The Hua equations imply (96). The converse question and a more explicit description of the kernel $\nu^{-1}(0)$ are of considerable interest. Compare Ch. I, Theorem 3.23.

The dual transform to (89) would be

$$\check{\varphi}(gP_\Gamma) = \int\limits_{P_\Gamma/L_o} \varphi(gqK_o)dq_{L_o} \qquad \varphi \in \mathcal{D}(G_o/K_o).$$

The left invariant measure on $L_oA_oN_o$ is just $d\ell_o da dn$ and up to a constant

$$\int\limits_X \varphi(x)dx = \int\limits_{A_oN_o} \varphi(an\cdot o)dadn.$$

Hence the dual transform just maps φ into the constant function

$$\check{\varphi}(gP_\Gamma) = \int\limits_X \varphi(x)dx.$$

Since

$$(\widehat{\nu}(D)\varphi)^\vee = \nu(D)\check{\varphi}$$

we see that if D is in the kernel of ν then $\widehat{\nu}(D)\mathcal{D}(G_o/K_o)$ is in the kernel of the dual transform $\varphi \longrightarrow \check{\varphi}$. Here one could inquire about analogs of and perhaps sharpening of Ch. I, Theorem 3.24. A result in this spirit is the following result which is included among the exercises in this chapter:

If $X = G/K$ is a symmetric space of rank one then

$$L_X(\mathcal{D}(X)) = \{f \in \mathcal{D}(X) : \int\limits_X f(x)dx = 0, \qquad \int\limits_X f(x)P(x,b)dx \equiv 0\},$$

P being the Poisson kernel on $X \times B$.

§5. THE WAVE EQUATION ON SYMMETRIC SPACES.

1. Introduction. Huygens' Principle.

It is a familiar phenomenon from daily life that waves propagate quite differently in 2 and 3 dimensions. When a pebble falls in water at a certain point P, circular ripples around P are formed. A given point Q near P will be hit by an initial ripple and later by residual waves.

In three dimensions the situation is quite different. A flash of light at a point P has an effect on the surface of a sphere around P after a certain time interval but then no more. There are no residual waves as those present on the water surface. The same is the case with sound waves; one has pure propagation without residual waves; thus music can exist in $\mathbf{R}^3$ but not in $\mathbf{R}^2$.

There is a mathematical explanation of these phenomena. The propagation of waves in $\mathbf{R}^n$ is governed by the wave equation

$$(1)\qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}, \qquad u(x,0) = f_o(x),\ u_t(x,0) = f_1(x).$$

Here $u_t = \partial u/\partial t$. The problem of solving (1) in terms of the initial data $f_o, f_1 \in \mathcal{E}(\mathbf{R}^n)$ is a special case of the so called Cauchy problem for hyperbolic equations. Equation (1) can be solved by various methods; here we restate the solution obtained in [GGA], Ch. II, Exercise F1 by use of the Ásgeirsson mean value theorem. The result has different forms depending on the parity of n.

Case I n odd. With $(M^r f)(x)$ denoting as usual the mean value of a function f on the sphere $S_r(x)$ we put

$$(2)\qquad (I_r f)(x) = \left(\frac{\partial}{\partial(r^2)}\right)^{\frac{n-3}{2}} \left(r^{n-2}(M^r f)(x)\right).$$

Then the solution to (1) is given by

$$(3)\qquad u(x,t) = c_n \left[\frac{\partial}{\partial t}(I_t f_o(x)) + (I_t f_1)(x)\right],$$

where c_n is the constant

$$c_n = \frac{\frac{1}{2}\Omega_n}{[\frac{1}{2}(n-3)]!\Omega_{n-1}}.$$

Case II n even. Here we put

$$(4)\qquad (J_r f)(x) = \left(\frac{\partial}{\partial(r^2)}\right)^{\frac{n-2}{2}} \left(r^{n-2}(M^r f)(x)\right).$$

Then the solution to (1) is given by

(5)
$$u(x,t) = C_n\left[\frac{\partial}{\partial t}\int_o^t r(t^2-r^2)^{-\frac{1}{2}}(J_r f_o)(x)dr + \int_o^t r(t^2-r^2)^{-\frac{1}{2}}(J_r f_1)(x)\right]$$

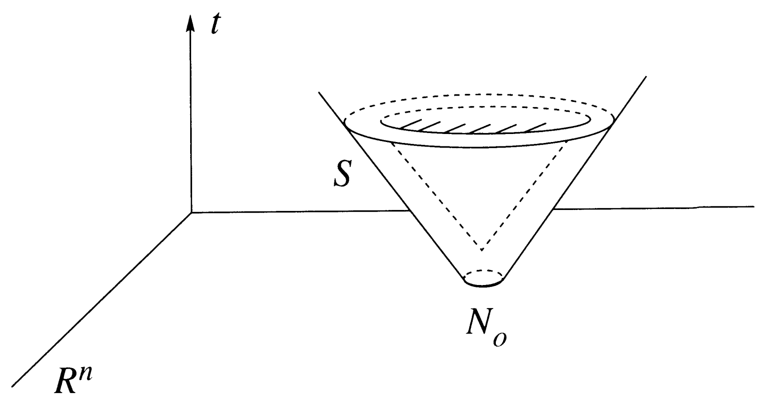
where C_n is the constant

$$C_n = \frac{1}{[\frac{1}{2}(n-2)]!}.$$

In Case II formula (5) shows that $u(x,t)$ is determined by the initial data f_o, f_1 in a ball $B_{t+\epsilon}(x)$ (for $\epsilon > 0$ arbitrarily small). In Case I a stronger statement holds:

(6) $u(x,t)$ is determined by the values of the initial data in an arbitrarily thin shell around $S_t(x)$.

To relate this better to light waves and sound waves suppose f_o and f_1 have support in a small ball $N_o = B_\epsilon(x_o)$ and consider the conical shell (see figure)



$S = \bigcup_{x\in N_o} C_x$ where C_x denotes the surface of the forward light cone with vertex x. Then Property (6) can also be stated in the equivalent form:

For all $\epsilon > 0$ the support of the function $(x,t) \longrightarrow u(x,t) \quad (t > 0)$ is contained in the conical shell

(6′) $S = \{(x,t) \| \, t - \epsilon < |x_o - x| < t + \epsilon\}$,

whenever f_o and f_1 have support in $B_\epsilon(x_o)$.

If (6′) holds then for a fixed $x \notin \overline{N}_o$ the function $t \longrightarrow u(x,t)$ is supported in a short time interval. This means pure propagation without residual effects.

For $n = 2$ formula (5) only implies $\mathrm{supp}(u) \subset \widetilde{S} = \bigcup_{x\in N_o} D_x \ (t > 0)$ where D_x is the solid forward light cone with vertex x. Thus if $x \notin \overline{N}_o$ is fixed, the function $t \longrightarrow u(x,t)$ will have support in an interval $t \geq |x - x_o| - \epsilon$ as illustrated by the water waves.

Property (6) is an instance of the so called *Huygens' principle* (Hadamard's "Minor Premise"), which for general Riemannian manifolds X can be formulated as follows. Let L_X denote the Laplace operator on X and consider the Cauchy problem,

$$(7) \qquad \frac{\partial^2 u}{\partial t^2} = (L_X + c)u, \qquad u(x,0) = f_o(x), \quad u_t(x,0) = f_1(x),$$

c being some constant. *Huygens' principle* is said to hold for (7) if the solution u satisfies (6). Hadamard raised the question of finding all X for which (7) satisfies Huygens' principle. He proved that $\dim X =$ odd is a necessary condition. More generally, the problem was raised with the operator $\partial^2/\partial t^2 - L_X$ replaced with the Laplacian on a Lorentzian manifold and the surface $t = 0$ by a spacelike hypersurface ([GGA], Ch. I, §6).

2. Huygens' Principle for Compact Groups and Symmetric Spaces $X = G/K$ (G complex).

To begin with let $X = G/K$ be an arbitrary symmetric space of the noncompact type and Exp:$\mathfrak{p} \longrightarrow X$ the usual exponential mapping. As in [GGA], Ch. II, §3 let J denote the function on $\mathfrak{p}$ defined by

$$(8) \qquad \int_{G/K} f(x)dx = \int_{\mathfrak{p}} f(\mathrm{Exp}X)J(X)dX$$

and put

$$(9) \qquad \Omega(\mathrm{Exp}Y) = J^{-\frac{1}{2}}(Y)L_{\mathfrak{p}}(J^{\frac{1}{2}})(Y).$$

Proposition 5.1. *Let E be a K-invariant distribution on $\mathfrak{p}$ and define $\mathcal{E}_E \in \mathcal{D}'(X)$ by*

$$(10) \qquad \mathcal{E}_E(f) = E(J^{\frac{1}{2}}(f \circ \mathrm{Exp})), \qquad f \in \mathcal{D}(X).$$

Then

$$(11) \qquad (L_X + \Omega)\mathcal{E}_E = \mathcal{E}_{L_{\mathfrak{p}}E}.$$

Proof. We know form [GGA], Ch. II, §3 that

$$(12) \qquad L_X^{\mathrm{Exp}^{-1}} F = (L_{\mathfrak{p}} + \text{ grad } (\log J))F$$

if F is a K-invariant function on $\mathfrak{p}$. Since

$$L_{\mathfrak{p}}(uv) = uL_{\mathfrak{p}}(v) + 2 \text{ grad } u(v) + vL_{\mathfrak{p}}(u)$$

(12) can be written,

$$(L_X+\Omega)(F^{\mathrm{Exp}}) = \left(J^{-\frac{1}{2}}L_{\mathfrak{p}}(J^{\frac{1}{2}}F)\right)^{\mathrm{Exp}}.$$

Hence

$$\begin{aligned}&((L_X+\Omega)(\mathcal{E}_E))\,((J^{-\frac{1}{2}}F)^{\mathrm{Exp}}) = \mathcal{E}_E\left\{(L_X+\Omega)\left((J^{-\frac{1}{2}}F)^{\mathrm{Exp}}\right)\right\}\\ &= \mathcal{E}_E\left\{\left(J^{-\frac{1}{2}}L_{\mathfrak{p}}(F)\right)^{\mathrm{Exp}}\right\}\\ &= E\left\{J^{\frac{1}{2}}(J^{-\frac{1}{2}}L_{\mathfrak{p}}(F))^{\mathrm{Exp}}\circ\mathrm{Exp}\right\} = E(L_{\mathfrak{p}}(F)) =\\ &(L_{\mathfrak{p}}E)(F) = (J^{\frac{1}{2}}(L_{\mathfrak{p}}E))(J^{-\frac{1}{2}}F) = (J^{\frac{1}{2}}L_{\mathfrak{p}}E)^{\mathrm{Exp}}(J^{-\frac{1}{2}}F)^{\mathrm{Exp}}.\end{aligned}$$

Since both $(L_X+\Omega)(\mathcal{E}_E)$ and $(J^{\frac{1}{2}}L_{\mathfrak{p}}E)^{\mathrm{Exp}}$ are K-invariant the calculation above implies that they coincide. But if $f\in\mathcal{D}(X)$ then

$$\left(J^{\frac{1}{2}}L_{\mathfrak{p}}E\right)^{\mathrm{Exp}}(f) = (L_{\mathfrak{p}}E)\left(J^{\frac{1}{2}}(f\circ\mathrm{Exp})\right) = \mathcal{E}_{L_{\mathfrak{p}}E}(f)$$

proving the proposition.

Now suppose G is complex. Then the K-radial part of $L_{\mathfrak{p}}$ ([GGA], Ch. II, §3) is given by

$$\Delta(L_{\mathfrak{p}}) = \pi^{-1}L_{\mathfrak{a}}\circ\pi, \tag{13}$$

where $\pi = \prod_{\alpha\in\Sigma^+}\alpha$. Also

$$J(H) = \prod_{\alpha\in\Sigma^+}(\sinh\alpha(H))^2\Big/\pi(H)^2. \tag{14}$$

Since

$$\prod_{\alpha\in\Sigma^+}2\sinh\alpha(H) = \sum_{s\in W}\epsilon(s)e^{s\rho(H)} \tag{15}$$

we thus deduce from (9), (13)–(14) that

$$\Omega(\exp H) = \langle\rho,\rho\rangle \qquad\qquad H\in\mathfrak{a}$$

so by K-invariance, $\Omega\equiv\langle\rho,\rho\rangle$. Because of Prop. 5.1 we thus take $c=\langle\rho,\rho\rangle$ in (7) and take the shifted wave equation

$$\frac{\partial^2}{\partial t^2} = \left(L_X+|\rho|^2\right)u,\qquad u(x,0)=f_o(x),\qquad u_t(x,0)=f_1(x) \tag{16}$$

as the natural one on X. (The so-called Klein-Gordon equation has opposite sign on the constant term).

Proposition 5.2. *Define the distribution E_t as follows:*

$$E_t(\varphi) = \begin{cases} c_n(I_t\varphi)(0), & n \text{ odd} \\ C_n \int\limits_o^t r(t^2-r^2)^{-\frac{1}{2}}(J_r\varphi)(0)dr, & n \text{ even} \end{cases} \tag{17}$$

where $\varphi \in \mathcal{D}(\mathfrak{p})$ and $n = \dim X$. Then the solution to

$$\frac{\partial^2 u}{\partial t^2} = L_{\mathfrak{p}} u, \qquad u(x,0) = f_o(x), \quad u_t(x,0) = f_1(x) \tag{18}$$

is given by

$$u(x,t) = (f_o * E_t')(x) + (f_1 * E_t)(x). \tag{19}$$

Here $*$ denotes convolution on $\mathfrak{p}$ and $'$ denotes d/dt.

Proof. This is just a restatement of (3) and (5).
In accordance with Prop. 5.1 we put

$$\mathcal{E}_t(f) = E_t(J^{\frac{1}{2}}(f \circ \operatorname{Exp})) \qquad f \in \mathcal{D}(X). \tag{20}$$

Theorem 5.3. *Let $X = G/K$ be a symmetric space of the noncompact type, G complex. The solution to the shifted wave-equation (16) with $f_o, f_1 \in \mathcal{E}(X)$ is given by*

$$u(x,t) = (f_o \times \mathcal{E}_t')(x) + (f_1 \times \mathcal{E}_t)(x), \tag{21}$$

where $\times$ denotes the convolution on X.
If $\dim X$ is odd then Huygens' principle holds for (16).

Proof. Apply $L_{\mathfrak{p}}$ as well as $\partial^2/\partial t^2$ to (19) and equate the results. Since f_o and f_1 are arbitrary, we deduce that

$$L_{\mathfrak{p}} E_t = \frac{\partial^2 E_t}{\partial t^2}, \tag{22}$$

Thus, by Prop. 5.1,

$$\frac{\partial^2 \mathcal{E}_t}{\partial t^2} = (L_X + |\rho|^2)\mathcal{E}_t. \tag{23}$$

Applying L_X to (21) we get by [GGA], Ch. II, Theorem 5.5,

$$(L_X)_x(u(x,t)) = (f_o \times L_X \mathcal{E}_t')(x) + (f_1 \times L_X \mathcal{E}_t)(x)$$

so by (23)

$$\left(L_X + |\rho|^2\right)_x (u(x,t)) = f_o \times \frac{\partial^2 \mathcal{E}_t'}{\partial t^2} + f_1 \times \frac{\partial^2 \mathcal{E}_t}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}.$$

Secondly

$$\lim_{t\to 0} \mathcal{E}_t'(f_o) = \lim_{t\to 0} E_t'\left(J^{\frac{1}{2}}(f_0 \circ \mathrm{Exp})\right) = f_0(o) \tag{24}$$

and since E_t' is K-invariant and invariant under the symmetry $Y \longrightarrow -Y$, the lift $\widetilde{\mathcal{E}}_t'$ of $\mathcal{E}_t'$ to G is invariant under the inversion $g \longrightarrow g^{-1}$. Hence

$$(f_0 \times \mathcal{E}_t')(g \cdot o) = \int f_0(g \cdot x) d\mathcal{E}_t'(x), \tag{25}$$

which by (24) has limit $f_0(g \cdot o)$ as $t \longrightarrow 0$. Similarly

$$f_0 \times \mathcal{E}_t'' \longrightarrow 0 \text{ and } f_1 \times \mathcal{E}_t \longrightarrow 0 \tag{26}$$

as $t \longrightarrow 0$. This proves that (21) does indeed satisfy the shifted wave equation.

Finally, if $\dim X$ is odd, E_t and E_t' have support contained in $S_t(0)$ and by (20), $\mathcal{E}_t$ and $\mathcal{E}_t'$ have support in $S_t(o)$. Since

$$u(g \cdot o, t) = \int_X f_0(g \cdot x) d\mathcal{E}_t'(x) + \int_X f_1(g \cdot x) d\mathcal{E}_t(x), \tag{27}$$

the value $u(g \cdot o, t)$ is determined by the values of f_0 and f_1 in an arbitrary shell around $g \cdot S_t(o) = S_t(g \cdot o)$. This concludes the proof.

Remark. The constant $\langle \rho, \rho \rangle$ in (16) is given by

$$\langle \rho, \rho \rangle = \frac{1}{12} \dim X. \tag{28}$$

To see this we extend $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}^{\mathbf{C}}$ of the complexification $\mathfrak{g}^{\mathbf{C}}$. Let $\Delta(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$ be the corresponding set of nonzero roots. We order the roots in such a way that if β is a positive root then the restriction $\bar{\beta} = \beta|\mathfrak{a}$ belongs to Σ^+ (cf. [DS], Ch. VI, Lemma 6.1). Let Δ_+ be the set of positive elements in $\Delta(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$ and let $\widetilde{\rho} = \frac{1}{2} \sum_{\beta \in \Delta_+} \beta$. Extending the Cartan involution θ of $\mathfrak{g}$ to a complex automorphism of $\mathfrak{g}^{\mathbf{C}}$ we know that θ leaves $\mathfrak{h}^{\mathbf{C}}$ invariant ([DS], Ch. VI, lemma 3.2) and by the compatibility of the orderings, $\beta \longrightarrow -\theta\beta$ is a permutation of Δ_+. Hence $\widetilde{\rho} = -\theta\widetilde{\rho}$ so $\widetilde{\rho}$ vanishes on $\mathfrak{h}^{\mathbf{C}} \cap \mathfrak{k}$ whence $\widetilde{\rho} = \rho$. The Killing forms B and $B^{\mathbf{C}}$ of $\mathfrak{g}^{\mathbf{C}}$ considered as a real (respectively complex) Lie algebra are related by $B = 2ReB^{\mathbf{C}}$. On the other hand, if G is simple we have by the "strange formula" of Freudenthal-deVries (cf. [GGA], p. 544) that

$$B^{\mathbf{C}}(\widetilde{\rho}, \widetilde{\rho}) = \frac{1}{24} \dim_{\mathbf{C}} G = \frac{1}{24} \dim X \tag{29}$$

so

$$B(\rho,\rho) = 2B^{\mathbf{C}}(\widetilde{\rho},\widetilde{\rho}) = \frac{1}{12}\dim X$$

if G is simple. In the general semisimple case, X is a product and both sides of this equation are additive. This proves (28) in general.

Next we shall prove an analog of Theorem 5.3 for compact semisimple Lie groups K. Let $R > 0$ be such that the mapping $\exp : \ \mathfrak{k} \longrightarrow K$ is a diffeomorphism of $B_R(0)$ onto $B_R(e)$. The group K is furnished with an invariant Riemannian metric given by the negative of the Killing form Q of $\mathfrak{k}$.

Again let J be the volume element ratio given by

$$\int_K f(k)dk = \int_{B_R(0)} f(\exp Y)J(Y)dY$$

and if E is a distribution on $B_R(0) \subset \mathfrak{k}$ invariant under $Ad(K)$ we put (as in (10))

$$\mathcal{E}_E(f) = E(J^{\frac{1}{2}}(f \circ \exp)) \qquad\qquad f \in \mathcal{D}(B_R(e)).$$

In analogy with (20) we put

$$\mathcal{E}_t(f) = E_t(J^{\frac{1}{2}}(f \circ \exp)) \qquad \text{for} \qquad f \in \mathcal{D}(B_R(e)), \tag{30}$$

where E_t is defined in (17) and $|t| < R$.

Theorem 5.4. *Let K be a simply connected compact semisimple Lie group with the bi-invariant metric given by the negative of the Killing form. Then the solution to the shifted wave equation on K*

$$\frac{\partial^2 u}{\partial t^2} = (L_K - \frac{\dim K}{24})u, \qquad u(k,0) = f_0(k), \quad u_t(k,0) = f_1(k) \tag{31}$$

where $f_0, f_1 \in \mathcal{D}(B_R(e))$ has the solution

$$u(k,t) = (f_o * \mathcal{E}'_t)(k) + (f_1 * \mathcal{E}_t)(k), \tag{32}$$

where $*$ denotes convolution on K.

If $\dim K$ is odd, then Huygens' principle holds for (31).

