched symmetric monoidal categories to enriched multicategories, read Sections 5.3 and 6.3. For the categorically-enriched multicategory of small permutative categories, read Sections 6.5 and 6.6. For the associative operad, the Barratt-Eccles operad, the braid operad, and the n-fold monoidal category operad, read Definitions 11.1.1, 11.4.10, 12.1.2, and 13.1.12. Elmendorf-Mandell K-theory is an enriched multifunctor, which can transport operadic algebras in small permutative categories to the same type of operadic algebras in symmetric spectra. This material is necessary to describe the enriched multifunctoriality of Elmendorf-Mandell K-theory and its applications to highly structured ring spectra. References for multicategories and operads include [**Fre17, MSS02, May72, Yau16**].

Ring, Bipermutative, Braided Ring, and E_n-Monoidal Categories. For the definitions of ring, bipermutative, braided ring, and E_n-monoidal categories, read Definitions II.9.1.2, II.9.3.2, II.9.5.1, II.10.1.1, and II.10.7.2. For their coherence theory, read Corollaries II.9.1.19, II.9.1.20, II.9.3.12, and II.9.3.13, Theorem II.10.6.8, and Question A.2.1. For their description in terms of operads, read Theorems 11.2.16, 11.5.5, 12.4.5, and 13.4.12. The fact that these monoidal categories with extra structures are algebras over operads is the precise reason why they are sent by Elmendorf-Mandell K-theory to structured ring spectra of the corresponding types.

Basic Homotopy Theory. For a brief review of simplicial objects and the nerve construction, read Sections 7.1 and 7.2. To review symmetric spectra, read Sections 7.3 and 7.4. For the smash product of symmetric spectra, read Sections 7.5 and 7.6. Model category theory is reviewed in Sections 7.7 and 7.8, but later chapters do not use model categories in any way. While we assume some basic homotopy theory, we do not assume any prior knowledge of symmetric spectra. Our discussion of symmetric spectra is very gentle and contains a lot of details that are not explicitly available elsewhere in the literature. The main point of both Segal K-theory and Elmendorf-Mandell K-theory is the construction of symmetric spectra from purely categorical data. A detailed understanding of symmetric spectra and their smash product is necessary to fully appreciate the K-theory constructions of Segal and Elmendorf-Mandell.

References for homotopy theory include [**BR20, May99, MP12, Mil20, Ric20, Rie14**]. References for simplicial homotopy theory include [**Cur71, GZ67, GJ09, May92**]. References for further background and a broader perspective on algebraic K-theory include [**Mil71, Qui73, Ros95, Wal85, Wei13**].

Segal K-Theory. To review the passage from Γ-simplicial sets to symmetric spectra, read Definitions 7.4.5, 8.1.8, and 8.2.5. For the passage from small permutative categories to Γ-categories, read Definitions 8.1.17, 8.3.1, 8.3.6, 8.3.9, and 8.3.12 and Proposition 8.4.8. To review Segal's K-theory construction, from small permutative categories to symmetric spectra via Γ-categories, read Definitions 8.4.10 and 8.5.1 and Theorem 8.5.2. Although Segal K-theory does not generally preserve multiplicative structures, it is a fundamental tool in this subject and the main motivation for Elmendorf-Mandell K-theory.

Elmendorf-Mandell K-Theory. To review the passage from $\mathcal{G}_*$-simplicial sets to symmetric spectra, read Sections 9.1 through 9.3. Its symmetric monoidality is discussed in Theorem 9.4.9. For the passage from small permutative categories to $\mathcal{G}_*$-categories, read Section 10.4. For Elmendorf-Mandell K-theory, from small permutative categories to symmetric spectra via $\mathcal{G}_*$-categories, read Lemma 10.2.14 and Corollary 10.3.24. For the comparison between Segal K-theory and Elmendorf-Mandell K-theory, read Theorem 10.6.10. For the passage from categorical data to highly structured ring spectra via Elmendorf-Mandell K-theory, read Theorem 10.3.33 and Corollaries 11.3.16, 12.5.3, and 13.5.2. This material is the main point of Volume III; all preceding chapters of Volume III are preparation for it.