

PREFACE TO SECOND EDITION

This book has been unavailable for some time and I am happy to follow the publisher's suggestion for a new edition.

While a related forthcoming book, “**Integral Geometry and Radon Transforms**” (here denoted [IGR]) deals with several examples of homogeneous spaces in duality with corresponding Radon transforms, the present work follows the direction of the first edition and concentrates on analysis on Riemannian symmetric spaces $X = G/K$. We develop further the theory of the Fourier transform and horocycle transform on X , also taking into account tools developed by Eguchi for the Schwartz space $\mathcal{S}(X)$. These transforms provide the principal methods for analysis on X , existence and uniqueness theorems for invariant differential equations on X , explicit solution formulas, as well as geometric properties of the solutions, for example the harmonic functions and the wave equation on X . On the space X there is a canonical hyperbolic system on X , introduced by Semenov-Tian-Shansky, which is multitemporal in the sense that the time variable has dimension equal to the rank of X . The solution has remarkable analogies to the classical wave equation on \mathbf{R}^n , summarized in a table in Chapter V, §5.

My intention has been to make the exposition easily accessible to readers with some modest background in Lie group theory which by now is rather widely known. To facilitate self-study and to indicate further developments each chapter concludes with a section “Exercises and Further Results”. Solutions and references are collected at the end of the book. The harder problems are starred. Occasionally results and proofs rely on material from my previous books “**Differential Geometry, Lie Groups and Symmetric Spaces**” abbreviated [DS] and “**Groups and Geometric Analysis**”, abbreviated [GGA].

Once again I wish to express my gratitude to my friends and collaborators, Adam Korányi, Gestur Ólafsson, François Rouvière and Henrik Schlichtkrull and especially to my long-term colleague David Vogan for significant help at specified spots in the text. Finally, I thank Brett Cooney and Jan Wetzel for their invaluable help in the production and the editor Dr. Edward Dunne for his interest in the work and his patient and accommodating cooperation.

I would also like to express my thanks for the following permissions of partial quotations:

- (i) To Academic Press concerning my papers [1970a], [1976], [1980a], [1992b], and [1992d] quoted at the end of the Preface to the first edition.
- (ii) To Elsevier concerning my paper [2005].
- (iii) To John Wiley and Sons concerning my paper [1998a] and my paper with Schlichtkrull [1999].

PREFACE

Among Riemannian manifolds the symmetric spaces in the sense of É. Cartan form an abundant supply of elegant examples whose structure is particularly enhanced by the rich theory of semisimple Lie groups. The simplest examples, the classical 2-sphere S^2 and the hyperbolic plane H^2 , play familiar roles in many fields in mathematics.

On these spaces, global analysis, particularly integration theory and partial differential operators, arises in a canonical fashion by the requirement of geometric invariance. On R^n these two subjects are related by the Fourier transform. Also harmonic analysis on *compact* symmetric spaces is well developed through the Peter-Weyl theory for compact groups and Cartan's refinement thereof. For the *noncompact* symmetric spaces, however, we are presented with a multitude of new and natural problems.

The present monograph is devoted to geometric analysis on noncompact Riemannian symmetric spaces X . (The Euclidean case and the compact case are also briefly investigated in Chapter III, §§7-9, and Chapter IV, §5, but from an unconventional point of view). A central object of study is the algebra $D(X)$ of invariant differential operators on the space. A simultaneous diagonalization of these operators is provided by a certain Fourier transform $f \rightarrow f^\sim$ on X which is the subject of Chapter III. Just as is the case with R^n the symmetric space X turns out to be self-dual under the mentioned Fourier transform; thus range questions like the intrinsic characterization of $(C_c^\infty(X))^\sim$ in analogy with the classical Paley-Wiener theorem in R^n become natural and their answers useful.

Chapters II and IV are devoted to the theory of the Radon transform on X , particularly inversion formulas and range questions. The space Ξ of horocycles in X offers many analogies to the space X itself and this gives rise to the study of conical functions and conical distributions on Ξ which are the analogs of the spherical functions on X . They have interesting connections with the representation theory of the isometry group G of X , discussed in Chapter II, §4, and in Chapter VI, §3, where the conical distributions furnish intertwining operators for the spherical principal series. In Corollary 3.9, Ch. VI, these intertwining operators are explicitly related to the above-mentioned Fourier transform on X .

While the Fourier transform theory in Chapter III gives rise to an explicit simultaneous diagonalization of the algebra $D(X)$, the Radon transform theory in Chapter II is considered within the framework of a general integral transform theory for *double fibrations* in the sense of Chapter I, §3. This viewpoint is extremely general: *two dual integral transforms arise whenever we are given two subgroups of a given group G* . In the introduction to Chapter I we stress this point by indicating five such examples

arising in this fashion from the single group $G = \mathbf{SU}(1, 1)$ of the conformal maps of the unit disk, namely the *X-ray transform*, the *horocycle transform*, the *Poisson integral*, the *Pompeiu problem*, *theta series*, and *cuspidal forms*. When range results are considered, this viewpoint of the Poisson integral as a Radon transform offers a very interesting analogy with the X-ray transform in \mathbf{R}^3 (Chapter I, §3, No. 5).

With the tools developed in Chapters I–IV we study in Chapter V some natural problems for the invariant differential operators on X , solvability questions, the structure of the joint eigenfunctions, with emphasis on the harmonic functions, as well as the solutions to the invariant wave equation on X . In Chapter VI we consider in some detail the representations of G which naturally arise from the joint eigenspaces of the operators in the algebra $\mathbf{D}(X)$ and the algebra $\mathbf{D}(\Xi)$.

The length of this book is a result of my wish to make the exposition easily accessible to readers with some modest background in semisimple Lie group theory. In particular, familiarity with representation theory is not needed. To facilitate self-study and to indicate further developments each chapter is concluded with a section “Exercises and Further Results”. Solutions and references are given towards the end of the book. The harder problems are starred. Occasionally, results and proofs rely on material from my earlier books, “*Differential Geometry, Lie groups, and Symmetric Spaces*” and “*Groups and Geometric Analysis*”. In the text these books are denoted by [DS] and [GGA].

Some of the material in this book has been the subject of courses at MIT over a number of years and feedback from participants has been most beneficial. I am particularly indebted to Men-chang Hu, who in his MIT thesis from 1973 determined the conical distributions for X of rank one. His work is outlined in Chapter II, §6, No. 5–6, following his thesis and in greater detail than in his article Hu [1975]. I am also deeply grateful to Adam Korányi for his advice and generous help with the material in Chapter V, §§3–4, as explained in the notes to that chapter. Similarly, I am grateful to Henrik Schlichtkrull for beneficial discussions and for his suggestions of Proposition 8.6 in Chapter III and Corollary 5.11 in Chapter V, indicated in the text. I have also profited in various ways from expert suggestions from my colleague David Vogan. I am grateful to the National Science Foundation for support during the writing of this book.

Many people have read at least parts of the manuscript and have furnished me with helpful comments and corrections; of these I mention Fulton Gonzalez, Jeremy Orloff, An Yang, Werner Hoffman, Andreas Juhl, François Rouvière, Sönke Seifert, and particularly Frank Richter. I thank them all. Finally, I thank Judy Romvos for her expert and conscientious \TeX -setting of the manuscript.

A good deal of the material in this monograph has been treated in earlier papers of mine. While subsequent consolidation has usually led to a rewriting of the proofs, texts of theorems as well as occasional proofs

have been preserved with minimal change. I thank Academic Press for permission to quote from the following journal publications of mine, listed in the bibliography: [1970a], [1976], [1980a], [1992b], [1992d], as well as the book [1962a].