Proof. Let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal abelian subalgebra, $\mathfrak{t}^{\mathbf{C}} \subset \mathfrak{k}^{\mathbf{C}}$ the corresponding complexifications, $\Delta(\mathfrak{k}^{\mathbf{C}}, \mathfrak{t}^{\mathbf{C}})$ the corresponding set of roots and 2ρ the sum of the corresponding set of positive roots relative to a specific ordering. Under the action of K on itself by conjugation $T = \exp \mathfrak{t}$ is a transversal submanifold in the sense of [GGA], Ch. II, Lemma 3.3. As a consequence of *loc. cit.*, Prop. 3.12, the corresponding radial part of the Laplacian L_K is given by

$$\Delta(L_K) = \delta^{-\frac{1}{2}} L_T \circ \delta^{\frac{1}{2}} + |\rho|^2, \tag{33}$$

where

$$\delta^{\frac{1}{2}}(\exp H) = \sum_{s \in W} (\det s) e^{s\rho(H)} \qquad (H \in \mathfrak{t}), \tag{34}$$

W being the Weyl group $W(\mathfrak{k}, \mathfrak{t})$. On the other hand, under the adjoint action of K of $\mathfrak{k}$ with $\mathfrak{t}$ as transversal submanifold the radial part $\Delta(L_{\mathfrak{k}})$ given by Harish-Chandra's formula

$$\delta(L_{\mathfrak{k}}) = \pi^{-1} L_{\mathfrak{t}} \circ \pi, \tag{35}$$

where π is the product of the positive roots. On the positive Weyl chamber in $\mathfrak{t}$ J equals the ratio $(\delta \circ \exp)/\pi^2$. We can now prove the following result.

Lemma 5.5. *Let* $F \in C_c^{\infty}(B_R(0))$ *be invariant under* $\mathrm{Ad}(K)$. *Then*

$$L_K^{\exp^{-1}} F = \left(J^{-\frac{1}{2}} L_{\mathfrak{k}} \circ J^{\frac{1}{2}} \right) F + |\rho|^2 F\,. \tag{36}$$

Also, if $E \in \mathcal{D}'(B_R(0))$ *is* $\mathrm{Ad}(\mathfrak{k})$*-invariant, and*

$$\mathcal{E}_E(f) = E(J^{\frac{1}{2}}(f \circ \exp)) \qquad f \in \mathcal{D}(B_R(e)) \tag{37}$$

then

$$\left(L_K - |\rho|^2\right) \mathcal{E}_E = \mathcal{E}_{L_{\mathfrak{k}} E} \qquad \text{on } B_R(e)\,. \tag{38}$$

Proof. For (36) it suffices to verify that both sides agree on $\mathfrak{t}$. If $H \in \mathfrak{t}$ we obtain from (33) and (35) if $f = F^{\exp}$

$$\begin{aligned} &\left(L_K^{\exp^{-1}} F\right)(H) = (L_K f)(\exp H) \\ &= \delta^{-\frac{1}{2}}(\exp H) L_T(\delta^{\frac{1}{2}} f)(\exp H) + |\rho|^2 f(\exp H). \\ &\left(J^{-\frac{1}{2}} L_{\mathfrak{k}}(J^{\frac{1}{2}} F)\right)(H) = J^{-\frac{1}{2}}(H) \pi^{-1}(H) \left(L_{\mathfrak{t}}(\pi J^{\frac{1}{2}} F)\right)(H) \\ &= \delta^{-\frac{1}{2}}(\exp H) L_{\mathfrak{t}} \left((\delta^{\frac{1}{2}} \circ \exp)(f \circ \exp)\right)(H), \end{aligned}$$

so (36) follows since $L_{\mathfrak{t}}(g \circ \exp) = L_T(g) \circ \exp$ for $g \in \mathcal{D}(\mathfrak{t} \cap B_R(0))$.

For (38) let $f \in \mathcal{D}(B_R(e))$ be invariant under K-conjugation. Then using (36),

$$\begin{aligned} &\left((L_K - |\rho|^2)\mathcal{E}_E\right)(f) = E\left(J^{\frac{1}{2}}(L_K - |\rho|^2) f \circ \exp\right) \\ &= E\left(L_{\mathfrak{k}}(J^{\frac{1}{2}}(f \circ \exp))\right) = (\mathcal{E}_{L_{\mathfrak{k}} E})(f). \end{aligned}$$

Next we can apply (29) to the simple components of the complexification of K and deduce

$$|\rho|^2 = \frac{1}{24}\dim K.$$

To finish the proof of Theorem 5.4, we note that by (24) and (38)

$$\begin{aligned}\frac{\partial^2 \mathcal{E}_t(f)}{\partial t^2} &= \frac{\partial^2 E_t(J^{\frac{1}{2}}(f\circ \exp))}{\partial t^2} = (L_{\mathfrak{k}} E_t)(J^{\frac{1}{2}}(f\circ\exp)) \\ &= \mathcal{E}_{L_{\mathfrak{k}}E_t}(f) = \left((L_K - |\rho|^2)\mathcal{E}_t\right)(f).\end{aligned}$$

Thus $u(x,t)$ as defined by (32) is a solution to the shifted wave equation. The initial conditions in (31) are verified in the same way as in Theorem 5.3. Again $\mathcal{E}_t$ and $\mathcal{E}_t'$ are invariant under the inversion $k \longrightarrow k^{-1}$ so

$$u(k,t) = \int_K f_o(kh)d\mathcal{E}_t'(h) + \int_K f_1(kh)d\mathcal{E}_t(h).$$

If $\dim K$ is odd, $\mathcal{E}_t'$ and $\mathcal{E}_t$ are supported in $S_t(e)$ $(t < R)$. Thus if k is such that $kS_t(e) \subset B_R(e)$, $u(k,t)$ is determined by the restrictions of f_0 and f_1 to an arbitrarily thin shell around $S_t(e)$. With these restrictions on t and k, the Huygens' principle (6) is verified and Theorem 5.4 proved.

3. Huygens' Principle and Cartan Subgroups.

In this subsection we shall study the solution of the shifted wave equation

$$\text{(39)} \qquad \frac{\partial^2 u}{\partial t^2} = \left(L_X + |\rho|^2\right)u, \qquad u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x)$$

for arbitrary symmetric spaces $X = G/K$ of the noncompact type, particularly as regards Huygens' principle. While Theorem 5.3 resulted from the special structure of L_X for G complex here we shall use the Fourier transform on X. We shall temporarily assume $f_0, f_1 \in \mathcal{D}(X)$.

Since the function

$$e_{\lambda,b}(x) = e^{(i\lambda+\rho)(A(x,b))} \qquad x \in X$$

satisfies

$$L_X e_{\lambda,b} = -\left(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\right)e_{\lambda,b}$$

we see that if $f \in \mathcal{D}(X)$,

$$(L_X f)^{\sim}(\lambda,b) = -\left(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle\right)\widetilde{f}(\lambda,b).$$

Thus taking formally the Fourier transform of (39) we would get, assuming the Fourier transform

$$\text{(40)} \qquad \widetilde{u}(\lambda,b,t) = \int_X u(x,t)e^{(-i\lambda+\rho)(A(x,b))}dx$$

defined for u as well as for the derivatives of u,

$$\widetilde{u}_{tt}(\lambda, b, t) + \langle \lambda, \lambda \rangle \widetilde{u}(\lambda, b, t) = 0. \tag{41}$$

For $\lambda \in \mathfrak{a}^*$ this would give

$$\widetilde{u}(\lambda, b, t) = \widetilde{f}_0(\lambda, b) \cos |\lambda| t + \widetilde{f}_1(\lambda, b) \frac{\sin |\lambda| t}{|\lambda|}. \tag{42}$$

Lemma 5.6. *The function $\lambda \longrightarrow |\lambda|^{-1} \sin(|\lambda| t)$ on $\mathbf{R}^n$ is the Fourier transform*

$$\frac{\sin(|\lambda| t)}{|\lambda|} = \int_{\mathbf{R}^n} e^{-i(x,\lambda)} dT_t(x), \tag{43}$$

where $T_t \in \mathcal{E}'(\mathbf{R}^n)$ has support in the closed ball $|x| \leq t$. If n is odd and > 1 then $\operatorname{supp}(T_t) \subset S_t(0)$.

Proof. While this could be proved directly (see Remark below) the lemma is here essentially contained in Prop. 5.2. In fact, take $f_0 \equiv 0$ in (18) and take Fourier transform in the x variable. Then

$$\widetilde{u}(\lambda, t) = \widetilde{f}_1(\lambda) \frac{\sin |\lambda| t}{|\lambda|}$$

so comparing with (19) we get $E_t = T_t$ whence the lemma.

Remark. For a more direct proof of Lemma 5.6 consider the integral over $S_t(0) \subset \mathbf{R}^n$ as the distribution

$$\tau_t(\varphi) = \int_{S_t(0)} \varphi(x)\, dw_t(x) = t^{n-1} \int_{S_1(0)} \varphi(tw)\, dw\,, \quad dw = dw_1\,.$$

Then ([GGA], Introduction, Lemma 3.6) we have with $r = t|x|$,

$$\frac{1}{t} \int_{\mathbf{R}^n} e^{-i\langle x,u \rangle}\, d\tau_t(u) = t^{n-2} \int_{\mathbf{S}^{n-1}} e^{-i\langle x,w \rangle t}\, dw = ct^{n-2} \frac{J_{(n-2)/2}(t|x|)}{(t|x|)^{(n-2)/2}}$$

$$= c|x|^{2-n} r^{(n-2)/2} J_{(n-2)/2}(r)$$

with $c = (2\pi)^{n/2}$. If n is odd, say $n = 2m + 3$ then $(n-2)/2 = m + \frac{1}{2}$ and we have the classical formula

$$\left(\frac{d}{z\, dz} \right)^m (z^{m+\frac{1}{2}} J_{m+\frac{1}{2}}(z)) = z^{\frac{1}{2}} J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi} \right)^{1/2} \sin z$$

(which follows easily from the power series expansion for $J_k(z)$). With $r = t|x|$ we have

$$\frac{d}{t\,dt} = |x|^2 \frac{d}{r\,dr}\,.$$

Thus, putting $\sigma_t = \left(\frac{d}{t\,dt}\right)^m \left(\frac{1}{t}\tau_t\right)$ we obtain

$$\int\limits_{\mathbf{R}^n} e^{-i\langle x,u\rangle}\, d\sigma_t(u) = \text{const.}\frac{\sin t|x|}{|x|}\,.$$

This readily implies (43) for n odd and > 1, even with T_t supported in the *sphere* $S_t(0)$. To prove (43) for n replaced by $n-1$ we use the "method of descent" and simply restrict λ to the subspace $\mathbf{R}^{n-1} \times 0$. If T_t^* denotes the distribution

$$f \longrightarrow \int\limits_{\mathbf{R}^n} f(x_1, \ldots, x_{n-1}) dT_t(x), \qquad f \in \mathcal{E}(\mathbf{R}^{n-1}),$$

then $T_t^*(f) = 0$ for $\operatorname{supp}(f)$ disjoint from the *ball* $|x| \leq t$ in $\mathbf{R}^{n-1}$. But then if $\mu \in \mathbf{R}^{n-1}$,

$$\frac{\sin|\mu|t}{|\mu|} = \int\limits_{\mathbf{R}^{n-1}} e^{-i(x,\mu)} dT_t^*(x), \tag{44}$$

proving (43) in the even dimensional case too.

Because of Lemma 5.6 we deduce from Euclidean Fourier transform theory that the function

$$\lambda \longrightarrow |\lambda|^{-1} \sin(|\lambda| t)$$

on $\mathfrak{a}^*$ extends to a holomorphic function on $\mathfrak{a}^*_{\mathbf{C}}$ which is of exponential type and slow growth in the sense of (40) Ch. III, §5. As a result of Cor. 5.9 in Ch. III there exists a distribution $\mathcal{E}_t \in \mathcal{E}'(X)$ with support in $\overline{B_t(o)}$ such that for $\lambda \in \mathfrak{a}^*_{\mathbf{C}}, b \in B$,

$$\frac{\sin(|\lambda| t)}{|\lambda|} = \int\limits_X e^{(-i\lambda+\rho)(A(x,b))} d\mathcal{E}_t(x). \tag{45}$$

While we arived at the "propagator" $\mathcal{E}_t$ by assuming f_o and f_1 of compact support we can now solve (39) without this assumption.

Theorem 5.7. *With $f_0, f_1 \in \mathcal{E}(X)$ the function*

$$u(x,t) = (f_0 \times \mathcal{E}'_t)(x) + (f_1 \times \mathcal{E}_t)(x) \tag{46}$$

is a solution of the shifted wave equation (39).

If $\dim X$ *is odd and if* G *has all its Cartan subgroups conjugate, then Huygens' principle holds.*

Proof. We have

$$(L_X u)(x,t) = (f_o \times L_X \mathcal{E}_t')(x) + (f_1 \times L_X \mathcal{E}_t)(x)$$

and by (45), $\mathcal{E}_0 = 0$, $\mathcal{E}_0' = \delta$. For (39) it thus remains to prove

$$(L_X + |\rho|^2)\mathcal{E}_t = \frac{\partial^2}{\partial t^2}\mathcal{E}_t.$$

Both sides give the same result when applied to the function $e_{\lambda,b}$ (Prop. 3.14, Ch. II) so they must coincide.

For the last statement of the theorem we first rewrite (43) for A :

$$\frac{\sin|\lambda|t}{|\lambda|} = \int_A e^{-i\lambda(\log a)} dT_t(a), \tag{47}$$

where $\operatorname{supp}(T_t) \subset \overline{B_t(e)}$ and $\operatorname{supp}(T_t) \subset S_t(e)$ if $\dim A$ is odd and > 1.

Lemma 5.8. *In terms of the product decomposition* $\Xi = (K/M) \times A$ *we have for the Radon transform* $\widehat{\mathcal{E}_t}$,

$$\widehat{\mathcal{E}_t} = 1 \otimes e^{\rho} T_t.$$

Proof. Because of the uniqueness in (45), $\mathcal{E}_t$ is invariant under K. Integrating (45) over b we get

$$\frac{\sin|\lambda|t}{|\lambda|} = \int_X \varphi_{-\lambda}(x) d\mathcal{E}_t(x), \tag{48}$$

where $\varphi_{-\lambda}$ is the spherical function. If $g = kan$ relative to the decomposition $G = KAN$ we put as usual, $a = \exp H(g)$. Let $\alpha \in \mathcal{E}(K/M)$, $\beta(a) = e^{-(i\lambda+\rho)(\log a)}$ and $\alpha^\natural$ the integral of α over K/M. Then by the definition of $\widehat{\mathcal{E}_t}$ (Ch. I, §3, (12)) and the K-invariance we have with $(\alpha \otimes \beta)(kM, a) = \alpha(kM)\beta(a)$,

$$\widehat{\mathcal{E}_t}(\alpha \otimes \beta) = \alpha^\natural \widehat{\mathcal{E}_t}(1 \otimes \beta) = \alpha^\natural \mathcal{E}_t((1 \otimes \beta)^\vee),$$
$$(1 \otimes \beta)^\vee (g \cdot o) = \int_K (1 \otimes \beta)(gk \cdot \xi_o) dk = \int_K \beta(\exp H(gk)) dk = \varphi_{-\lambda}(g)$$

so by (47)–(48),

$$\widehat{\mathcal{E}_t}(\alpha \otimes \beta) = \alpha^\natural \mathcal{E}_t(\varphi_{-\lambda}) = \alpha^\natural T_t(e^{-i\lambda}) = (1 \otimes e^{\rho} T_t)(\alpha \otimes \beta).$$

Since the functions $\alpha\otimes\beta$ span a dense subspace of $\mathcal{E}(\Xi)$ the lemma follows.

Let as in Ch. IV, §1,

$$\beta_R = \{\xi \in \Xi :\ d(o,\xi) < R\}\,, \quad \sigma_R = \{\xi \in \Xi :\ d(o,\xi) = R\}\,.$$

Lemma 5.9. *Suppose G has all its Cartan subgroups conjugate. Then if $T \in \mathcal{E}'(X)$,*

$$\operatorname{supp}(\widehat{T}) \subset \sigma_R \Longrightarrow \operatorname{supp}(T) \subset S_R(o).$$

Proof. From Ch. II, Theorem 3.13 we have the inversion formula for the Radon transform which we write in the form

$$f = (L\widehat{f})^\vee \tag{49}$$

and under our assumption, L is a differential operator on Ξ.

Let $\epsilon > 0$ and suppose f satisfies $\operatorname{supp}(f) \subset B_{R-\epsilon}(o)$. Then $\operatorname{supp}(\widehat{f}) \subset \bar{\beta}_{R-\epsilon}$ and since L in (49) is a differential operator, $\operatorname{supp}(L\widehat{f}) \subset \bar{\beta}_{R-\epsilon}$. Hence

$$T(f) = T\left((L\widehat{f})^\vee\right) = \widehat{T}(L\widehat{f}) = 0$$

so $\operatorname{supp}(T) \cap B_{R-\epsilon}(o) = \emptyset$. Since ϵ is arbitrary, $\underline{\operatorname{supp}(T)} \cap B_R(o) = \emptyset$. On the other hand, by Ch. IV, Cor. 1.2, $\operatorname{supp}(T) \subset \overline{B_R(o)}$ so $\operatorname{supp}(T) \subset S_R(o)$ as desired.

The assumption on G implies that all restricted root subspaces are even-dimensional so since $\dim X$ is odd, $\dim A$ is odd. Assuming also $\dim A > 1$, T_t in (47) has support in $S_t(e)$ so by Lemma 5.8–5.9, $\operatorname{supp}(\mathcal{E}_t) \subset S_t(o)$. Since $\varphi_{-\lambda}(g\cdot o) = \varphi_\lambda(g^{-1}\cdot o)$ formula (48) implies that the lift of $\mathcal{E}_t$ to G is invariant under $g \longrightarrow g^{-1}$. Thus the solution formula (46) can be written

$$u(g\cdot o,t) = \int\limits_X f_o(g\cdot x)\,d\mathcal{E}'_t(x) + \int\limits_X f_1(g\cdot x)\,d\mathcal{E}_t(x)$$

and now Huygens' principle is obvious.

Finally, if X is of odd dimension and of rank one, X is an odd-dimensional hyperbolic space. In this case we have for the Euclidean Fourier transform

$$2\frac{\sin|\lambda|t}{|\lambda|} = \left(\Phi_{B_{t(e)}}\right)^\sim(\lambda),\ \ 2\cos|\lambda|t = \left(\Phi_{S_{t(e)}}\right)^\sim(\lambda),$$

Φ_E denoting the characteristic function of a subset $E \subset A$, so the argument above gives Huygens' principle for (39) in the case when $f_1 \equiv 0$. However, it holds also in the general case because of the solution formula

$$u(x,t) = c_n\left[\frac{\partial}{\partial t}(J_t f_o)(x) + (J_t f_1)(x)\right] \tag{50}$$

with c_n as in (3) and

$$(J_t f)(x) = \left(\frac{\partial}{\partial(2\operatorname{ch} t)}\right)^{\frac{n-3}{2}} \left(\operatorname{sh}^{n-2} t (M^t f)(x)\right). \tag{51}$$

Here $\mathbf{H}^n$ is taken with metric g with sectional curvature -1 (cf. [GGA], pp. 343, 577), and the shifted wave equation considered is

$$\frac{\partial^2 v}{\partial s^2} = \left(L + (\tfrac{n-1}{2})^2\right) v, \qquad v(x,0) = v_o(x),\ v_s(x,0) = v_1(x). \tag{52}$$

For completeness this should be related to (39). We know from [DS], p. 566 that $g = B/2(n-1)$, where B is the Killing form. From [GGA], p. 582, we have for the single restricted root α, $|\alpha|^2 = \frac{1}{2} m_\alpha^{-1} = (2(n-1))^{-1}$ so

$$|\rho|^2 = |\tfrac{1}{2}(n-1)\alpha|^2 = \tfrac{1}{8}(n-1).$$

Thus

$$L_X + |\rho|^2 = \frac{1}{2(n-1)} L + \frac{1}{8}(n-1) = \frac{1}{2(n-1)} \left(L + (\tfrac{n-1}{2})^2\right).$$

Thus the solutions to (39) and (52) correspond with

$$v(x,s) = u(x, (2(n-1))^{\frac{1}{2}} s).$$

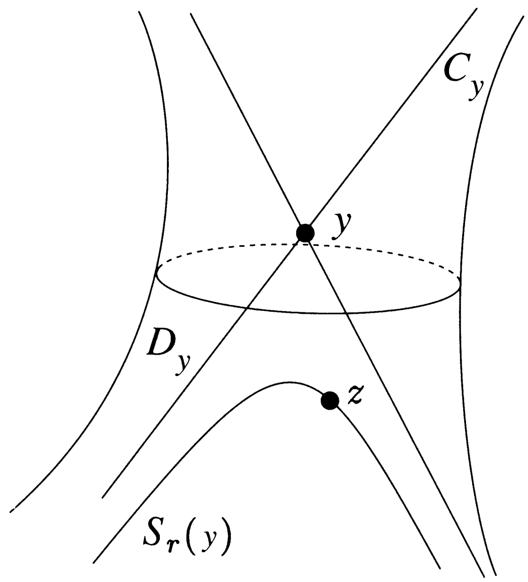
4. Orbital Integrals and Huygens' Principle.

Let X be an n-dimensional Lorentzian manifold, that is a pseudo-Riemannian manifold whose pseudo-Riemannian structure has signature $(1, n-1)$. We assume X has constant sectional curvature $\varkappa = 0, -1$ or $+1$ and therefore locally isometric to the respective spaces

$$\mathbf{O}_o(1,n-1)\mathbf{R}^n/\mathbf{O}_o(1,n-1),\ \mathbf{O}_o(1,n)/\mathbf{O}_o(1,n-1),\ \mathbf{O}_o(2,n-1)/\mathbf{O}_o(1,n-1), \tag{53}$$

the subscript o denoting identity component (cf. [GGA], Ch. I, §6).

Let $X = G/H$ be one of the spaces (53). For $y \in X$ let $\mathbf{C}_y \subset X$ be the *light cone* with vertex y and $\mathbf{D}_y \subset \mathbf{C}_y$ the *retrograde cone* with vertex y. Note that in [GGA], I, §6 $\mathbf{D}_y$ denotes the solid retrograde cone but here $\mathbf{D}_y$ denotes the boundary of the solid retrograde cone. On the figure we consider the case $\mathbf{O}_o(1,2)/\mathbf{O}_o(1,1)$ which is identified with the hyperboloid of one sheet in $\mathbf{R}^3$.



The group $G = \mathbf{O}_o(1,2)$ acts transitively on the hyperboloid. Here the light cone $\mathbf{C}_y$ consists of two lines through y and $\mathbf{D}_y$ of two half lines.

If $n > 2$ the isotropy subgroup of G at y acts transitively on the connected set $\mathbf{D}_y - \{y\}$. Let o denote the origin eH in X, $Y_1, \cdots, Y_n$ a basis of the tangent space $(G/H)_o$ in which the Lorentzian structure g_o has the form

$$g_o(Y,Y) = y_1^2 - \cdots - y_n^2 \quad \text{if} \quad Y = \Sigma y_i Y_i. \tag{54}$$

Then if Exp : $(G/H)_o \longrightarrow G/H$ is the usual exponential mapping the retrograde cone $\mathbf{D}_o$ is given by $\mathbf{D}_o = \text{Exp}(D_o)$ where D_o is the retrograde cone in $(G/H)_o$:

$$y_1^2 - y_2^2 - \cdots - y_n^2 = 0, \qquad y_1 \leq 0.$$

The component of the hyperboloid $g_o(Y,Y) = r^2$ which lies inside D_o is denoted $S_r(o)$; if $y \in X$ and if $g \in G$ is chosen such that $g \cdot o = y$ we put $S_r(y) = g \cdot S_r(o)$. This is a valid definition because $h \cdot D_o \subset D_o$ for $h \in H$ by the connectedness of H. Finally, we put

$$\mathbf{S}_r(y) = \text{Exp}\ S_r(y) \tag{55}$$

for $r > 0$; for the last case in (53) we limit r to the interval (o, π) in order for Exp to be injective (see [GGA], I, §6).

Let dh denote the bi-invariant measure on the unimodular group H. Let $y \in X, r > 0$ and select $g \in G$ such that $g \cdot o = y$ and let $x \in \mathbf{S}_r(o)$. We can then define the operator M^r (*the orbital integral*) by

$$(M^r u)(y) = \int_H u(gh \cdot x) dh, \qquad u \in C_c(X). \tag{56}$$

As shown *loc. cit.*, Lemma 6.11, the measure dh can be normalized such that

$$(M^r u)(y) = \frac{1}{A(r)} \int_{\mathbb{S}_r(y)} u(z) d\boldsymbol{w}_r(z) \tag{57}$$

where $d\boldsymbol{w}_r$ is the Riemannian measure on $\mathbf{S}_r(y)$ and

$$A(r) = r^{n-1}, \qquad (\sinh r)^{n-1}, \qquad (\sin r)^{n-1} \tag{58}$$

in the three cases in (53). We then have the following result (*loc. cit.* §6).

Theorem 5.10. *Let X be one of the Lorentzian manifolds (53) of constant curvature $\varkappa = 0, -1, +1$ and dimension $n > 2$.*

(i) *If $v \in C_c(X)$ then*

$$a = \lim_{r \to o} r^{n-2} (M^r v)(y) \text{ exists and is } \not\equiv 0. \tag{59}$$

(ii) $M^r(Lu) = L_r(M^r u) \qquad u \in \mathcal{D}(X)$ *where* $L_r = d^2/dr^2 + A'(r)/A(r) d/dr$ *is the radial part of the Laplacian L.*

(iii) *Suppose n is even, $n = 2m$, and put*

$$Q(L) = (L - \varkappa(n-3)2)(L - \varkappa(n-5)4) \cdots (L - \varkappa(1)(n-2)).$$

Then if $u \in \mathcal{D}(X)$,

$$u(y) = c \lim_{r \to o} r^{n-2} Q(L_r)((M^r u)(y)) \qquad (n > 2) \tag{60}$$

where the constant c is given by

$$c^{-1} = (4\pi)^{m-1}(m-2)!.$$

Also if f is such that $f \circ \text{Exp}$ has support in a ball $B_A: \ \Sigma y_i^2 < A^2 < 1$ in X_o then (*loc. cit.* §6, No. 4) we have

$$|r^{n-2}(M^r f)(o)| \leq CA \sup |f|, \tag{61}$$

where C is a constant independent of r. If φ is the "smoothed-out" characteristic function of B_A we write with $u \in \mathcal{D}(X)$, $u_1 = u\varphi$, $u_2 = u(1-\varphi)$

$$r^{n-2}(M^r u)(o) = r^{n-2}(M^r u_1)(o) + r^{n-2}(M^r u_2)(o). \tag{62}$$

Take $y = o$ in (59) and $v \in C_c(X - \{o\})$. The limit (59) is a positive, linear and H-invariant functional on $C_c(X - \{o\})$ with support in the H-orbit $\mathbf{D}_o - \{o\}$; hence it is "the" H-invariant measure on $\mathbf{D}_o - \{o\}$ and we denote it by μ. Then we have

$$\begin{aligned}&|r^{n-2}(M^r u)(o) - \int_{\mathbf{D}_o} u(z)d\mu(z)| \\ &\quad \leq |r^{n-2}(M^r u_1)(o) - \int_{\mathbf{D}_o} u_1(z)d\mu(z)| + |r^{n-2}(M^r u_2)(o) - \int_{\mathbf{D}_o} u_2(z)d\mu(z)|.\end{aligned}$$

By (61) the first term on the right is $O(A)$ uniformly in r and the second tends to 0 as $r \to 0$. Since A is arbitrary we conclude

$$\lim_{r\to o} r^{n-2}(M^r u)(o) = \int_{\mathbf{D}_o} u(z)d\mu(z). \tag{63}$$

Corollary 5.11. *Let $n > 2$ be even and δ the delta distribution at o. Then*

$$\delta = c\,Q(L)\mu \tag{64}$$

In fact, using (ii) and (63) equation (60) becomes

$$u(o) = c \lim_{r\to o} r^{n-2}(M^r(Q(L)u))(o) = c\int_{\mathbf{D}_o} (Q(L)u)(z)d\mu(z),$$

which is (64).

Note that by (57), formula (63) can also be written

$$\lim_{r\to o} \frac{1}{r}\boldsymbol{w}_r = \mu. \tag{63$'$}$$

Remark. Formula (64) shows that each factor

$$L_m = L - \varkappa(n-m)(m-1) \qquad (m = 3, 5, \cdots, n-1)$$

in $Q(L)$ has a fundamental solution supported on the retrograde cone $\mathbf{D}_o$. It is known that this is equivalent to Huygens' principle for the Cauchy problem for the equation $L_m u = 0$ with initial data given on some spacelike hypersurface (see Günther [1991] and [1988], Ch. IV, Cor. 1.13).

For the operator $L - \frac{\varkappa}{4}n(n-2)$ which is L_m for $m = 1 + 2\left[\frac{n}{4}\right]$ this is *a priori* clear by the conformal invariance of Huygens' principle (Ørsted [1981]). Our space X is in fact conformally equivalent to $\mathbf{R}^n$ so by the classical transformation formula for the Laplacian under conformal diffeomorphisms (see e.g. [GGA], p. 332) the null spaces of L_m and $L_{\mathbf{R}^n}$ correspond. Huygens' principle for $\mathcal{L}_{\mathbf{R}^n}$ explained in No. 1 of this section thus implies the same for L_m on X.

5. Energy Equipartion.

Let X be an oriented manifold with Riemannian structure $\langle , \rangle$ and let L_X denote the Laplace-Beltrami operator on X. For a vector field Y on X let ω_Y be the 1-form given by $\omega_Y(Z) = \langle Y, Z\rangle$ and extend $\langle , \rangle$ to 1-forms by $\langle \omega_Y, \omega_Z\rangle = \langle Y, Z\rangle$. The star operator $*$ on the Grassmann algebra $\mathfrak{A}(X)$ maps p-forms to $(n-p)$-forms (if $n = \dim X$), it operates pointwise, i.e. $*(f\omega) = f(*\omega)$ if $f \in \mathcal{E}(X)$ and it has the property that $*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \pm\omega_{j_i} \wedge \cdots \wedge \omega_{j_{n-p}}$ if (ω_i) is an orthonormal basis of 1-forms and $\{i_1, \ldots, i_p, j_1, \ldots j_{n-p}\}$ is a permutation of $\{1, \ldots, n\}$, the sign being $+$ or $-$ depending on whether the permutation is even or odd (cf. [DS], p. 142 and [GGA], p. 330). The operator δ is the linear operator on $\mathfrak{A}(X)$ given by

$$\delta\omega = (-1)^{np+n+1} * d * \omega, \qquad \omega \in \mathfrak{A}_p(X),$$

d denoting as usual the exterior derivative. Then we know ([GGA], Ch. II, Ex. A.1) that for $f \in \mathcal{E}(X)$, Y a vector field

$$\delta(\omega_Y) = -\text{ div } Y, \qquad \omega_{\text{ grad } f} = df \tag{65}$$

and on $\mathcal{E}(X)$, $L_X = -\delta d$. The familiar formula

$$\text{div }(fY) = f \text{ div } Y + Yf$$

thus becomes

$$\delta(f\omega) = f\delta\omega - \langle df, \omega\rangle, \qquad \omega \ 1-\text{form}, \ f \in \mathcal{E}(X). \tag{66}$$

Consider now the shifted wave equation (7) with real valued initial data f_0 and f_1. From general theory it is known that if f_0 and f_1 have compact support then so does the function $x \longrightarrow u(x,t)$. More precisely, if $\text{supp}(f_i) \subset B_R(x_o)$ then for each $t_o > 0$ the solution $x \longrightarrow u(x, t_o)$ has support in $B_{R+t_o}(x_o)$. Thus if $f_0, f_1 \in \mathcal{D}(X)$ we can consider the *energy*

$$(\mathcal{E}_c u)(t) = \int_X (E_c u)(x,t)dx, \tag{67}$$

where

$$(E_c u)(x,t) = \frac{1}{2}\left(u_t^2 + \langle du, du\rangle - cu^2\right). \tag{68}$$

Lemma 5.12. *(Conservation of energy).*

$$(\mathcal{E}_c u)(t) \text{ is independent of } t,$$

if u is a solution to (7).

Proof. From (68) we get

$$\begin{aligned}(E_c u)_t &= u_t u_{tt} + \langle du, du_t\rangle - cuu_t \\ &= u_t(u_{tt} - cu) + \langle du, du_t\rangle = u_t(-\delta du) + \langle du, du_t\rangle \\ &= -\delta(u_t du)\end{aligned}$$

by (66). Since by (65) this last expression is a divergence $\operatorname{div} Z$ and since $E_c u$ has compact support in the x-variable we have

$$\partial_t \int_X (E_c u)(x,t)dx = \int_X (E_c u)_t(x,t)dx = \int_X \operatorname{div} Z = 0,$$

proving the lemma.

We now specialize X to the spaces considered in Theorem 5.7 and take $c = |\rho|^2$ so our shifted wave equation is

$$\frac{\partial^2 u}{\partial t^2} = (L_X + |\rho|^2)u, \qquad u(x,0) = f_0(x), \quad u_t(x,0) = f_1(x) \tag{69}$$

with f_0, f_1, u real-valued and $f_0, f_1 \in \mathcal{D}(X)$. We split the energy $\mathcal{E} = \mathcal{E}_{|\rho|^2}$ in two parts, the *kinetic energy* and the *potential energy,*

$$(\mathcal{K}u)(t) = \tfrac{1}{2}\int_X u_t^2 dx, \tag{70}$$

$$\mathcal{P}u(t) = \tfrac{1}{2}\int_X (\langle du, du\rangle - |\rho|^2 u^2)dx. \tag{71}$$

Since

$$\langle du, du\rangle = u\,\delta du - \delta(u\,du)$$

and since by (65), $\int_X \delta(u\,du)dx = 0$ (71) can also be written

$$(\mathcal{P}u)(t) = -\frac{1}{2}\int_X u(L_X + |\rho|^2)u\ dx. \tag{72}$$

We can now state the energy equipartition theorem.

Theorem 5.13. *Let X be a symmetric space G/K of the noncompact type. We assume* $\dim X$ *odd and that all Cartan subgroups of* G *are conjugate. Then if* $\operatorname{supp} f_0$, $\operatorname{supp} f_1 \subset B_R(o)$ *we have*

$$\mathcal{K}u(t) = \mathcal{P}u(t) \qquad \text{for} |t| \geq R .$$

Proof. Again we use the Fourier transform on X which, by (42), gives for the solution,

$$\widetilde{u}(\lambda, b, t) = \Phi(\lambda, b) \cos pt + \Psi(\lambda, b) p^{-1} \sin pt, \tag{73}$$

where $\Phi = \widetilde{f_0}$, $\Psi = \widetilde{f_1}$, $p = |\lambda|$. Hence

$$\widetilde{u}_t(\lambda, b, t) = -p\,\Phi(\lambda, b) \sin pt + \Psi(\lambda, b) \cos pt$$

so by the Plancherel formula,

$$\begin{aligned} 2w(\mathcal{K}u)(t) = & \int\limits_{\mathfrak{a}^* \times B} \cos^2 pt |\Psi(\lambda, b)|^2 |\mathbf{c}(\lambda)|^{-2} d\lambda\; db \\ & + \int\limits_{\mathfrak{a}^* \times B} \sin^2 pt\; p^2 |\Phi(\lambda, b)|^2 |\mathbf{c}(\lambda)|^2 d\lambda\; db \\ & - \int\limits_{\mathfrak{a}^* \times B} \sin pt \cos pt\; p(\Phi\overline{\Psi} + \Psi\overline{\Phi})(\lambda, b) |\mathbf{c}(\lambda)|^{-2} d\lambda\; db. \end{aligned}$$

Hence, by the double-angle identities,

$$\begin{aligned} 2w(\mathcal{K}u)(t) = & \tfrac{1}{2}\|\Psi\|^2 + \tfrac{1}{2}\|p\Phi\|^2 \\ & + \tfrac{1}{2} \int\limits_{\mathfrak{a}^* \times B} \cos 2pt (|\Psi|^2 - |p\Phi|^2) |\mathbf{c}(\lambda)|^{-2} d\lambda\; db \\ & - \tfrac{1}{2} \int\limits_{\mathfrak{a}^* \times B} \sin 2pt\; p(\Phi\overline{\Psi} + \Psi\overline{\Phi}) |\mathbf{c}(\lambda)|^{-2} d\lambda\; db\,, \end{aligned} \tag{74}$$

the norms being taken in $L^2(\mathfrak{a}^* \times B, |\mathbf{c}(\lambda)|^{-2} db\; d\lambda)$. On the other hand, by (70), (72),

$$2(\mathcal{E}u)(t) = \int\limits_X u_t^2 dx - \int\limits_X u(L_X + |\rho|^2) u\; dx,$$

which by Lemma 5.12 and the Plancherel formula equals

$$2(\mathcal{E}u)(0) = (\|\Psi\|^2 + \|p\Phi\|^2) w^{-1}.$$

Since $\mathcal{E}u(t) = (\mathcal{K}u)(t) + (\mathcal{P}u)(t)$ it will suffice to prove that

$$\mathcal{I}(t) = 0 \qquad \text{for } |t| > R,$$

where $\mathcal{I}(t)$ is the sum of the two last terms in (74). Actually each of the terms will be shown to vanish for $|t| > R$. Now we have by Ch. III, (25) in §5,

$$\begin{aligned}\Phi(\lambda, kM) &= \int_X f_0(x) e^{(\rho - i\lambda)(A(x,kM))} dx \\ &= \int_A \widehat{f}_0(ka \cdot \xi_o) e^{(\rho - i\lambda)(\log a)} da, \quad \lambda \in \mathfrak{a}^*.\end{aligned}$$

Here we put $q = \lambda/p$ for the angular spherical coordinate in $\mathfrak{a}^*$ and write

$$\begin{aligned}\Phi(p; q, kM) = \Phi(\lambda, kM) &= \int_X f_0(x) e^{(\rho - ipq)(A(x,kM))} dx \\ &= \int_A \widehat{f}_0(ka \cdot \xi_o) e^{(\rho - ipq)(\log a)} da.\end{aligned}$$

For (q, kM) fixed this defines an extension of $p \longrightarrow \Phi(p; q, kM)$ from $\mathbf{R}^+$ to a holomorphic function $\Phi^*(p, q, kM)$ of a single complex variable $p = \sigma + i\tau$. The last formula shows, since $\widehat{f}_0(ka \cdot \xi_o) = 0$ for $|\log a| > R$ and $|q| \leq 1$, that $p \longrightarrow \Phi(p; q, kM)$ is an entire function on $\mathbf{C}$ of exponential type R, uniformly in (q, kM). The complex conjugate $\overline{\Phi}(\bullet, q, kM)$ has a similar extension to an entire function $p \longrightarrow \overline{\Phi}^*(p; q, kM)$ of exponential type R. Since f_0 is real-valued we actually have

$$\overline{\Phi}^*(p; q, b) = \Phi^*(p; -q, b).$$

Similar construction is carried out for Ψ. Recall also ([GGA], Ch. IV, Cor. 6.15) that under our assumption on G, $\mathbf{c}(\lambda)^{-1}$ is a polynomial $\gamma(\lambda)$ in $i\lambda$ with real coefficients. We put $\gamma(p; q) = \gamma(pq)$ for $p \in \mathbf{R}^+, |q| = 1$. Now

$$|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(\lambda)^{-1}\mathbf{c}(-\lambda)^{-1}, \qquad \lambda \in \mathfrak{a}^*$$

so for a fixed $q \in \mathfrak{a}^*$, $|q| = 1$, $(|\mathbf{c}(\lambda)|^{-2})|\mathfrak{a}^*$ has the holomorphic extension $\gamma(p; q)\gamma(p; -q)$ to $p \in \mathbf{C}$. We now switch to polar coordiantes (p, q) in the integrals in (74) so $d\lambda = p^{\ell-1} dp\; dq$ where dq is the (suitably normalized) measure on the unit sphere S_1 in $\mathfrak{a}^*$. Then

$$\begin{aligned}\mathcal{I}(t) &= \int_o^\infty \Big(p^{\ell-1} \cos 2pt \int_{S_1 \times B} Y_o(p, q, b) db\; dq\Big) dp \\ &\quad + \int_o^\infty \Big(p^{\ell-1} \sin 2pt \int_{S_1 \times B} Y_1(p, q, b) db\; dq\Big) dp,\end{aligned}$$

where

$$2Y_o(p;q,b) =$$
$$\left[\Psi^*(p;q,b)\Psi^*(p,-q,b) - p^2\Phi^*(p;q,b)\,\Phi^*(p;-q,b)\right]\gamma(p;q)\gamma(p;-q)$$
$$2Y_1(p;q,b) =$$
$$-\,p\left[\Phi^*(p;q,b)\Psi^*(p;-q,b) + \Phi^*(p;-q,b)\Psi^*(p;q,b)\right]\gamma(p;q)\gamma(p;-q).$$

These functions Y_0, Y_1 are for fixed (q,b) entire functions of p of exponential type $2R$. By the uniformity mentioned, the same is the case for the functions

$$Z_j(p) = p^{\ell-1}\int\limits_{S_1\times B} Y_j(p;q,b)db\;dq, \qquad (j=0,1)$$

and the expression for $\mathcal{I}(t)$ becomes

$$\mathcal{I}(t) = \int\limits_0^\infty Z_o(p)\cos 2pt\;dp + \int\limits_o^\infty Z_1(p)\sin 2pt dp.$$

On the other hand, $\Phi^*(-p;q,b) = \Phi^*(p;-q,b)$ and similarly for Ψ^*. The assumptions on X and G imply that $\dim A$ is odd so

$$Z_o(-p) = Z_0(p), \qquad Z_1(-p) = -Z_1(p),$$

whence

$$\mathcal{I}(t) = \frac{1}{2}\int\limits_{-\infty}^\infty (Z_0(p) + iZ_1(p))e^{-2ipt}dp.$$

Here Z_0 and Z_1 are entire functions of exponential type $2R$ so by the classical Paley-Wiener theorem $\mathcal{I}(t) = 0$ for $|2t| > 2R$. This proves Theorem 5.13.

6. The Flat Case Revisited.

We shall now write down the solution to the Cauchy Problem (1) in a different form. We first discuss the Radon transform version and will use the inversion formula from Ch. I, §1, (9).

Lemma 5.14. *If $h \in C^2(\mathbf{R})$ and $\omega \in \mathbf{S}^{n-1}$ then the function*

$$v(x,t) = h(\langle x,\omega\rangle + t) \tag{75}$$

satisfies $Lv = \partial^2/\partial t^2 v$.

The proof is obvious. It is now easy on the basis of the inversion formula and (75) to write down the solution to the Cauchy problem (74).

Theorem 5.15. *The solution to (1) with $f_0, f_1 \in \mathcal{D}(\mathbf{R}^n)$ is given by*

$$u(x,t) = \int_{\mathbf{S}^{n-1}} (Sf)(\omega, \langle x, \omega\rangle + t)\, d\omega\,, \tag{76}$$

where

$$Sf = \begin{cases} c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ odd}, > 1\,. \\ c[\mathcal{H}(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)]\,, & n \text{ even}\,. \end{cases} \tag{77}$$

Here $\partial = \partial/\partial p$, and the constant c equals

$$c = \tfrac{1}{2}(2\pi i)^{1-n}\,.$$

Proof. Because of Lemma 5.14 we just have to check the initial conditions in (1).

(i) If $n > 1$ is odd then $\omega \to (\partial^{n-1}\widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is an even function on $\mathbf{S}^{n-1}$ but $\omega \to (\partial^{n-2}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$ is odd. Thus $u(x, 0)) = f_0(x)$ by the inversion formula. Applying $\partial/\partial t$ to the right hand side of (76) and putting $t = 0$ gives $u_t(x, 0) = f_1(x)$, this time because the function $\omega \to (\partial^n \widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is odd and $\omega \to (\partial^{n-1}\widehat{f}_1)(\omega, \langle, x, \omega\rangle)$ is even.

(ii) n even. Here the proof is the same if one remarks that $c\mathcal{H}$ interchanges even and odd functions on $\mathbf{R}$.

Definition. For the initial data $f = \{f_0, f_1\}$ we shall refer to the function Sf in (77) on $\mathbf{S}^{n-1} \times \mathbf{R}$ as the *source*.

Definition. (Lax–Phillips) The wave $u(x,t)$ is

outgoing if $u(x,t) = 0$ in the *forward cone* $|x| < t$

incoming if $u(x,t) = 0$ in the *backward cone* $|x| < -t$.

Corollary 5.16. *The solution $u(x,t)$ to (1) is*

(i) *outgoing if and only if $(Sf)(\omega, s) = 0$ for $s > 0$.*

(ii) *incoming if and only if $(Sf)(\omega, s) = 0$ for $s < 0$.*

Proof. For (i) suppose $(Sf)(\omega, s) = 0$ for $s > 0$. Then if $|x| < t$ we have $\langle x, \omega\rangle + t > -|x| + t > 0$ so by (76) u is outgoing. Conversely, suppose

$u(x,t) = 0$ for $|x| < t$. Let $t_0 > 0$ be arbitrary and $\varphi \in \mathcal{D}(t_0, \infty)$. Then if $|x| < t_0$ we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}} u(x,t)\varphi(t)\,dt = \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, \langle x, \omega\rangle + t)\varphi(t)\,dt \\ &= \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, p)\varphi(p - \langle x, \omega\rangle)\,dp = 0\,. \end{aligned}$$

Taking $\partial^{|k|}/\partial x_{i_1}, \ldots \partial x_{i_k}$ at $x = 0$ we deduce

$$\int_{\mathbf{R}} \Big(\int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \ldots \omega_{i_k}\,d\omega \Big) (\partial^{|k|}\varphi)(p)\,dp = 0$$

for each k and each $\varphi \in \mathcal{D}(t_0, \infty)$. Integrating by parts in the **R**-integral we deduce that the function

$$p \to \int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \ldots \omega_{i_k}\,d\omega \tag{78}$$

has its kth derivative $\equiv 0$ for $p > t_0$. Having compact support if n is odd and being $O(\log |p|)$ if n is even we deduce that the function (78) vanishes identically for $p > t_0$. Now varying k we see that $(Sf)(\omega, p)$ vanishes for $p > t_0$. Since $t_0 > 0$ was arbitrary this proves (i). Part (ii) is proved in the same way.

Since f_0 and f_1 have compact support we can rephrase the corollary as follows:

For n odd the solution $u(x,t)$ is

(i) *outgoing if and only if* $(\partial \widehat{f}_0 + \widehat{f}_1)(w, s) = 0$ *for* $s > 0$.

(ii) *incoming if and only if* $(\partial \widehat{f}_0 + \widehat{f}_1)(w, s) = 0$ *for* $s < 0$.

The Cauchy Problem (1) can also be solved by using the Fourier transform

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x, \zeta\rangle}\,dx\,.$$

Assuming the function $x \to u(x,t)$ in $\mathcal{S}(\mathbf{R}^n)$ for a given t we obtain

$$\widetilde{u}_{tt}(\zeta, t) + (\zeta, \zeta)\widetilde{u}(\zeta, t) = 0\,.$$

Solving this ordinary differential equation with the intial data in (1) we get

$$\widetilde{u}(\zeta, t) = \widetilde{f}_0(\zeta)\cos(|\zeta| t) + \widetilde{f}_1(\zeta)\frac{\sin(|\zeta| t)}{|\zeta|}\,.$$

By Lemma 5.6 (and subsequent Remark) we deduce

$$u(x,t) = (f_0 * T_t')(x) + (f_1 * T_t)(x)\,, \qquad T_t' = \frac{d}{dt}T_t \tag{79}$$

where T_t has support in $B_{|t|}(0)$ and

$$\frac{\sin(|\zeta|t)}{|\zeta|} \int\limits_{\mathbf{R}^n} e^{-\langle \zeta, x\rangle}\, dT_t(x)\,, \quad \cos(|\zeta|t) = \int\limits_{\mathbf{R}^n} e^{-\langle \zeta, x\rangle}\, dT_t'\,. \tag{80}$$

Formula (79) gives a solution to (1) for any $f_0, f_1 \in \mathcal{E}(\mathbf{R}^n)$.
Since

$$LT_t = \frac{d^2}{dt^2}T_t\,, \quad LT_t' = \frac{d^2}{dt^2}T_t' \tag{81}$$

$$(T_t)_{t=0} = 0\,, (T_t')_{t=0} = \delta_0\,, \tag{82}$$

one refers to the pair $\{T_t, T_t'\}$ as the *fundamental solution* to the wave equation.

For n odd, (76) implies immediately the classical Huygens' principle in the form $(6')$.

In fact, for n odd (76) implies

$$u(0,t) = 0 \quad \text{for } |t| \geq R\,. \tag{83}$$

If $y \in \mathbf{R}^n$ the translated initial data f_0^{-y}, f_1^{-y} give the solution $x \to u(x+y,t)$. Thus by (83) since $\operatorname{supp}(f_i^{-y}) \subset B_{R+|y|}(0)$,

$$u(y,t) = 0 \quad \text{for } t \geq R + |y|\ .$$

On the other hand, since T_t above has support in $B_{|t|}(0)$, (79) implies that u has support in the region $|x| \leq |t| + R$.

For $n > 1$ odd one has also the following limit theorem of Friedlander.

Theorem 5.17. *For n odd the solution u to (1) satisfies*

$$\lim_{|t|\to\infty} t^{\frac{n-1}{2}} u_t((t+p)\omega, t) = (-\partial^{(n+1)/2}\widehat{f}_0 + \partial^{(n-1)/2}\widehat{f}_1)(\omega, p)\,.$$

For a proof see the book Lax-Phillips, [15, p.108] which also contains the proof above for Cor. 5.16 for n odd.

7. The Multitemporal Wave Equation On $X = G/K$

While the wave equation can be considered on any Riemannian manifold (see (7)), Semenov-Tjan-Shanski in [1976] introduced a kind of a

Cauchy problem on the symmetric space $X = G/K$ in which the time variable is multidimensional.

For convenience we review some of the notation from Chapter III whose results will be extensively used. We have the space $X = G/K$, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the corresponding Cartan involution θ, the Killing form B and we put $\langle Y, Z\rangle = B(Y, Z)$ for $Y, Z \in \mathfrak{p}$. We fix the maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, its dual $\mathfrak{a}^*$, $\sum = \sum(\mathfrak{g}, \mathfrak{a})$ the system of roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ and $\sum_0$ the set of indivisible roots.

The Weyl group W of $\sum$ acts on $\mathfrak{a}$ and $\mathfrak{a}^*$ by

$$(\sigma \cdot \lambda)(H) = \lambda(\sigma^{-1}H) = \lambda^\sigma(H) \quad \text{for} \quad \sigma \in W, \lambda \in \mathfrak{a}^*, H \in \mathfrak{a}$$

and this action extends to automorphisms of the symmetric algebras $S(\mathfrak{a})$ and $S(\mathfrak{a}^*)$.

For $\lambda \in \mathfrak{a}$, let $H_\lambda \in \mathfrak{a}$ be determined by $\langle H_\lambda, H\rangle = \lambda(H)$ for $H \in \mathfrak{a}$. For a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ let $\mathfrak{a}^*_+$ correspond to $\mathfrak{a}^+$ under $\lambda \to H_\lambda$. Let $\sum^+$ be the set of positive roots corresponding to $\mathfrak{a}^+$, let 2ρ be their sum (with multiplicity) and $\sum_0^+ = \sum^+ \cap \sum_0$.

Let $G = NAK$, $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ be the Iwasawa decompositions corresponding to our ordering. Let M be the centralizer of A in K and put $B = K/M$, $\overline{N} = \theta N$. For $H \in \mathfrak{a}$ let $\partial(H)$ be the corresponding directional derivative. The map $H \to \partial(H)$ extends to an isomorphism of $S(\mathfrak{a})$ (resp. $S^{\mathbf{C}} = \mathbf{C} \otimes S(\mathfrak{a})$) onto the algebra of all differential operators on $\mathfrak{a}$ with constant real (respectively complex) coefficients. The map $\lambda \to H_\lambda$ extends to an isomorphism of $S^{\mathbf{C}}(\mathfrak{a}^*)$ onto $S^{\mathbf{C}}(\mathfrak{a})$.

Let $I(\mathfrak{a})$ denote the algebra of W-invariants in $S(\mathfrak{a})$ and $I_+(\mathfrak{a})$ the set of $q \in I(\mathfrak{a})$ without constant term. A polynomial $p \in S(\mathfrak{a}^*)$ is said to be W-harmonic if $\partial(q)p = 0$ for all $q \in I_+(\mathfrak{a})$. Let $H(\mathfrak{a}^*)$ denote the space of W-harmonic polynomials and let $H(\mathfrak{a}) \subset S(\mathfrak{a})$ correspond to $H(\mathfrak{a}^*)$ under the extension of $\lambda \to H_\lambda$. It is well known that (see e.g. [GGA], Ch. III, §3)

$$S(\mathfrak{a}) = S(\mathfrak{a})\, I_+(\mathfrak{a}) \oplus H(\mathfrak{a}) \tag{84}$$

and

$$\dim H(\mathfrak{a}) = w. \tag{85}$$

For the adjoint action of K on $\mathfrak{p}$ we have the analogous notions $S(\mathfrak{p})$, $S(\mathfrak{p}^*)$, $I(\mathfrak{p})$, $I(\mathfrak{p}^*)$, $H(\mathfrak{p})$ and $H(\mathfrak{p}^*)$. The analog of (84) is also valid (loc. cit. III, §1).

As usual let $\pi \in S(\mathfrak{a})$ be given by $\pi = \prod_{\alpha \in \sum_0^+} H_\alpha$, and $\epsilon(\sigma)$ given by $\pi^\sigma = \epsilon(\sigma)\pi$, $\sigma \in W$.

Spectral Properties of $H(\mathfrak{a})$.

With $s_1, \ldots, s_w$ running through W and $p_1 = 1, p_2, \ldots, p_w$ being a homogeneous basis of $H(\mathfrak{a})$, we consider the $w \times w$ matrix

$E = (e_{ij})$ given by

$$e_{ij}(\lambda) = p_j(s_i\lambda) \qquad \lambda \in \mathfrak{a}^*. \tag{86}$$

For simplicity we write S, I and H for $S(\mathfrak{a}), I(\mathfrak{a})$ and $H(\mathfrak{a})$. We shall need the following properties of I and H from Chevalley [1955]. The proofs can also be found in [GGA], Ch. III, §3. The algebra I is generated by ℓ algebraically independent homogeneous generators $j_1, \ldots, j_\ell$ of degrees $d_1, \ldots, d_\ell$. The mapping $j \otimes h \to jh$ extends to a linear bijection of $I \otimes H$ onto S. With H^k denoting the space of homogeneous elements of H of degree k the Poincaré series of H, $\sum_{k \geq 0} (\dim H^k) t^k$, has the form

$$P_H(t) = \prod_{1 \leq i \leq \ell} (1 + t + \cdots + t^{d_i - 1}). \tag{87}$$

Moreover,

$$w = d_1 \cdots d_\ell \quad \text{and} \quad \sum_i^\ell (d_i - 1) = r, \tag{88}$$

where r is the number of reflections in W. The sum $\sum_1^w \deg(p_i)$ is clearly $\sum_0^w k \dim H^k$, which equals $P_H'(1)$. Taking the logarithmic derivative of $P_H(t)$ at $t = 1$ we get

$$P_H'(1)/P_H(1) = \frac{1}{2} \sum_1^\ell (d_i - 1) = \frac{r}{2}, \quad P_H(1) = w. \tag{89}$$

Given $\lambda \in \mathfrak{a}^*$ put

$$S_\lambda = \{p \in S : p(\lambda) = 0\}, \quad I_\lambda = S \cap I.$$

The latter has codimension 1 in I. Also

$$S = IH \approx I \otimes H, \quad SI_\lambda = I_\lambda H \approx I_\lambda \otimes H$$

so we see from the above remark that SI_λ is an ideal of codimension w in S and that the p_i are linearly independent mod SI_λ. If λ belongs to the set $\mathfrak{a}'$ of regular elements in $\mathfrak{a}$ then the points $s\lambda$ $(S \in W)$ are all different and S/SI_λ has w distinct homomorphisms into $\mathbf{C}$. Following a method kindly provided by David Vogan we shall prove that

$$\bigcap_{s \in W} S_{s\lambda} = SI_\lambda. \tag{90}$$

The inclusion $\supset$ being obvious it suffices to prove that the left hand side has codimension at least w in S. For this we use the following result.

Proposition. *If R is a commutative ring and $\alpha_1, \dots, \alpha_w$ distinct homomorphisms onto $\mathbf{C}$ then the direct sum*

$$\alpha_1 \oplus \cdots \oplus \alpha_w : R \to \mathbf{C} + \cdots + \mathbf{C}$$

is surjective.

In order to deduce (90) we take α_i as the homomorphism $p \to p(s_i\lambda)$ ($p \in S$). Then the kernel of the direct sum $\oplus_i \alpha_i$ equals the left hand side of (90). Thus the proposition implies that the codimension is w.

For the proposition we observe the following easy result.

Lemma. *Suppose R is a commutative ring, M a maximal ideal and $I \subset R$ an ideal not contained in M. Then the natural map of*

$$R/I \cap M \to R/I + R/M$$

is an isomorphism.

The proposition follows by induction taking M_i as the kernel of α_i and taking $I = m_1 \cap \cdots \cap m_{i-1}$ and $M = m_i$.

Lemma 5.18. $\mathrm{Det}(E) = c\pi^{\frac{1}{2}w}$ where $c \neq 0$ is a constant.

Proof. We first claim that $\det(E(\lambda)) \neq 0$ if $\lambda \in \mathfrak{a}'$. If to the contrary $\sum_{j=1}^{w} c_j p_j(s_i(\lambda)) = 0$ for some constants c_j (for all i) then $p = \sum c_j p_j$ vanishes on $W \cdot \lambda$ so by (90) $p \in SI_\lambda$. But $p \in H$ and $H \cap SI_\lambda = 0$ so $p = 0$. If $s_\alpha \in W$ is a reflection the map $s \to ss_\alpha$ permutes the rows in E. Since $p_j(ss_\alpha)(\lambda) = p_j(s\lambda)$ whenever $H_\alpha(\lambda) = 0$ we see that the difference between the ss_α row and the s row is divisible by H_α. There being $\frac{1}{2}w$ such pairs for each α, $\det E$ is divisible by $H_\alpha^{\frac{1}{2}w}$. Since α is arbitrary, $\det E$ is divisible by $\pi^{\frac{1}{2}w}$. The same argument shows that each cofactor in E is divisible by $\pi^{\frac{1}{2}w-1}$.

On the other hand $\deg(\det E) = \sum_j \deg p_j$, which equals $\sum_1^w j \dim H^j$ and by (88)–(89) this equals $r\frac{w}{2}$. This proves the lemma.

Consider now the quotient fields $C(S)$ and $C(I)$ and the bilinear form $\langle\ ,\ \rangle$ on $C(S) \times C(S)$ with values in $C(I)$ given by

$$\langle a, b\rangle = \sum_{\sigma \in W} a^\sigma b^\sigma. \tag{91}$$

Given $(p_j) \subset H$ as above we can construct $q^j \in C(S)$ explicitly satisfying

$$\langle p_j, q^i \rangle = \delta_{ij}$$

by making $f_{k\ell}(\lambda) = q^k(s_\ell\,\lambda)$ the inverse of $e_{\ell j}(\lambda) = p_j(s_\ell\,\lambda)$. In fact

$$\delta_{kj} = \sum_\ell f_{k\ell}\,e_{\ell j} = \sum_\ell q^k(s_\ell\lambda)\,p_j(s_\ell\,\lambda) = \langle p_j, q^k \rangle .$$

Lemma 5.19. *Each q^j satisfies $\pi q^j \in S$.*

In fact by Lemma 5.18 and its proof q^j is the ratio of a cofactor divided by $\det E$. The lemma follows.

Lemma 5.20.

(i) $S = \bigoplus_{i=1}^{w} I p_i$

(ii) $\frac{1}{\pi} S = \{x \in C(S(\mathfrak{a})) : \langle x, y\rangle \in I \text{ for } y \in S\}$.

Proof. Part (i) is already proved. For (ii) let $\check{I}$ denote the right hand side. Writing an $\check{\imath} \subset \check{I}$ in terms of the dual basis q^k, $\check{\imath} = \sum_k \frac{i_k}{j_k} q^k$, we have $\langle \check{\imath}, p_\ell\rangle = i_\ell / j_\ell$ so $\check{\imath} \in \bigoplus_k I q^k$. Thus by Lemma 5.19 $\pi \check{I} \subset S$. On the other hand, let $u \in S$. Then for $y \in S$, $\left\langle \frac{u}{\pi}, y\right\rangle = \sum_\sigma \frac{u^\sigma}{\pi} \epsilon(\sigma) y^\sigma = \frac{1}{\pi} \sum_\sigma \epsilon(\sigma)(uy)^\sigma$ which belongs to S since sum is skew and thus divisible by π. Hence $\frac{u}{\pi} \in \check{I}$.

8. The Multitemporal Cauchy Problem

The Iwasawa decomposition $G = NAK$ gives a decomposition (Ch. III, §2, (18))

$$\mathbf{D}(G) = \mathbf{D}(A) \oplus (\mathfrak{n}\mathbf{D}(G) + \mathbf{D}(G)\mathfrak{k}) \tag{92}$$

which provides a projection $D \to q^D$ of $\mathbf{D}(G)$ onto $\mathbf{D}(A)$. Restricting this to $\mathbf{D}_K(G)$, the algebra of K-right-invariant members of $\mathbf{D}(G)$, the mapping

$$D \to e^{-\rho} q^D \circ e^\rho$$

is a homomorphism of $\mathbf{D}_K(G)$ onto $I(\mathfrak{a})$ with kernel $\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k}$ and this in turn gives an isomorphism Γ of $\mathbf{D}(G/K)$ onto $I(\mathfrak{a})$. A member L of $I(\mathfrak{a})$ will be regarded as a differential operator $\partial(L)$ on $\mathfrak{a}$.

We now fix a real homogeneous basis $p_1 = 1, p_2, \ldots, p_n$ of $H(\mathfrak{a})$. For fixed $f_i \in \mathcal{D}(X)$ $(1 \le i \le w)$ we now consider the "**multitemporal Cauchy problem**" for $u \in \mathcal{E}(X \times \mathfrak{a})$:

$$Du = \partial(\Gamma(D))u, \quad D \in \mathbf{D}(G/K) \tag{93}$$

$$(\partial(p_i)u)(x, 0) = f_i(x), \quad 1 \le i \le w \tag{94}$$

In (93) D operates on the first variable, $\partial(\Gamma(D))$ on the second variable. As remarked by Schlichtkrull, the system (94) of initial data is equivalent to

$$(\partial(p)u)(x, 0) = F(p) \quad p \in H(\mathfrak{a}),$$

where $F \in \mathrm{Hom}_{\mathbf{R}}(H(\mathfrak{a}), \mathcal{D}(X)), \quad F(p_i) = f_i$.

We shall now use the Fourier theory in Chapter III to study the system (93)–(94). We use the Q^δ polynomial matrices given by III, §2 (22)–(22′). The representation $\check{\delta}$ contragredient to δ operates on the dual space $V'_\delta = V_{\check{\delta}}$. For $v \in V_\delta$ let $v' \in V'_\delta$ be determined by $v'(u) = \langle u, v\rangle$ for $u \in V_\delta$. If $C_\delta : V_{\check{\delta}}^M \to E_\delta$ is the linear transformation determined by $C_\delta v'_j = \epsilon_j$ $(1 \leq j \leq \ell(\delta))$ then

$$Q^\delta(\lambda)C_\delta v'_j = \sum_{i=1}^{\ell(s)} Q^\delta(\lambda)_{ij} v'_i. \tag{95}$$

We now prove a uniqueness theorem for the system (93)–(94).

Proposition 5.21. *Suppose u' and u'' are two solutions to (93)–(94) and that for each $H \in \mathfrak{a}$, the functions $x \to u'(x,H)$ and $x \to u''(x,H)$ have compact support. Then $u' = u''$.*

Proof. Let $u = u' - u''$. Then u satisfies (93)–(94) with $f_j \equiv 0$ for all j. Consider the Fourier transform $\widetilde{u}(\lambda, b\,; H)$ of u in the x-variable. Then if $D \in \mathbf{D}(G/K)$ and D^* its adjoint,

$$\begin{aligned} \partial(\Gamma(D))_H(\widetilde{u}(\lambda,b\,;H)) &= \Gamma(D^*)(-i\lambda)\widetilde{u}(\lambda,b\,;H) \\ &= \Gamma(D)(i\lambda)\widetilde{u}(\lambda,b\,;H) \end{aligned} \tag{96}$$

by Ch. III §1 (5). Also,

$$\{\partial(p_i)_H(u(x,H))\}_{H=0} \equiv 0\,. \tag{97}$$

Because of (84)–(85) each $p \in S(\mathfrak{a})$ can be written $p = \sum_{j=1}^w q_j p_j \quad (q_j \in I(\mathfrak{a}))$. Then using (93) and (96) we get

$$\begin{aligned} \partial(p)_H(\widetilde{u}(\lambda,b\,;H)) &= \textstyle\sum_j \partial(p_j)\partial(q_j)_H(\widetilde{u}(\lambda,b\,;H)) \\ &= \textstyle\sum_j \partial(p_j)_{\mathcal{H}}(q_j(i\lambda)\widetilde{u}(\lambda,b\,;H)) \end{aligned}$$

Using the inversion formula for $u(x,H)$ and (97) we deduce

$$\{\partial(p)_H(u(x,H))\}_{H=0} = 0\,, \qquad p \in S(\mathfrak{a})\,. \tag{98}$$

On the other hand, if $\lambda \in \mathfrak{a}^*$ is regular (96) implies by a result of Steinberg and Harish–Chandra [GGA], III, Theorem 3.13 that for constants C_s

$$\widetilde{u}(\lambda,b;H) = \sum_{s\in W} C_s(\lambda,b)e^{is\lambda(H)}\,.$$

Thus the equation $\{\partial(p)_H(\widetilde{u}(\lambda,b\,;H))\}_{H=0} = 0$ for all p implies

$$\sum_{s\in W} p(is\lambda)C_s(\lambda,b) = 0\,, \qquad p \in S(\mathfrak{a})\,.$$

For $s_o \in W$ fixed we choose p such that $p(is\lambda) = 0$ for all $s \neq s_0$ in W and $p(is_0\lambda) = 1$. Then we get $C_{s_0}(\lambda) = 0$. Thus $\widetilde{u}(\lambda, b; H) = 0$ for all $\lambda \in \mathfrak{a}^*$ regular and $b \in B$ so by the inversion formula, $u(x, H) \equiv 0$, as desired.

Next we prove existence of the solution with an explicit formula.

Theorem 5.22. *Given $f_1, \ldots, f_w \in \mathcal{E}(X)$ the system*

$$Du = \partial(\Gamma(D))u \qquad D \in \mathbf{D}(G/K) \tag{99}$$

with initial data

$$(\partial(p_j)u)(x, 0) = f_j(x) \qquad (1 \leq j \leq w) \tag{100}$$

has a solution given by the convolution

$$u(x, H) = \sum_{j=1}^{w} (f_j \times S_H^j)(x) \tag{101}$$

where for each j, $H \in \mathfrak{a}$, S_H^j is in $\mathcal{E}'(X)$ and is for all b given by

$$\sum_{\sigma \in W} q^j(i\sigma\lambda) e^{(i\sigma\lambda)(H)} = \int_X e^{(-i\lambda+\rho)(A(x,b))} \, dS_H^j(x)\,. \tag{102}$$

Remark. Because of Prop. 5.21 we shall refer to (101) as *the* solution to the multitemporal Cauchy Problem.

Because of Lemma 5.19, $q^j = h^j/\pi$ where $h^j \in S(\mathfrak{a})$ so the left hand side of (102) equals

$$\frac{1}{\pi(i\lambda)} \sum_{\sigma} h^j(i\sigma\lambda)\epsilon(\sigma) e^{i\sigma\lambda(H)} \tag{103}$$

where $\pi(\sigma\lambda) = \epsilon(\sigma)\pi(\lambda)$ and $\epsilon(\sigma) = \pm 1$. The sum in (103) is skew and is thus divisible by $\pi(i\lambda)$ so (103) is holomorphic in λ. Also if $\lambda = \xi + i\eta \quad (\xi, \eta \in \mathfrak{a}^*)$ we have

$$\left| e^{i\sigma\lambda(H)} \right| \leq e^{|H||\eta|}$$

so (103) is an entire function of exponential type of polynomial growth for λ real. By the Paley–Wiener theorem extended to $\mathcal{E}'(X)$ (Ch. III, Corollary 5.9) there exist unique $S_H^j \in \mathcal{E}'(X)$ satisfying (102) and

$$\operatorname{supp}(S_H^j) \subset B_{|H|}(o)\,. \tag{104}$$

Next we prove that S_H^j depends smoothly on $H \in \mathfrak{a}$, that is, for each $f \in \mathcal{D}(X)$, $S_H^j(f)$ is smooth in H. Since S_H^j is K-invariant, $(f \times S_H^j)\widetilde{\ } = \widetilde{f}(S_H^j)\widetilde{\ }$ (Ch. III, Lemma 1.4) so (102) implies for the lift $\widetilde{S}_J^j$ to G

$$\int_G f(g^{-1} \cdot 0) \, d\widetilde{S}_H^j(g) = \int_{\mathfrak{a}^* \times B} \widetilde{f}(\lambda, b) \sum_{\sigma} q^j(-i\sigma\lambda) e^{-i\sigma\lambda(H)} \, d\mu(\lambda, b)$$

so the smoothness in H follows.

Defining $u(x,H)$ by (101) we shall now verify (99)–(100). Since $D(f \times S) = f \times DS$ for $f \in \mathcal{E}(X), S \in \mathcal{E}'(X)$ we consider DS^j_H and shall prove

$$DS^j_H = \partial(\Gamma(D))_H(S^j_H)\,. \tag{105}$$

For this we observe that by (102)

$$(DS^j_H)\widetilde{\ }(\lambda,b) \quad = \quad \Gamma(D^*)(-i\lambda)(S^j_H)\widetilde{\ }(\lambda,b) = \Gamma(D)(i\lambda)(S^j_H)\widetilde{\ }(\lambda,b)\,.$$

On the other hand, applying $\partial(\Gamma(D))$ to (102) we see by the W-invariance of $\Gamma(D)$ that

$$(\partial(\Gamma(D))_H(S^j_H))\widetilde{\ }(\lambda,b) = \Gamma(D)(i\lambda)(S^j_H)\widetilde{\ }(\lambda,b)\,.$$

These formulas imply (105), so (99) follows. Secondly, applying $\partial(p_k)_H$ to (102) we get

$$\sum_\sigma q^j(i\sigma\lambda)p_k(i\sigma\lambda)e^{i\sigma\lambda(H)} = (\partial(p_k)_H(S^j_H))\widetilde{\ }(\lambda,b)\,.$$

Putting here $H=0$ and using $\langle q^j, p_k\rangle = \delta_{jk}$, we obtain

$$\Big\{(\partial(p_k)_H(S^j_H))\widetilde{\ }(\lambda,b)\Big\}_{H=0} = \delta_{jk} \tag{106}$$

for all λ, b. By the injectivity of the Fourier transform this means

$$\Big\{\partial(p_k)S^j_H\Big\}_{H=0} = \delta_{jk}\delta_o\,, \tag{107}$$

where δ_o the delta distribution of X at o. Now (100) follows immediately.

Remark. Relations (105)–(107) are the analogs to (81)–(82) and one can thus consider the family

$$\{S^1_H, \dots, S^w_H\}$$

as the *fundamental solution* to (99)-(100).

Example. G/K of rank one. Let H_0 be the vector in $\mathfrak{a}^+$ of length 1. Then with $p_1 = 1, p_2 = H_0$ we have $q^1 = 1/2, q^2 = 1/2H_0$. Also $\Gamma(L) = H_0^2 - |\rho|^2$. Writing $v(x,t) = u(x,tH_0)$ equations (99) and (100) become

$$(L+|\rho|^2)v = \frac{\partial^2}{\partial t^2}v\,, \quad v(x,0) = f_0(x)\,, v_t(x,0) = f_1(x)$$

Writing S^j_t instead of $S^j_{tH_0}$ equation (102) becomes (since $\lambda(H_0) = \pm|\lambda|$)

$$\cos|\lambda|\,t = \int_X e^{(-i\lambda+\rho)(A(x,b))}\,dS^1_t(x)\,,$$

$$\frac{\sin|\lambda|\,t}{|\lambda|} = \int_X e^{(-i\lambda+\rho)(A(x,b))}\,dS^2_t(x)$$

in exact analogy with (80).

From Theorem 5.22 we can now deduce the following result stating informally that the speed of propagation is ≤ 1.

Corollary 5.23. *If* $\mathrm{supp}(f_j) \subset B_R(x_0)$ *for* $1 \leq j \leq w$ *then for each* $H \in \mathfrak{a}$,

$$u(x, H) = 0 \quad \text{for } x \notin B_{R+|H|}(x_0)\,. \tag{108}$$

By the group invariance we can take $x_0 = o$. Then the result is an immediate consequence of (101), (104) and the following simple lemma.

Lemma 5.24 *Let* $f \in \mathcal{E}(X), S \in \mathcal{E}'(X)$. *Then*

$$\mathrm{supp}(f) \subset \overline{B_r(o)}\,,\ \mathrm{supp}(S) \subset \overline{B_s(o)} \Rightarrow \mathrm{supp}(f \times S) \subset \overline{B_{r+s}(o)}\,. \tag{109}$$

Proof. By approximation we may assume S is a continuous function. If $\widetilde{S}$ denotes the lift of S to G ($\widetilde{S}(g) = S(g \cdot o)$) then

$$(f \times S)(g \cdot o) = \int_G f(gh^{-1} \cdot o)\widetilde{S}(h)\, dh\,.$$

If this is $\neq 0$ then for some h, $d(gh^{-1} \cdot o, o) \leq r$ and $d(h \cdot o, o) \leq s$ whence,

$$d(o, g \cdot o) = d(g^{-1} \cdot o, o) \leq d(g^{-1} \cdot o, h^{-1} \cdot o) + d(h^{-1} \cdot o, o) \leq r + s\,,$$

proving (109).

We can now reformulate the property in Cor. 5.23 as follows.

Proposition 5.25 *Let* $f_j \in \mathcal{D}(X)(1 \leq j \leq w)$ *and* $u(x, H)$ *the corresponding solution to (99)–(100). Then* if

$$\overline{B_\rho(x_0)} \cap \bigcup_{j=1}^{w} \mathrm{supp}(f_j) = \emptyset \tag{110}$$

we have for each $H \in \mathfrak{a}$,

$$u(x, H) = 0 \quad \text{for } x \in B_{\rho-|H|}(x_0)\,. \tag{111}$$

Proof. The compact set $C = \bigcup_{j=1}^{w} \mathrm{supp}(f_j)$ can be covered by finitely many balls $B_\epsilon(x_i)$ (ϵ fixed) such that

$$\overline{B_\rho(x_0)} \cap \overline{B_\epsilon(x_i)} = \emptyset \quad \text{for each } i\,. \tag{112}$$

Choose a corresponding partition of unity i.e., $\varphi_i \in \mathcal{D}(B_\epsilon(x_i)), \varphi_i \geq 0, \sum_i \varphi_i \leq 1$ with equality in a neighborhood of C. Then $f_j = \sum_i f_j^i\ (1 \leq$

$j \leq w$) where $f_j^i = \varphi_i f_j$ has support in $B_\epsilon(x_i)$. Let $u^i(x, H)$ be the solution to (99)–(100) corresponding to $f_j^i (1 \leq j \leq w)$. By Cor. 5.23,

$$u^i(x, H) = 0 \quad \text{for } x \notin B_{\epsilon+|H|}(x_i)\,. \tag{113}$$

However, $B_{\epsilon+|H|}(x_i)$ is disjoint from $\overline{B_{\rho-|H|}(x_0)}$; in fact a common point y would satisfy $d(x_i, y) \leq \epsilon + |H|$, $d(x_0, y) \leq \rho - |H|$ whence $d(x_i, x_0) \leq \epsilon + \rho$, contradicting (112). Thus by (113), $u^i(x, H) = 0$ for $x \in B_{\rho-|H|}(x_0)$ so (111) follows since $u = \sum_i u^i$.

We shall now derive some other formulas for the solution. If $S \in \mathcal{E}'(X)$ is K-invariant and $f \in \mathcal{D}(X)$ we have

$$(f \times S)\tilde{} = \tilde{f}\tilde{S}$$

so by (101) and the inversion formula for the Fourier transform,

$$u(x, H) = \frac{1}{w} \int_{\mathfrak{a}^*} \sum_{j,\sigma} q^j(i\sigma\lambda) e^{i\sigma\lambda(H)} (\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1} \int_B \tilde{f}_j(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} db\, d\lambda\,.$$

Since $(\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1}$ as well as the integral over B are W-invariant in λ (cf. Ch. III, Lemma 1.2), the terms in $\sum_\sigma$ all have the same integral over $\mathfrak{a}^*$. Thus we can replace the average $\frac{1}{w}\sum_\sigma$ by a single term. For each $\sigma \in W$ we thus obtain,

$$u(x, H) = \int_{\mathfrak{a}^* \times B} \sum_k j_k^\sigma(\lambda) \tilde{f}_k(\lambda, b) e^{i\lambda(A(x,b)+\sigma H)} e^{\rho(A(x,b))}\, d\lambda\, db\,, \tag{114}$$

where

$$j_k(\lambda) = q^k(i\lambda)/\mathbf{c}(\lambda)\mathbf{c}(-\lambda) \tag{115}$$

and as usual, $j_k^\sigma(\lambda) = j_k(\sigma^{-1}\lambda)$. Consider now the Euclidean Fourier transform

$$F^*(\lambda) = \int_A e^{-i\lambda(\log a)} F(a)\, da$$

and let J_k denote the operator on $\mathcal{S}(A)$ determined by

$$(J_k F)^*(\lambda) = j_k(\lambda) F^*(\lambda)\,. \tag{116}$$

Since $q^k(\lambda)\pi(\lambda)$ is a polynomial and $(\pi(\lambda)\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1}$ has all its derivatives bounded by a polynomial, J_k does indeed map $\mathcal{S}(A)$ into itself. Also, by a simple computation,

$$(J_k^\sigma F)^*(\lambda) = j_k^\sigma(\lambda) F^*(\lambda)\,.$$

Consider now the Radon transform

$$\widehat{f}(\xi) = \int_{\xi} f(x)\, dm(x)\,,$$

where ξ is a horocycle $gNg^{-1} \cdot x$ and dm the measure on ξ derived from dn. More explicitly, if $\xi_0 = N \cdot o$,

$$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o)\, dn\,.$$

Since each ξ equals $ka \cdot \xi_0$ where $kM \in K/M$ and $a \in A$ are unique we can write $\widehat{f}(\xi)$ in the form $\widehat{f}(kM, a)$. Then J_k operates on functions on the horocycle space Ξ. Also by Ch. III, §5, (25)

$$\widetilde{f}(\lambda, b) = \int_A \widehat{f}(b, a) e^{(-i\lambda+\rho)(\log a)}\, da = (e^{\rho}\widehat{f})^*(\lambda, b)\,. \tag{117}$$

Thus by (114)

$$u(x, H) = \int_{\mathfrak{a}^* \times B} \sum_k (J_k^{\sigma}(e^{\rho}\widehat{f}_k))^*(\lambda, b) e^{i\lambda(A(x,b)+\sigma H)} e^{\rho(A(x,b))}\, d\lambda\, db$$

so by the inversion formula for the Euclidean Fourier transform,

$$u(x, H) = \int_B \sum_k J_k^{\sigma}(e^{\rho}\widehat{f}_k)(b, \exp(A(x,b) + \sigma H)) e^{\rho(A(x,b))}\, db\,. \tag{118}$$

Here e^{ρ} is the function on Ξ given by $e^{\rho}(kM, a) = e^{\rho(\log a)}$. As in Ch. II, §3 we introduce the operators

$$\Lambda_{k,\sigma} = e^{-\rho} J_k^{\sigma} \circ e^{\rho} \qquad (1 \le k \le w)$$

on the space $\mathcal{S}_{\rho}(\Xi) = \{\varphi \in \mathcal{E}(\Xi) : e^{\rho}\varphi \in \mathcal{S}(\Xi)\}$, where $\mathcal{S}(\Xi) = \mathcal{S}(K/M \times A)$. Then the space $\mathcal{S}_{\rho}(\Xi)$ and the operators $\Lambda_{k,\sigma}$ are G-invariant (Ch. II, Lemma 3.7.)

Theorem 5.26 *Fix $\sigma \in W$. The solution to (99)-(100) can be written*

$$u(g \cdot o, H) = e^{\rho(\sigma H)} \int_{K/M} \sum_j (\Lambda_{j,\sigma}\widehat{f}_j)(gk \exp \sigma H \cdot \xi_0)\, dk_M\,. \tag{119}$$

Proof. If $g = e$ the right hand side becomes

$$e^{\rho(\sigma H)} \int_B \sum_j (e^{-\rho} J_j^{\sigma}(e^{\rho}\widehat{f}_j))(b, \exp \sigma H)\, db = \int_B \sum_j (J_j^{\sigma}(e^{\rho}\widehat{f}_j))(b, \exp \sigma H)\, db\,,$$

which agrees with the right hand side of (118) for $x = o$. By the G-invariance of (99)-(100) in the x-variable the initial data $f_j^{\tau(g^{-1})}$ generate the solution $u^{\tau(g^{-1})}$. Since the Radon transform as well as the operators $\Lambda_{j,\sigma}$ commute with the G-action we deduce from the above

$$u(g\cdot o,H) = u^{\tau(g^{-1})}(o,H) = e^{\rho(\sigma H)}\int_{K/M}\sum_j(\Lambda_{j,\sigma}(f_j^{\tau(g^{-1})})\widehat{\ })(k\exp\sigma H\cdot\xi_0)\,dk_M$$

$$= e^{\rho(\sigma H)}\int_{K/M}\sum_j(\Lambda_{j,\sigma}\widehat{f}_j)(gk\exp\sigma H\cdot\xi_0)\,dk_M$$

as claimed.

Remark. For the case rank $(X) = 1$, Theorem 5.26 gives an alternative solution to the shifted wave equation in Theorem 5.7.

Definition. Given $\sigma \in W$ the function

$$(S_\sigma f)(\xi) = \sum_j(\Lambda_{j,\sigma}\widehat{f}_j)(\xi)\,, \qquad \xi\in\Xi$$

will be called the σ-*source* of the solution u.

Corollary 5.27. *The solution to (99)–(100) is*

$$u(x,H) = e^{\rho(\sigma H)}\int_B (S_\sigma f)(b,\exp(A(x,b)+\sigma H))e^{2\rho(A(x,b))}\,db \tag{120}$$

and this formula is the symmetric space analog to (76).

From (120) it is easy to deduce the analog of Huygens' principle in §5, (6′).

Corollary 5.28. *Assume G has all its Cartan subgroups conjugate. Then if* $\operatorname{supp}(f_j)\subset B_R(o)(1\le j\le w)$ *the solution $u(x,H)$ has support in the region*

$$|H| - R \le d(x,o) \le |H| + R\,. \tag{121}$$

In fact, the assumption on G is equivalent to all $\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})$ having even multiplicity ([DS], Ch. IX, §6). This in turn implies by the Harish–Chandra and Gindikin–Karpelevic formula for $\mathbf{c}(\lambda)$ that $\mathbf{c}(\lambda)^{-1}$ and even $(\pi(\lambda)\mathbf{c}(\lambda))^{-1}$ are polynomials. In addition, $\pi(\lambda)q^j(i\lambda)$ is a polynomial so the operators $\Lambda_{j,\sigma}$ are differential operators. Since

$$d(o,a\cdot o)\le d(o,na\cdot o)\,, \qquad a\in A\,, n\in N\,, \tag{122}$$

we thus deduce from (120) that $u(x,H)=0$ if $|H+A(x,b)|\geq R$ for $b\in B$. But if $x=gK$, $b=kM$, $k^{-1}g=nak_0$, (122) implies

$$|A(x,b)|=\left|A(k^{-1}g)\right|=|\log a|\leq d(o,na\cdot o)\leq d(o,x)\,. \tag{123}$$

Thus if $|H|>R+d(o,x)$ we have for $b\in B$,

$$|H+A(x,b)|\geq|H|-|A(x,b)|\geq R$$

so

$$u(x,H)=0 \quad \text{for} \quad d(o,x)<|H|-R\,.$$

This, together with (108) implies the corollary.

The Case G complex. In this case we can give a simpler formula for the solution of the multitemporal Cauchy Problem. For h^j as in (103) we have $\partial(h^j(e^{i\sigma\lambda}))=h^j(i\sigma\lambda)e^{i\sigma\lambda}$ so by (102)–(103)

$$\widetilde{S^j_H}(\lambda)=\frac{1}{\pi(i\lambda)}\partial(h^j_H)\bigg(\sum_\sigma \epsilon(\sigma)e^{i\sigma\lambda(H)}\bigg)\,. \tag{124}$$

The density δ is given by

$$\delta(\exp H)=\prod_{\alpha\in\Sigma^+}(e^\alpha-e^{-\alpha})^{m_\alpha}$$

and now the multilpicities m_α equal 2 so we have (see e.g. [GGA], Ch. IV, §5)

$$\delta^{\frac{1}{2}}(a)=\sum\epsilon(\sigma)e^{\sigma\rho(\log a)}$$

and the spherical function φ_λ equals

$$\varphi_\lambda(a)=\frac{\pi(\rho)}{\pi(i\lambda)}\,\frac{\sum_\sigma\epsilon(\sigma)e^{i\sigma\lambda(\log a)}}{\delta^{\frac{1}{2}}(a)}\,.$$

Hence

$$\widetilde{S^j_H}(\lambda)=\frac{1}{\pi(\rho)}\partial(h^j)_H(\delta^{\frac{1}{2}}(\exp H)\varphi_\lambda(\exp H))\,. \tag{125}$$

Theorem 5.29. *For G complex the solution to (99)–(100) can be written*

$$u(x,H)=w\sum_{j=1}^{w}\partial(h^j)_H(\delta^{\frac{1}{2}}(\exp H)(M^{\exp H}f_j)(x))\,, \tag{126}$$

where M^x is the mean value operator (§2, (1)).

Remark. The analogy with the odd-dimensional flat case (3) is striking.

Proof. The mean value operator satisfies

$$M^h(e^{(i\lambda+\rho)(A(x,b))}) = \varphi_\lambda(h)e^{(i\lambda+\rho)(A(x,b))}$$

(Ch. II, Prop. 2.6) so by the inversion formula (Ch. III, Theorem 1.3)

$$(M^{\exp H} f_j) = \frac{1}{w} \int_{\mathfrak{a}^* \times B} \varphi_\lambda(\exp H) \widetilde{f}_j(\lambda, b) e^{(i\lambda+\rho)(A(x,b))}\, d\mu(\lambda, b) \tag{127}$$

so by (125)

$$\frac{1}{\pi(\rho)} \int_{\mathfrak{a}^*} \partial(h^j)_H(\delta^{\frac{1}{2}} \exp H)\varphi_\lambda(\exp H)) \int_B e^{(i\lambda+\rho)(A(x,b))} \widetilde{f}_j(\lambda, b)\, db\, d\mu(\lambda, b)$$
$$= \int_{\mathfrak{a}^* \times B} \widetilde{S^j_H}(\lambda) e^{(i\lambda+\rho)A(x,b)} \widetilde{f}_j(\lambda, b)\, d\mu(\lambda, b) = w(f_j \times S^j_H)(x)\,.$$

In view of (127) and the solution formula (101) this proves the theorem.

For a hyperbolic space $\mathbf{H}^n$ (n odd) the shifted wave equation (52) is a special case of our rank-one example. The explicit solution as given by (50)–(51) is quite analogous to the solution (126). In the case $n = 3$, the isometry group G of $\mathbf{H}^3$ is the complex group $\mathbf{SL}(2, \mathbf{C})$ and in this case the two formulas will coincide.

9. Incoming Waves and Supports.

In this section we shall prove the symmetric space analog of Cor. 5.16, relating incoming waves to support properties of the sources S_σ.

As usual, we put

$$^+\mathfrak{a} = \left\{H \in \mathfrak{a} : \langle H, H' \rangle > 0 \quad \text{for } H' \in \mathfrak{a}^+\right\}$$

and $\mathfrak{a}^- = -\mathfrak{a}^+$. Note that $^+\mathfrak{a} \cup \{0\}$ is closed ([GGA], Ch. IV, §6, (13)) and $\mathfrak{a}^+ \subset {}^+\mathfrak{a}$.

Definition. Fix $\sigma \in W$. The wave $u(x, H)$ in (120) is said to be σ-*incoming* if

$$u(x, H) = 0 \quad \text{for } d(o, x) < |H|\,, \quad \sigma H \in \mathfrak{a}^-\,. \tag{128}$$

To distinguish from balls in X, an open ball in $\mathfrak{a}$ of radius r and center H is denoted $V_r(H)$.

We shall now, in analogy with Cor. 5.16, characterize σ-incoming waves in terms of their σ-sources.

Theorem 5.30. *Fix $\sigma \in W$ and $f_j \in \mathcal{D}(X)$, $(1 \leq j \leq w)$. Then u is σ-incoming if and only if*

$$\operatorname{supp}(S_\sigma f)(b, \cdot) \subset \exp(\overline{{}^+\mathfrak{a}}) \quad \text{for all } b \in B\,. \tag{129}$$

Naturally, formula (120) is basic to the proof. First we prove (129) $\Rightarrow$ (128). Suppose $H \in \mathfrak{a}$, $\sigma H \in \mathfrak{a}^-$ and $d(o,x) < |H|$. Then by (123), $A(x,b) \in V_{|H|}(0)$. Now let $H_0 \in {}^+\mathfrak{a} + (-\sigma H)$. Then $\langle H_0 + \sigma H, -\sigma H\rangle \geq 0$ so

$$-\,|H_0|\,|H| \leq \langle H_0, \sigma H\rangle \leq -\,|H|^2$$

so $H_0 \notin V_{|H|}(0)$. Hence $A(x,b) \notin {}^+\mathfrak{a} + (-\sigma H)$ so by (129) and (120), $u(x,H) = 0$. This proves (129) $\Rightarrow$ (128).

For the converse we assume (129). For $\varphi \in \mathcal{D}(\mathfrak{a})$ we consider the integral

$$v(x) = \int_{\mathfrak{a}} u(x,H)\varphi(H)\,dH\,. \tag{130}$$

Then by (114) we obtain

$$v(x) = \int_{\mathfrak{a}^* \times B} F^\sigma_\lambda(b)\psi^*(\lambda)e^{(i\lambda+\rho)(A(x,b))}\,d\lambda\,db\,, \tag{131}$$

where

$$\psi(H) = \varphi(-\sigma^{-1}H) \quad \text{and} \quad F^\sigma_\lambda(b) = \sum_k j^\sigma_k(\lambda)\widetilde{f}_k(\lambda,b)\,.$$

Note that

$$F^\sigma_\lambda(b) = h_\sigma(b,\cdot)^*(\lambda)\,,$$

the Fourier transform (in A) of the function

$$h_\sigma(b,a) = \sum_k J^\sigma_k(e^\rho \widehat{f}_k)(b,a)\,, \quad b \in B\,, a \in A\,. \tag{132}$$

Let $\sigma^* \in W$ be the element interchanging $\mathfrak{a}^+$ and $\mathfrak{a}^-$ and put $\tau = \sigma^{-1}\sigma^*$. Then $\sigma H \in \mathfrak{a}^- \iff H \in \tau\mathfrak{a}^+$ so by (128)

$$u(x,H) = 0 \quad \text{for } d(o,x) < |H|\ , H \in \tau\mathfrak{a}^+\,. \tag{133}$$

Now fix $H_0 \in \tau\mathfrak{a}^+$ and consider the solution $(x,H) \longrightarrow u(x, H+H_0)$ to (93) whose initial data are

$$\{\partial(p_i)_H(u(x,H+H_0))\}_{H=0} = (\partial(p_i)u)(x,H_0)\,.$$

We denote these by $F_i(x)\,(1 \leq i \leq w)$. Then (133) implies $F_i(x) = 0$ for $d(o,x) < |H_0|$, i.e., $\operatorname{supp}(F_i) \cap B_{|H_0|}(o) = \emptyset$. Hence by Prop. 5.25,

$$u(x, H+H_0) = 0 \quad \text{for } x \in B_{|H_0|-|H|}(o)\,. \tag{134}$$

Of course, $H+H_0 \in V_{|H_0|-\,\epsilon}(H_0) \Longleftrightarrow |H| < |H_0|-\epsilon$ which in turn implies

$$B_{|H_0|-|H|}(o) \supset B_\epsilon(o)\,.$$

Thus (134) implies the following result.

Lemma 5.31. *Let $\tau = \sigma^{-1}\sigma^*$, fix $H_0 \in \tau\mathfrak{a}^+$ and let $\epsilon < |H_0|$. Then*

$$u(x, H + H_0) = 0 \quad \text{for } d(o,x) < \epsilon \qquad \text{if } H + H_0 \in V_{|H_0|-\,\epsilon}(H_0)\,.$$

Now let

$$\mathfrak{a}_\epsilon(\tau) = \left\{H \in \mathfrak{a} : \langle H, H'\rangle > \epsilon\,|H'| \ \text{ for some } H' \in \tau\mathfrak{a}^+\right\}\,. \tag{135}$$

Then $\mathfrak{a}_\epsilon(\tau) = \tau \cdot \mathfrak{a}_\epsilon(e)$. Let H_0 in Lemma 5.31 run along a fixed ray $\{te_0 : t > 0, |e_0| = 1\}$ in $\tau\mathfrak{a}^+$. The balls $V_{|H_0|-\epsilon}(H_0)$ will then fill up the half space $\langle H, e_0\rangle > \epsilon$. As H_0 varies in $\tau\mathfrak{a}^+$ (and $|H_0| > \epsilon$) these half spaces have union $\mathfrak{a}_\epsilon(\tau)$. Thus Lemma 5.31 implies the following result.

Lemma 5.32. *Let $\tau = \sigma^{-1}\sigma^*$, $\epsilon > 0$ and $H \in \mathfrak{a}_\epsilon(\tau)$. Then*

$$u(x,H) = 0 \quad \text{for } d(o,x) < \epsilon\,.$$

Now let φ in (130) satisfy $\varphi \in \mathcal{D}(\mathfrak{a}_\epsilon(\tau))$. Then

$$v(x) = 0 \quad \text{for } d(o,x) < \epsilon\,, \tag{136}$$

which by (131) amounts to

$$v(x) = \int_{\mathfrak{a}^*} \psi^*(\lambda)\mathcal{P}_\lambda(F_\lambda^\sigma)(x)\,d\lambda = 0\,, \tag{137}$$

where $\mathcal{P}_\lambda$ is the Poisson transform defined by

$$(\mathcal{P}_\lambda F)(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}F(b)\,db\,. \tag{138}$$

As in the proof in Ch. III, §2 of the injectivity criterion for the Poisson transform we exploit (137) by means of the equation

$$\{D_g(v(g\cdot o))\}_{g=e} = 0 \tag{139}$$

for all $D \in \mathbf{D}(G)$, the algebra of left-invariant differential operators on G. The kernel in (138) is given by

$$e^{(i\lambda+\rho)(A(gK,kM))} = \zeta_{-\lambda}(k^{-1}g)\,,$$

where $\zeta_\lambda(g) = e^{(-i\lambda+\rho)(A(g))}$. Fix $\delta \in \widehat{K}_M$ (Ch. III, §2) and consider the operators $\epsilon_j(v_i)$ there. By the proof of Theorem 2.6 in Ch. III, they satisfy

$$(\epsilon_j(v_i)\xi_{-\lambda})(k^{-1}) = \sum_{p=1}^{\ell(\delta)} \langle v_i, \delta(k)v_p\rangle Q^\delta(-\lambda)_{pj}\,. \tag{140}$$

Writing (131) and (137) in the form (with $dk_M = db$)

$$v(g\cdot o) = \int_{\mathfrak{a}^*} d\lambda \int_{K/M} \zeta_{-\lambda}(k^{-1}g)F(\lambda, kM)\, dk_M\,, \tag{141}$$

where $F(\lambda, b) = h_\sigma(b,\cdot)^*(\lambda)\psi^*(\lambda)$, we thus have

$$0 = \{\epsilon_j(v_i)_g(v(g\cdot o))\}_{g=e} = \int_{\mathfrak{a}^*} \sum_{p=1}^{\ell(\delta)} Q^\delta(-\lambda)_{pj} F^\delta_{pi}(\lambda)\, d\lambda\,.$$

Here

$$F^\delta_{pi}(\lambda) = \int_{K/M} \langle v_i, \delta(k)v_p\rangle F(\lambda, kM)\, dk_M\,.$$

Now $F(\lambda, b) = (h_\sigma(b,\cdot)\times\psi)^*(\lambda)$, $\times$ denoting convolution on A (and $\mathfrak{a}$). Thus, defining

$$\zeta_{pi}(H) = \int_{K/M} h_\sigma(kM, \exp H)\langle v_i, \delta(k)v_p\rangle\, dk_M \quad (H \in \mathfrak{a})$$

we have for $1 \le i, j \le \ell(\delta)$,

$$\sum_{p=1}^{\ell(\delta)} \int_{\mathfrak{a}^*} Q^\delta(-\lambda)_{pj}(\zeta_{pi}\times\psi)^*(\lambda)\, d\lambda = 0\,. \tag{142}$$

Define $Q^\delta_{pj} \in S(\mathfrak{a})$ by $Q^\delta_{pj}(\lambda) = Q^\delta(\lambda)_{pj}$ and $P^\delta_{pj} \in S(\mathfrak{a})$ by $P^\delta_{pj}(\lambda) = Q^\delta_{pj}(i\lambda)$. Then, since $P^\delta(i\lambda) = Q^\delta(-\lambda)$, (142) implies by the Fourier inversion formula,

$$\sum_{p=1}^{\ell(\delta)} \left[(\partial(P^\delta_{pj})\zeta_{pi})\times\psi\right](0) = 0\,,$$

so, if $\check{\psi}(H) = \psi(-H)$,

$$\sum_{p=1}^{\ell(\delta)} \int_{\mathfrak{a}} (\partial(P^\delta_{pj})\zeta_{pi})(H)\check{\psi}(H)\, dH = 0\,. \tag{143}$$

This holds whenever $\varphi \in \mathcal{D}(\mathfrak{a}_\epsilon(\tau))$, and $\epsilon > 0$.

Let $\mathfrak{a}_0(\tau) = \lim_{\epsilon\to 0} \mathfrak{a}_\epsilon(\tau)$. Then $\mathfrak{a}_0(\tau) = \tau\mathfrak{a}_0(e)$ so the complement of $\mathfrak{a}_0(\tau)$ is $-\tau({}^+\mathfrak{a}) \cup \{0\} = \sigma^{-1}({}^+\mathfrak{a}) \cup \{0\}$. Since $\breve{\psi}(H) = \varphi(\sigma^{-1}H)$, (143) implies

$$\sum_{p=1}^{\ell(\delta)} \partial(P^\delta_{pj})\zeta_{pi} = 0 \quad \text{on } \mathfrak{a}\diagdown{}^+\mathfrak{a}\,. \tag{144}$$

The matrix P^δ with entries $P^\delta_{pj} \in S(\mathfrak{a})$ satisfies

$$\det(P^\delta) = P_c P^\delta\,, \tag{145}$$

where P_c is a matrix whose entries $P^c_{pi} \in S(\mathfrak{a})$ are the cofactors of P^δ. Applying $\partial(P^c_{pj})$ to (144) and summing on j we get by (145),

$$(\partial(\det P^\delta)\zeta_{pi})(H) = 0 \quad \text{for } H \in \mathfrak{a} \setminus {}^+\mathfrak{a}\,.$$

Now $\det P^\delta(\lambda) = \det Q^\delta(i\lambda)$ which by Ch. III, Theorem 4.2 and Cor. 11.3 is a product of factors $\langle\lambda,\alpha\rangle + c_{j,\alpha}$ where $c_{j,\alpha} > 0$. Here $\alpha \in \sum_0^+$ and j runs through a certain finite set. With $H_\alpha \in \mathfrak{a}$ given by $\langle H_\alpha, H\rangle = \alpha(H)$ we thus have for $F = \zeta_{pi}$, which belongs to $\mathcal{S}(\mathfrak{a})$,

$$\prod_{\alpha,j}(\partial(H_\alpha) + c_{j,\alpha})F = 0 \quad \text{on } \mathfrak{a}\diagdown{}^+\mathfrak{a}\,. \tag{146}$$

We pull out a single factor $\partial(H_\alpha) + c$ $(c > 0)$ and consider the equation

$$(\partial(H_\alpha) + c)f = 0 \quad \text{in } \mathfrak{a}\diagdown{}^+\mathfrak{a}\,,\ f \in \mathcal{S}(\mathfrak{a})\,.$$

Then the function

$$g(H) = e^{c_0\alpha(H)}f(H) \quad c_0 = c/\langle\alpha,\alpha\rangle \tag{147}$$

satisfies

$$\partial(H_\alpha)g = 0 \quad \text{in } \mathfrak{a} \setminus {}^+\mathfrak{a}\,. \tag{148}$$

Fix $H_0 \in \mathfrak{a} \setminus \mathfrak{a}^+$. Since $-H_\alpha \in \mathfrak{a} \setminus {}^+\mathfrak{a}$ and since $\mathfrak{a} \setminus {}^+\mathfrak{a}$ is star-shaped with respect to 0 we see that $H_0 - tH_\alpha = t(\frac{H_0}{t} - H_\alpha)$ belongs to $\mathfrak{a} \setminus {}^+\mathfrak{a}$ for $t > 0$ sufficiently large. Then (148) implies that the function $t \to g(H_0 - tH_\alpha)$ is a constant C for large $t > 0$ so by (147)

$$f(H_0 - tH_\alpha) = Ce^{-C_0\alpha(H_0)}e^{C_0\alpha(H_\alpha)t}\,. \tag{149}$$

This contradicts $f \in \mathcal{S}(\mathfrak{a})$ unless $f \equiv 0$ on $\mathfrak{a} \setminus {}^+\mathfrak{a}$. Iterating this argument for the factors in (146) we deduce

$$\zeta_{pi} \equiv 0 \quad \text{on } \mathfrak{a} \setminus {}^+\mathfrak{a}\,, \quad 1 \le i\,,\, p \le \ell(\delta)\,. \tag{150}$$

Observe now that for each $k_0 \in K$ assumption (128) is valid for the function $u_{k_0} : (x, H) \to u(k_0 \cdot x, H)$. Since $A(k_0 \cdot x, b) = A(x, k_0^{-1} \cdot b)$ it is clear from (118) that replacing u by u_{k_0} amounts to replacing h_σ by the function $(b, a) \to h_\sigma(k_0 \cdot b, a)$. Thus (150) implies

$$(151) \qquad \int_{K/M} h_\sigma(k_0 kM, \exp H)\langle v_i, \delta(k)v_p\rangle \, dk_M = 0 \,, \quad H \in \mathfrak{a} \setminus {}^+\mathfrak{a} \,,$$

and here we put $u = k_0 k \,, k = k_0^{-1}u$. Since the vectors $\delta(k_0)v_i \, (k_0 \in K, 1 \leq i \leq \ell(\delta))$ span V_δ, (151) shows that the function $uM \to h_\sigma(uM, \exp H)$ is orthogonal to all $\langle v, \delta(k)v_p\rangle (1 \leq p \leq \ell(\delta), v \in V_\delta)$. Since $\delta \in \widehat{K}_M$ is arbitrary this proves

$$h_\sigma(b, \exp H) = 0 \quad \text{for } b \in B \,, H \in \mathfrak{a} \setminus {}^+\mathfrak{a}$$

so (129) is proved.

10. Energy and Spectral Representation.

We now restate the Cauchy Problem (93)–(94) into a vector formulation and use this to generalize the classical energy for the wave equation to the present context. We recall the definition.

Let $p \in S(\mathfrak{a})$. By Lemma 5.20 we have a matrix $L_p = (L_p)_{ij}$ with entries in $I(\mathfrak{a})$ such that

$$pp_j = \sum_1^w (L_p)_{ij} p_i \,, \quad 1 \leq j \leq w \,.$$

Let $D_p = (D_p)_{ij}$ denote the corresponding matrix with entries in $\mathbf{D}(G/K)$, i.e., $\Gamma((D_p)_{ij}) = (L_p)_{ij}$. Let ${}^tD_p$ denote the transpose of D_p.

Lemma 5.33. *The Cauchy problem (93)–(94) is equivalent to the problem*

$$(152) \qquad {}^tD_p \, \mu = \partial(p)\mu \,, \quad p \in S(\mathfrak{a}) \,,$$

where the column vector μ is a smooth map from $X \times \mathfrak{a}$ to $\mathbf{C}^w$ and

$$(153) \qquad {}^t\mu(x, 0) = (f_1(x), \ldots, f_w(x)) \,.$$

More precisely, if u satisfies (93)–(94) then

$$(154) \qquad {}^t\mu(x, H) = ((\partial(p_1)u)(x, H), \ldots, (\partial(p_w)u)(x, H))$$

satisfies (152)–(153) above and if μ satisfies (152)–(153) then $u = \mu_1$ (first component) satisfies (93)–(94).

Proof. Since $p_k = p_k p_1 = \sum_i (L_{p_k})_{i1} p_i$ we have $(L_{p_k})_{i1} = \delta_{ik}$. Thus, if μ satisfies (152)–(153) then

$$\partial(p_k)\mu_1 = (\partial(p_k)\mu)_1 = ({}^tD_{p_k}\mu)_1 = \sum_i (D_{p_k})_{i1}\mu_i = \mu_k$$

so (94) holds. Also, if $D \in \mathbf{D}(G/K)$, $L_{\Gamma(D)}$ is a diagonal matrix, ${}^tD_{\Gamma(D)} = DI_w$ so (152) implies $D\mu_1 = \partial(\Gamma(D))\mu_1$. Thus $u = \mu_1$ satisfies (93).

On the other hand, assume u satisfies (93)–(94). Define μ by (154). Then if $p \in S(\mathfrak{a})$,

$$\begin{aligned}(\partial(p)\mu)_j &= \partial(p)\mu_j = \partial(p)\partial(p_j)u = \partial(pp_j)u \\ &= \partial\Big(\sum_i (L_p)_{ij}p_i\Big)(u) = \sum_i \partial(p_i)\partial(\Gamma(D_p)_{ij})u \\ &= \sum_i \partial(p_i)(D_p)_{ij}u = \sum_i (D_p)_{ij}\partial(p_i)u = ({}^tD_p\mu)_j\end{aligned}$$

so (152) holds; also (153) is obvious.

Remark. If (152) holds for $p \in \mathfrak{a}$ it holds for all $p \in S(\mathfrak{a})$.

In fact, the map $p \to L_p$ is an isomorphism of $S(\mathfrak{a})$ onto an algebra of $w \times w$ matrices with entries in $I(\mathfrak{a})$. Thus $\partial(pq)\mu = {}^tD_{pq}\mu$, justifying the remark.

Consider now the matrix $A = (A_{ij})$ with entries in $I(\mathfrak{a})$ given by

$$A_{ij} = (\pi q^j\,, \theta(\pi q^i))\,, \tag{155}$$

where $(\,,)$ is defined in (91). Let $\mathcal{A} = (\mathcal{A}_{ij})$ be the matrix with entries in $\mathbf{D}(G/K)$ given by $\Gamma(\mathcal{A}_{ij}) = A_{ij}$.

Definition. Given $u, v \in \mathcal{E}(X \times \mathfrak{a})$ the *energy form* is defined by

$$E(u,v;H) = \int_X ({}^t\mu\mathcal{A}\overline{\nu})(x,H)\,dx\,, \tag{156}$$

the integral assumed convergent. Here ${}^t\mu, \nu$ are given by (133) from u and v, respectively.

It is a routine matter to verify that the function ${}^t\mu\mathcal{A}\overline{\nu}$ is independent of the choice of basis of $H(\mathfrak{a})$. However, the homogeneity of each p_i plays a role in the proof of Theorem 5.34 and in Theorem 5.35.

In the rank-one example discussed before this reduces to

$$E(u,v;tH_0) = \frac{\langle\pi,\pi\rangle}{2}\int_X \big(u_t\overline{v}_t - u(L+|\rho|^2)\overline{v}\big)\,dx$$

which up to the factor $\langle\pi,\pi\rangle$ agrees with the usual formula in (70) and (72).

As used in the proof of (96) we have

$$\Gamma(D^*) = \theta\Gamma(D) \qquad D \in \mathbf{D}(G/K)\,.$$

Hence

$$\Gamma(\mathcal{A}_{ij}^*)(i\lambda) = \Gamma(\mathcal{A}_{ij})(-i\lambda) = A_{ij}(-i\lambda) = A_{ji}(i\lambda) = \Gamma(\mathcal{A}_{ji})(i\lambda)$$

so $\mathcal{A}_{ij}^* = \mathcal{A}_{ji}$. This implies the Hermitian symmetry,

$$E(u,v;H) = \overline{E(v,u;H)}\,.$$

While this relation is a simple formality, the positivity

$$E(u,u;H) \geq 0$$

which is a consequence of (162) below in connection with Theorem 5.34 is more subtle and seems to require the Fourier transform theory. Because of (162) we have a norm

$$\|F\| = \left\{ \int\limits_X ({}^tF\mathcal{A}\overline{F})(x)\,dx \right\}^{1/2}, \ {}^tF = (f_1, \ldots, f_w)$$

on $\mathcal{D}(X) \times \cdots \times \mathcal{D}(X)$ (w times) which by the injectivity in Theorem 5.36 is strictly positive for $F \not\equiv 0$.

Theorem 5.34. *Let u and v be the solutions to (93)–(94) with initial data $(f_i) \subset \mathcal{D}(X)$, $(g_j) \subset \mathcal{E}(X)$. (Theorem 5.22). Then the energy form $E(u,v;H)$ is constant in the time $H \in \mathfrak{a}$.*

Proof. Consider the matrix $B = (B_{ij})$ given by

$$B_{ij} = (\pi q^i, \pi q^j) \in I(\mathfrak{a})$$

and J the diagonal matrix whose i^{th} entry is $(-1)^{d_i}$ where $d_i = \deg p_i = -\deg(q^i)$. Then $A_{ij} = (\pi q^j, \theta(\pi q^i)) = (-1)^{\deg(\pi)+d_i} B_{ij}$ so if $\mathcal{B} = (\mathcal{B}_{ij})$ with $\Gamma(\mathcal{B}_{ij}) = B_{ij}$,

$$(-1)^{\deg(\pi)} JB = A\,, \quad (-1)^{\deg(\pi)} J\mathcal{B} = \mathcal{A}\,.$$

Taking inner product with q^m and p_k, respectively, we derive the relations

$$\sum_i B_{ik} p_i = \pi^2 q^k\,,\ pq^j = \sum_i (L_p)_{ji} q^i\,, \quad p \in S(\mathfrak{a})\,,$$

which imply quickly that the matrix L_pB is symmetric. Thus

$$L_pB = B\,{}^tL_p\,,\ D_p\mathcal{B} = \mathcal{B}\,{}^tD_p\,.$$

In particular, take $p = H \in \mathfrak{a}$ and observe that $\partial(p)\,{}^t\mu = {}^t(\partial(p)\mu)$. Then if u and v are solutions and μ, ν determined by (154) we have (by (152))

$$\begin{aligned}\partial(p)_H(E(u,v;H)) &= \int_X \left[{}^t({}^tD_p\mu)\mathcal{A}\overline{\nu} + {}^t\mu\mathcal{A}\partial(p)\overline{\nu}\right](x,H)\,dx \\ &= (-1)^{\deg(\pi)}\int_X \left[{}^t({}^tD_p\mu)J\mathcal{B}\overline{\nu} + {}^t\mu J\mathcal{B}\,{}^tD_p\overline{\nu}\right](x,H)\,dx\,.\end{aligned}$$

In the last term we replace $\mathcal{B}\,{}^tD_p$ by $D_p\mathcal{B}$ and put $\xi = \mathcal{B}\overline{\nu}$. The integrand is a sum of the terms

$$\begin{aligned}\sum_j ({}^t({}^tD_p\mu))_j(J\xi)_j &= \sum_{i,j}(D_p)_{i,j}\mu_i(-1)^{d_j}\xi_j \\ \sum_i ({}^t\mu)_i(JD_p\xi)_i &= \sum_{i,j}\mu_i(-1)^{d_i}(D_p)_{ij}\xi_j\end{aligned}$$

so it would suffice to prove

$$\int_X \left[(D_p)_{ij}\mu_i\xi_j + (-1)^{d_i-d_j}\mu_i(D_p)_{ij}\xi_j\right](x,H)\,dx = 0\,.$$

Since $pp_j = \sum_i (L_p)_{ij}p_i$, $(L_p)_{ij}$ is either 0 or homogeneous of degree $1 + d_j - d_i$. Also $\Gamma(D^*) = \theta\Gamma(D)$ as used before, so

$$\Gamma((D_p)_{ij}{}^*) = \theta((L_p)_{ij}) = (-1)^{1+d_j-d_i}\Gamma((D_p)_{ij})\,,$$

whence

$$(D_p)_{ij}{}^* = (-1)^{1+d_j-d_i}(D_p)_{ij}\,.$$

Thus the last integral equals

$$\int_X \left[(D_p)_{ij}\mu_i\xi_j - \mu_i(D_p)_{ij}{}^*\xi_j\right](x,H) = 0\,.$$

Hence $\partial(p)_H(E(u,v;H)) = 0$ as claimed.

We consider now for each $\sigma \in W$, $\lambda \in \mathfrak{a}^*_C$, $b \in B$ the particular solution $\mu^\sigma(x,H;\lambda,b)$ of (152) given by (154) for

$$u^\sigma(x,H;\lambda,b) = e^{i\lambda(H)+(i\sigma\lambda+\rho)(A(x,b))}\,. \tag{157}$$

This is indeed a solution for

$$(Du^\sigma) = \Gamma(i\sigma\lambda)u^\sigma = \Gamma(i\lambda)u^\sigma\,, \quad D \in \mathbf{D}(G/K)$$

and

$$\partial(\Gamma(D))u^\sigma = \Gamma(i\lambda)u^\sigma\,.$$

We consider then the linear map (*the spectral representation*)

$$\mathcal{E}^\sigma : F(x) \to \int_X ({}^tF\mathcal{A}\overline{\mu}^\sigma)(x,0\,;\lambda,b)\,dx\,, \tag{158}$$

where ${}^tF(x) = (f_1(x),\dots f_w(x))$, $f_i \in \mathcal{D}(X)$. Thus $\mathcal{E}^\sigma$ maps $\mathcal{D}(X)\times\dots\times\mathcal{D}(X)$ into a function space on $\mathfrak{a}^*\times B$. The following result shows that $\mathcal{E}^\sigma$ is intimately related to the Fourier transform on X.

Theorem 5.35. *For each* $\sigma\in W$

$$\mathcal{E}^\sigma(F)(\lambda,b) = \pi(\lambda)^2\sum_k q^k(i\lambda)\widetilde{f}_k(\sigma\lambda,b)\,.$$

Proof. For $\lambda\in\mathfrak{a}^*$ (real) we have

$$\mu_j^\sigma(x,0\,;\lambda,b)^- = p_j(-i\lambda)e^{(-i\sigma\lambda+\rho)(A(x,b))}$$

so

$$\mathcal{E}^\sigma(F) = \int_X\sum_{i,j} f_i(x)A_{ij}(-i\sigma\lambda)p_j(-i\lambda)e^{(-i\sigma\lambda+\rho)(A(x,b))}\,dx\,, \tag{159}$$

which by the W-invariance of A_{ij} equals

$$\sum_k\widetilde{f}_k(\sigma\lambda,b)\sum_j A_{kj}(-i\lambda)p_j(-i\lambda)\,. \tag{160}$$

Here the sum $\sum_j$ equals

$$\begin{aligned}&\sum_j\Big[\sum_{\tau\in W}(\pi q^j)(-\tau i\lambda)(\pi q^k)(\tau i\lambda)\Big]p_j(-i\lambda)\\ &=\sum_{\tau\in W}(\pi q^k)(\tau i\lambda)\sum_j(\pi q^j)(-\tau i\lambda)p_j(-i\lambda)\,.\end{aligned}$$

Now if $d_j =$ degree p_j then degree $q^j = -d_j$ so the last $\sum_j$ equals

$$(-1)^{\deg\pi}\sum_j(\pi q^j)(\tau i\lambda)p_j(i\lambda)\,.$$

Thus

$$\begin{aligned}\sum_j A_{kj}(-i\lambda)p_j(-i\lambda) &= (-1)^{\deg(\pi)}\sum_j p_j(i\lambda)\sum_\sigma(\pi q^k)(\sigma i\lambda)(\pi q^j)(\sigma i\lambda)\\ &= (-1)^{\deg(\pi)}\sum_j p_j(i\lambda)(\pi q^k\,,\pi q^j)(i\lambda)\,.\end{aligned}$$

However, $\sum_j p_j(\pi q^k, \pi q^j) = \pi^2 q^k$ as we see by taking inner product with q^m and noting that $(\pi^2 q^k, q^m) = (\pi q^k, \pi q^m)$. Putting these formulas together we see that expression (160) equals

$$\sum_k \widetilde{f}_k(\sigma\lambda, b)(-1)^{\deg \pi}\pi(i\lambda)^2 q^k(i\lambda)$$

and this proves the theorem.

Given $H_0 \in \mathfrak{a}$ let U_{H_0} denote the map of $\mathcal{D}(X) \times \ldots \times \mathcal{D}(X)$ into itself given by

$$U_{H_0} : \mu(x, 0) \to \mu(x, H_0)\,, \tag{161}$$

μ satisfying (152). Since $\partial(p)$ is invariant under translations of $\mathfrak{a}$ it follows that for each $H \in \mathfrak{a}$, U_{H_0} maps the function $x \to \mu(x, H)$ into the function $x \to \mu(x, H + H_0)$.

In view of Theorem 5.35 the following result can be viewed as an enlargement of the Plancherel theorem for the Fourier transform on X. In fact it will be proved by the tools described in Ch. III, §5.

Theorem 5.36. *For each $\sigma \in W$ the map $\mathcal{E}^\sigma$ is an injective norm-preserving map of $\mathcal{D}(X) \times \ldots \times \mathcal{D}(X)$ onto a dense subspace of $L^2(\mathfrak{a}^* \times B,\, d\lambda\, db/\left|\pi(\lambda)\mathbf{c}(\lambda)\right|^2)$. Thus*

$$\int_{\mathfrak{a}^* \times B} \mathcal{E}^\sigma(F)\overline{\mathcal{E}^\sigma(F)}(\lambda, b)\frac{d\lambda\, db}{\left|\pi(\lambda)\mathbf{c}(\lambda)\right|^2} = \int_X ({}^tF\mathcal{A}\overline{F})(x)\, dx\,. \tag{162}$$

Moreover, $\mathcal{E}^\sigma$ intertwines U_{H_0} and the endomorphism

$$e(H_0) : \varphi(\lambda, b) \to e^{i\lambda(H_0)}\varphi(\lambda, b) \quad \text{of } L^2(a^* \times B, \frac{d\lambda\, db}{\left|\pi(\lambda)\mathbf{c}(\lambda)\right|^2})\,,$$

i.e.,

$$\mathcal{E}^\sigma U_{H_0} = e(H_0)\mathcal{E}^\sigma\,.$$

Proof. We separate the various statements.

1. *The norm identity* (162). We have by Theorem 5.35,

$$\mathcal{E}^\sigma(F)\overline{\mathcal{E}^\sigma(F)} = \pi(\lambda)^4 \sum_{k,\ell} q^k(i\lambda)q^\ell(-i\lambda)\widetilde{f}_k(\sigma\lambda, b)\overline{\widetilde{f}_\ell(\sigma\lambda, b)} \tag{163}$$

and by (12) in Ch. III, §1,

$$\int \widetilde{f}_k(\sigma\lambda, b)\overline{\widetilde{f}_\ell(\sigma\lambda, b)}\, db$$

is independent of σ. Thus the left hand side of (162) equals

$$\frac{1}{w}\int\limits_{\mathfrak{a}^*\times B}\sum_{k,\ell}\pi(\lambda)^2\widetilde{f}_k(\lambda,b)\overline{\widetilde{f}_\ell(\lambda,b)}\sum_{\sigma\in W}q^k(\sigma i\lambda)q^\ell(-\sigma i\lambda)\frac{d\lambda db}{|\mathbf{c}(\lambda)|^2}\,.$$

By writing $\pi(\lambda)^2=\pi(\sigma i\lambda)\pi(-\sigma i\lambda)$ our expression becomes

$$\frac{1}{w}\int\limits_{\mathfrak{a}^*\times B}\sum_{k,\ell}\widetilde{f}_k(\lambda,b)\overline{\widetilde{f}_\ell(\lambda,b)}A_{\ell k}(i\lambda)\frac{d\lambda\,db}{|\mathbf{c}(\lambda)|^2}\,. \tag{164}$$

On the other hand if $g\in\mathcal{D}(X)$ we have the inversion formula

$$g(x)=\frac{1}{w}\int\limits_{\mathfrak{a}^*\times B}\widetilde{g}(\lambda,b)e^{(i\lambda+\rho)(A(x,b))}\,|\mathbf{c}(\lambda)|^{-2}\;d\lambda\,db\,.$$

Also $A_{\ell k}$ is real so $\mathcal{A}_{\ell k}$ has real coefficients and

$$(\mathcal{A}_{\ell k}g)^-=\mathcal{A}_{\ell k}\overline{g}\,,\quad \Gamma(\mathcal{A}_{\ell k})(i\lambda)=A_{\ell k}(i\lambda)\,. \tag{165}$$

By the Plancherel formula and (165)

$$\int\limits_X f(x)(\mathcal{A}_{k\ell}\overline{g})(x)\,dx=\frac{1}{w}\int\limits_{\mathfrak{a}^*\times B}\widetilde{f}(\lambda,b)\Big(A_{k\ell}(i\lambda)\widetilde{g}(\lambda,b)\Big)^-\;|\mathbf{c}(\lambda)|^2\;d\lambda\,db\,.$$

However, the definition (155) shows easily that for λ real, $\overline{A_{k\ell}(i\lambda)}=A_{\ell k}(i\lambda)$ and we introduce this in the last integral. Taking now $f=f_k$, $g=f_\ell$ and summing on k,ℓ we see that the right hand side of (162) equals (164). This proves (162).

2. *The injectivity.* For this suppose $\mathcal{E}^\sigma(F)=0$. Then by Theorem 5.35

$$\sum_k q^k(i\lambda)\widetilde{f}_k(\sigma\lambda,b)=0\,. \tag{166}$$

Here we multiply by $e^{(i\sigma\lambda+\rho)(A(x,b))}$ and integrate over B. Since this integral

$$\int\limits_B\widetilde{f}_k(\lambda,b)e^{(i\lambda+\rho)(A(x,b))}\,db \tag{167}$$

is W-invariant in λ we derive from (166) for each $s\in W$,

$$\sum_k q^k(si\lambda)\int\limits_B\widetilde{f}_k(\lambda,b)e^{(i\lambda+\rho)(A(x,b))}\,db=0\,.$$

We multiply by $p_\ell(si\lambda)$ and sum on s. This implies that (167) vanishes identically in x. By the injectivity (see Ch. III, Theorem 4.4) of the Poisson transform for $\lambda \in \mathfrak{a}^*$ we have $\widetilde{f}_k(\lambda, b) \equiv 0$ whence by the injectivity of the Fourier transform, $f_k \equiv 0$.

The proof has the follwing consequence (Schlichtkrull).

Corollary 5.37.

$$f_k(x) = \int\limits_{\mathfrak{a}^* \times B} p_k(i\lambda)\mathcal{E}^\sigma(F)(\lambda, b)e^{(i\sigma\lambda+\rho)(A(x,b))}\frac{d\lambda\, db}{\pi(\lambda)^2\,|\mathbf{c}(\lambda)|^2}\,.$$

In fact, substitute for $\mathcal{E}^\sigma(F)$ from Theorem 5.35 and use the W-invariance of (167). Again, replacing λ by $s\lambda$ and summing on s the right hand side reduces to $f_k(x)$ because of the inversion formula.

3. *The surjectivity.* Because of Theorem 5.35 this amounts to showing that if $g(\lambda, b)$ satisfies

$$\int\limits_{\mathfrak{a}^* \times B} \widetilde{f}(\sigma\lambda, b)\pi(\lambda)^2 q^j(i\lambda)g(\lambda, b)\frac{d\lambda\, db}{|\pi(\lambda)\mathbf{c}(\lambda)|^2} = 0$$

for all $f \in \mathcal{D}(X)$, and all $j (1 \leq j \leq w)$ then $g \equiv 0$. Putting $g^j(\lambda, b) = q^j(\sigma^{-1}i\lambda)g(\sigma^{-1}\lambda, b)$ we then have

$$\int\limits_{\mathfrak{a}^* \times B} \widetilde{f}(\lambda, b)g^j(\lambda, b)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda\, db = 0\,,\ f \in \mathcal{D}(X)\,,\ 1 \leq j \leq w\,. \tag{168}$$

If we had this relation on $\mathfrak{a}^*_+ \times B$ the problem would be solved since by the Plancherel theorem the functions $\widetilde{f}(\lambda, b)$ are dense in $L^2(\mathfrak{a}^*_+ \times B)$, $|\mathbf{c}(\lambda)|^{-2}\,d\lambda\, db$. The fact that (168) holds for all j is decisive in passing from $\mathfrak{a}^*_+ \times B$ to $\mathfrak{a}^* \times B$.

Once again we invoke the representations $\delta \in \widehat{K}_M$ and the results for the δ-spherical transform described in Ch. III, §2. We specialize f in (168) to $f \in \mathcal{D}_{\check{\delta}}(X)$ and put

$$g^j_\delta(\lambda) = \int\limits_K g^j(\lambda, kM)\delta(k)\,dk\,,\ g_\delta(\lambda) = \int\limits_K g(\lambda, kM)\delta(k)\,dk\,. \tag{169}$$

If A, $B(k) \in \mathrm{Hom}(V_\delta, V_\delta)$ then

$$\int\limits_K \mathrm{Tr}(AB(k))\,dk = \mathrm{Tr}\big(A\int\limits_K B(k)\,dk\big)\,.$$

Using in addition $\mathfrak{a}^* = \cup_{s\in W} s\mathfrak{a}^*_+$ as well as formula (48) (Ch. III, §5,) we derive from (168) the relation

$$\sum_{s\in W}\int_{\mathfrak{a}^*_+} \operatorname{Tr}(\widetilde{f}(s\lambda)g^j_\delta(s\lambda))\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda = 0\,. \tag{170}$$

Now according to Ch. III, Cor. 5.14 and (95) this section,

$$\widetilde{f}(s\lambda) = Q^{\check{\delta}}(s\lambda)F_\delta(\lambda) = Q^{\check{\delta}}(s\lambda)C_{\check{\delta}}C_{\check{\delta}}^{-1}F_\delta(\lambda)\,, \tag{171}$$

where F_δ is a W-invariant holomorphic function on $\mathfrak{a}^*_C$, rapidly decreasing on $\mathfrak{a}^*$ and with values in $\operatorname{Hom}(V_\delta, E_{\check{\delta}})$. We insert the expression (171) into (170). The map $Q^{\check{\delta}}(s\lambda)C_{\check{\delta}}$ is a linear transformation of V^M_δ into itself, but we extend it to V_δ defining it 0 on the orthocomplement of V^M_δ in V_δ. Then $g^j_\delta(s\lambda)$ can be shifted to the left in (170) and we obtain

$$\int_{\mathfrak{a}^*_+} \operatorname{Tr}(G_j(\lambda)F^o_\delta(\lambda))\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda = 0\,, \tag{172}$$

where

$$G_j(\lambda) = \sum_{s\in W} g^j_\delta(s\lambda)Q^{\check{\delta}}(s\lambda)C_{\check{\delta}}\,,\quad \lambda\in\mathfrak{a}^*\,. \tag{173}$$

$$F^o_\delta(\lambda) = C_{\check{\delta}}^{-1}F_\delta(\lambda)\,,\quad \lambda\in\mathfrak{a}^*\,. \tag{174}$$

Here $F^o_\delta(\lambda)$ maps V_δ into V^M_δ. By the Paley–Wiener theorem for the δ-spherical transform (Cor. 5.14, Ch. III) we can for each i and k $(1\le i\le \ell(\delta))$, $(1\le k\le d(\delta))$ take $F^o_\delta(\lambda) = \varphi(\lambda)P^{ik}$ where $\varphi(\lambda)$ is an arbitrary W-invariant holomorphic scalar-valued function of exponential type on $\mathfrak{a}^*_C$, rapidly decreasing on $\mathfrak{a}^*$, and $P^{ik}(v_m) = \delta_{km}v_i \quad (1\le m\le d(\delta))$. Then

$$\operatorname{Tr}(G_j(\lambda)F^o_\delta(\lambda)) = \varphi(\lambda)G_j(\lambda)_{ki}\,.$$

Thus by (172)

$$G_j(\lambda)_{ki} = 0 \quad 1\le i\le \ell(\delta)\,,\quad 1\le k\le d(\delta)\,, \tag{175}$$

for almost all $\lambda\in\mathfrak{a}^*_+$, hence by the W-invariance for almost all $\lambda\in\mathfrak{a}^*$. This means that $G_j(\lambda)$ maps V^M_δ into 0 and by definition it is 0 on $(V^M_\delta)^\perp$. Thus

$$G_j(\lambda) = 0 \text{ for almost all } \lambda\in\mathfrak{a}^*\,,\ 1\le j\le w\,. \tag{176}$$

We now have by (169)

$$g^j_\delta(s\lambda) = q^j(\sigma^{-1}si\lambda)g_\delta(\sigma^{-1}s\lambda)\,,\quad 1\le j\le w\,. \tag{177}$$

If $a_{mn}(\lambda)$ is an arbitrary matrix entry in $g_\delta(\sigma^{-1}\lambda)Q^{\check{\delta}}(\lambda)C_{\check{\delta}}$ and $s_1,\dots,s_w$ the elements in W then by (176)–(177),

$$\sum_{r=1}^{w} q^j(\sigma^{-1}s_r i\lambda)a_{mn}(s_r\lambda)=0\,,\quad 1\le j\le w\,. \tag{178}$$

By its definition the matrix $\big\{q^j(s_r\nu)\big\}_{1\le j,r\le w}$ is non-singular for regular ν so (178) implies

$$g_\delta(\sigma^{-1}\lambda)Q^{\check{\delta}}(\lambda)C_{\check{\delta}}=0\quad\text{for almost all }\lambda\,.$$

On $V_{\check{\delta}}^M$, $\det(Q^{\check{\delta}}(\lambda)C_{\check{\delta}})\not\equiv 0$ so

$$g_\delta(\sigma^{-1}\lambda)v=0\quad\text{for }v\in V_{\check{\delta}}^M\,. \tag{179}$$

However, if $u\in V_\delta$ we have for $m\in M$

$$\begin{aligned} g_\delta(\lambda)u &= \int_K g(\lambda,kM)\delta(k)u\,dk\\ &=\int_K g(\lambda,kM)\delta(km)u\,dk=\int_M g(\lambda,kM)\delta(k)\,dk\int_M \delta(m)u\,dm\end{aligned}$$

which vanishes by (179). Thus $g_\delta(\lambda)=0$ for all $\delta\in\widehat{K}_M$ whence $g(\lambda,b)=0$ for almost all $\lambda\in\mathfrak{a}^*$ as desired.

4. *Intertwining.* Putting ${}^tF=(f_1,\dots,f_w)$ we have by (161),

$$U_{H_0}(F)(x)={}^t\mu(x,H_0)=\big(\partial(p_1)u,\dots,\partial(p_w)u\big)(x,H_0)\,.$$

Taking Fourier transforms in (101) and then using (102) we get

$$\begin{aligned}(\partial(p_k)u)(\cdot,H_0)^{\sim}(\lambda,b)&=\partial(p_k)_{H_0}\big(\sum_j \widetilde{f}_j(\lambda,b)\widetilde{S}^j_{H_0}(\lambda,b)\big)\\ &=\sum_j\widetilde{f}_j(\lambda,b)\sum_{\tau\in W}q^j(i\tau\lambda)p_k(i\tau\lambda)e^{i\tau\lambda(H_0)}\,.\end{aligned}$$

Using Theorem 5.35 we thus find

$$\mathcal{E}^\sigma U_{H_0}(F)=\pi(\lambda)^2\sum_k q^k(i\lambda)\sum_j\widetilde{f}_j(\sigma\lambda,b)\sum_{\tau\in W}q^j(i\tau\sigma\lambda)p_k(i\tau\sigma\lambda)e^{i\tau\sigma\lambda(H_0)}\,. \tag{180}$$

By construction the matrices

$$e_{ij}=p_j(s_i\lambda)\qquad f_{k\ell}=q^k(s_\ell\lambda)$$

are inverses. Summing on the index k in the right hand side of (180) we see that

$$\sum_k q^k(i\lambda)p_k(i\tau\sigma\lambda) = 0 \quad \text{unless} \quad \tau\sigma = e\,.$$

Thus (180) implies by $\langle q^j, p_k\rangle = \delta_{jk}$,

$$\begin{aligned}\mathcal{E}^\sigma U_{H_0}(F)(\lambda, b) &= \pi(\lambda)^2 e^{i\lambda(H_0)} \sum_j \widetilde{f}_j(\sigma\lambda, b) q^j(i\lambda) \\ &= e(H_0)\mathcal{E}^\sigma(F)(\lambda, b)\end{aligned}$$

as stated.

Because of Theorem 5.35 it is natural to describe the image of the spectral representation $\mathcal{E}^\sigma$ by means of the Paley–Wiener theorem for the Fourier transform on X. Let $\mathcal{H}(\mathfrak{a}^* \times B)$ be the space of functions satisfying (2) in Ch. III, §5 for some R.

Theorem 5.38. *The range of $\mathcal{D}(X) \times \cdots \times \mathcal{D}(X)$ under $\mathcal{E}^\sigma$ is the set of functions $\varphi(\lambda, kM)$ on $\mathfrak{a}^*_{\mathbf{c}} \times B$ satisfying*

(i) $\varphi \in \mathcal{H}(\mathfrak{a}^* \times B)\pi$.

(ii) *For each $\delta \in \widehat{K}_M$ the function*

$$\lambda \to Q^{\check{\delta}}(\sigma\lambda)^{-1} \int_K \varphi(\lambda, kM)\delta(k^{-1})\, dk$$

is holomorphic on $\mathfrak{a}^*_{\mathbf{c}}$.

Proof. We first prove the theorem with $\sigma = e$. Let $\mathcal{E} = \mathcal{E}^e$. In the proof we invoke the δ-spherical transform (47) in Ch. III, §5. For $f \in \mathcal{D}(X)$ let $f^\delta \in \mathcal{D}(X, \mathrm{Hom}(V_\delta, V_\delta^M))$ and χ_s be as in Ch. III, §5. Then

$$\mathrm{Tr}(f^\delta) = d(\delta)\chi_\delta * f \in \mathcal{D}_{\check{\delta}}(X) \tag{181}$$

and

$$\widetilde{\mathrm{Tr}(f^\delta)}(\lambda) = (\widetilde{f})^\delta(\lambda)\,, \tag{182}$$

where on the left we have the δ-spherical transform and on the right we have defined for each function φ on $\mathfrak{a}^*_{\mathbf{c}} \times B$

$$\varphi^\delta(\lambda) = d(\delta) \int_K \varphi(\lambda, kM)\delta(k^{-1})\, dk\,.$$

Let $F = (f_1, \ldots, f_w) \in \mathcal{D}(X) \times \cdots \times \mathcal{D}(X)$ and put $\varphi = \mathcal{E}(F)$. Then (i) is obvious from Theorem 5.35. Moreover, (ii) is satisfied, since by Theorem 5.11, the function $Q^{\check{\delta}}(\lambda)^{-1}\widetilde{\mathrm{Tr}(f^\delta)}(\lambda)$ is holomorphic, and by (182) the latter function equals $Q^{\check{\delta}}(\lambda)^{-1}\widetilde{f}^\delta(\lambda)$.

Conversely, assume that a given function $\varphi(\lambda, kM)$ satisfies (i) and (ii). Define

$$f_j(x) = \int\limits_{\mathfrak{a}^* \times B} p_j(i\lambda)\varphi(\lambda, b)e^{(i\lambda+\rho)A(x,b)} \frac{d\lambda\, db}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2} \tag{183}$$

for $x \in X, j = 1, \ldots, w$. We claim that $f_j \in \mathcal{D}(X)$ and $\varphi = \mathcal{E}(F)$ where $F = (f_1, \ldots, f_w)$.

We know that $\lambda \mapsto \pi(\lambda)\mathbf{c}(\lambda)$ is nonzero on $\mathfrak{a}^*$ and that its reciprocal satisfies a polynomial estimate. It then follows from the uniform, rapid decrease of $\varphi(\cdot, b)$ that (183) defines a smooth function on X.

An easy computation shows that

$$f_j^\delta(x) = d(\delta)\int\limits_K f_j(kx)\delta(k^{-1})\, dk = \int\limits_{\mathfrak{a}^*} p_j(i\lambda)\Phi_{\lambda,\delta}(x)\varphi^\delta(\lambda)\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}$$

where $\Phi_{\lambda,\delta}(x)$ is the generalized spherical function. The function $\varphi^\delta(\lambda)$ is divisible by $Q^{\check{\delta}}(\lambda)$ because of our assumption (ii), and we can then write

$$f_j^\delta(x) = \int\limits_{\mathfrak{a}^*} p_j(i\lambda)\Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)Q^{\check{\delta}}(\lambda)^{-1}\varphi^\delta(\lambda)\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}.$$

Averaging over W, and observing that $\lambda \mapsto \Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)$ is W-invariant (cf. Ch. III, Theorem 5.15) we obtain

(184)

$$f_j^\delta(x) = \frac{1}{w}\int\limits_{\mathfrak{a}^*} \Phi_{\lambda,\delta}(x)Q^{\check{\delta}}(\lambda)\left[\sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda)\right]\frac{d\lambda}{\pi(\lambda)^2|\mathbf{c}(\lambda)|^2}.$$

In order to prove that $\mathrm{Tr}(f_j^\delta)$ is compactly supported, it now suffices, by Ch. III, Cor. 5.14 and Theorem 5.15 to prove that the function

$$\lambda \mapsto \left[\sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda)\right]\frac{1}{\pi(\lambda)^2} \tag{185}$$

belongs to the space $\mathcal{J}^\delta(\mathfrak{a}^*)$ of W-invariant functions in $\mathcal{H}(\mathfrak{a}_\mathbf{c}, \mathrm{Hom}(V_\delta, V_\delta^M))$. Clearly, (185) is W-invariant, and the expression in square brackets is holomorphic of exponential type, uniformly in δ. It remains to be seen that it is divisible by $\pi(\lambda)^2$. We rewrite (185) as follows:

$$\left[\sum_{s\in W} \epsilon(s)p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\frac{\varphi^\delta(s\lambda)}{\pi(s\lambda)}\right]\frac{1}{\pi(\lambda)}.$$

Here the expression in square brackets is holomorphic by (i) and (ii), since $\pi(\lambda)$ and the determinant of $Q^{\check{\delta}}(\lambda)$ have no common zeroes, cf. Ch. III, Theorem 4.2 and Cor. 11.3. Moreover, being skew, it is divisible by $\pi(\lambda)$. Hence $\mathrm{Tr}(f_j^\delta) \in \mathcal{D}(X)$ with support uniform in δ, and so $f_j = \sum_\delta \mathrm{Tr}(f_j^\delta) \in \mathcal{D}(X)$.

It remains to be seen that $\mathcal{E}(F) = \varphi$. It follows from the above that the δ-spherical Fourier transform $\widetilde{\mathrm{Tr}(f_j^\delta)}$ of $\mathrm{Tr}(f_j^\delta)$ is given by $Q^{\check{\delta}}(\lambda)$ times (185). By (182) $\widetilde{\mathrm{Tr}(f_j^\delta)} = \widetilde{f}_j^\delta$ and hence

$$\begin{aligned}\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f}_j^\delta(\lambda) &= \sum_j q^j(i\lambda)Q^{\check{\delta}}(\lambda) \sum_{s\in W} p_j(is\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda)\\ &= \sum_{s\in W}\left(\sum_j q^j(i\lambda)p_j(is\lambda)\right)Q^{\check{\delta}}(\lambda)Q^{\check{\delta}}(s\lambda)^{-1}\varphi^\delta(s\lambda).\end{aligned}$$

Now $\sum_j q^j(i\lambda)p_j(is\lambda) = 1$ for $s = e$, and otherwise it vanishes; hence we obtain

$$\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f}_j^\delta(\lambda) = \varphi^\delta(\lambda).$$

Since this expression holds for all δ, we conclude that

$$\pi(\lambda)^2 \sum_j q^j(i\lambda)\widetilde{f}_j(\lambda, b) = \varphi^\delta(\lambda, b)$$

for all $b \in B$, as claimed.

We now prove the theorem with arbitrary σ. We use the notation g^σ, φ^σ for $\lambda \mapsto g(\sigma^{-1}\lambda), (\lambda, b) \mapsto \varphi(\sigma^{-1}\lambda, b)$. The proof will be based on the observation that conditions (i) and (ii) are independent of the basis $p = (p_1, \ldots, p_w)$ of $H(a)$ used.

Let $\mathcal{E}_p^\sigma$ denote the mapping $\mathcal{E}^\sigma$ with the basis p. Thus

$$\mathcal{E}_p^e(\mathcal{D} \times \cdots \times \mathcal{D}) = \mathcal{E}_{p'}^\sigma(\mathcal{D} \times \cdots \times \mathcal{D})$$

if $p' = (p_1', \ldots, p_w')$ is any homogeneous basis of $H(a)$ with $p_1' = 1$. In particular,

$$\mathcal{E}_p^e(\mathcal{D} \times \cdots \times \mathcal{D}) = \mathcal{E}_{p^\sigma}^e(\mathcal{D} \times \cdots \times \mathcal{D}). \tag{186}$$

On the other hand

$$(\mathcal{E}_p^e F)(\lambda, b) = \sum_k (q^k)^\sigma(i\sigma\lambda)\widetilde{f}_k(\sigma\lambda, b)\mathcal{E}_{p^\sigma}^e(\sigma\lambda, b)$$

so

$$\mathcal{E}_p^\sigma F = (\mathcal{E}_{p^\sigma}^e F)^{\sigma-1}. \tag{187}$$

Using (186) and (187), the proof of (ii) is easily completed.

11. The Analog of the Friedlander Limit Theorem.

We shall now prove the analog for the multitemporal wave equation of Friedlander's Theorem 5.17. The proof is a reformulation of the proof in Phillips–Shahshahani [1993].

In the analogy with (116) let Q_k, J and $\bar{J}$ denote the operators on $\mathcal{S}(A)$ that under the Fourier transform on A correspond to multiplication by $q^k(i\lambda)\pi(i\lambda)$, $c(\lambda)^{-1}$ and $c(-\lambda)^{-1}$, respectively.

Theorem 5.39. *Let $\sigma \in W$ and $H, H_0 \in \mathfrak{a}^+$, and put $a_t = \exp(tH)$. Then if $k \in K$,*

$$\lim_{t\to+\infty} e^{t\rho(H)}(\partial(\pi)u)(ka_t\cdot o, -t\sigma H + \sigma H_0) = \big(\bar{J}\sum_k Q_k^{\sigma^{-1}}(e^{\rho}\widehat{f}_k)\big)(kM, H_0)\,.$$

Proof. Because of (114)

$$(\partial(\pi)u)(x,H) = \int\limits_{\mathfrak{a}^*\times B} F_\sigma(\lambda,b)e^{(i\lambda+\rho)(A(x,b))}e^{i\sigma\lambda(H)}\frac{d\lambda\,db}{|\mathbf{c}(\lambda)|^2}\,,$$

where

$$F_\sigma(\lambda,b) = \sum_k \widetilde{f}_k(\lambda,b)q^k(i\sigma\lambda)\pi(i\sigma\lambda)\,.$$

Thus if $\ell \in K$

$$\begin{aligned}(188)\qquad & e^{t\rho(H)}(\partial(\pi)u)(\ell a_t\cdot o\,,\, -t\sigma H + \sigma H_0)\\ &= e^{t\rho(H)}\int\limits_{\mathfrak{a}^*\times B} F_\sigma(\lambda,b)e^{(i\lambda+\rho)(A(a_t\cdot o,\ell^{-1}b))}e^{-it\lambda(H)+i\lambda(H_0)}\frac{d\lambda\,db}{|\mathbf{c}(\lambda)|^2}\,.\end{aligned}$$

By the K-invariance of db, we can replace

$$(189)\qquad F_\sigma(\lambda,b)\ \text{ by }\ F_\sigma(\lambda,\ell\cdot b)\,,\quad A(a_t\cdot o,\ell^{-1}\cdot b)\ \text{ by }\ A(a_t\cdot o,b)\,.$$

In the Iwasawa decomposition $G = KAN$, we write as before

$$(190)\qquad g = k(g)\exp H(g)n(g)\,,\qquad H(g)\in\mathfrak{a}\,.$$

Then $A(gK,kM) = -H(g^{-1}k)$. If $b = k'M$ and we define $k \in K$ by $k' = k(a_tk)$ then $kM \to k'M$ is a diffeomorphism and

$$(191)\qquad dk' = e^{-2\rho(H(a_tk))}\,dk\,.$$

Since $a_tk = k'\exp H(a_tk)n$ and A normalizes N we have

$$(192)\qquad H(a_{-t}k') = -H(a_tk)\,.$$

After the substitution (189) and $A(a_t \cdot o, k'M) = -H(a_{-t}k')$ relations (191)–(192) reduce the right hand side of (188) to

$$(193) \qquad e^{t\rho(H)} \int_{\mathfrak{a}^*} e^{-i\lambda(tH-H_0)} \int_B F_\sigma(\lambda, \ell k(a_t k)M) e^{(i\lambda-\rho)(H(a_t k))} \frac{d\lambda\, dk_M}{|\mathbf{c}(\lambda)|^2}\,.$$

Now we use the map $\overline{n} \to k(\overline{n})M$ of $\overline{N}$ into K/M under which $dk_M = e^{-2\rho(H(\overline{n}))}\, d\overline{n}$ (see e.g. [GGA], Ch. I, §5) to transfer (194) to $\overline{N}$. Since A normalizes N we have

$$a\overline{n} = ak(\overline{n}) \exp H(\overline{n}) n_1 = k(ak(\overline{n})) \exp H(ak(\overline{n})) \exp H(\overline{n}) n_2$$

so we have Harish–Chandra's relations

$$(194) \qquad k(a\overline{n}) = k(ak(\overline{n})) = k(\overline{n}^a)\,, H(a\overline{n}) = H(ak(\overline{n})) + H(\overline{n})$$

$$(195) \qquad H(a\overline{n}) - \log a = H(\overline{n}^a)\,,$$

where $\overline{n}^a = a\overline{n}a^{-1}$. Thus (193) becomes

$$(196) \qquad \int_{\mathfrak{a}^*} e^{-i\lambda(tH-H_0)} \int_{\overline{N}} F_\sigma(\lambda, \ell k(\overline{n}^{a_t})M) L(a_t, \overline{n}) \frac{d\lambda\, d\overline{n}}{\mathbf{c}(\lambda)\mathbf{c}(-\lambda)}\,,$$

where the kernel $L(a, \overline{n})$ equals

$$(197) \qquad L(a, \overline{n}) = e^{(i\lambda-\rho)(H(\overline{n}^a))} e^{-(i\lambda+\rho)(H(\overline{n}))}\,.$$

Let $\lambda = \xi + i\eta(\xi, \eta \in \mathfrak{a}^*)$ and $-\eta \in \mathfrak{a}^*_+$ so small that $(\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1}$ is holomorphic in $\mathfrak{a}^* + it\eta$ ($|t| \leq 1$). Let $\epsilon > 0$ be such that

$$0 < \epsilon < 1\,, \quad \rho + \epsilon\eta \in \mathfrak{a}^*_+\,.$$

Then by [GGA], Ch. IV, §6, (41),

$$|L(a_t, \overline{n})| \leq e^{(\epsilon\eta - \rho)(H(\overline{n}))}\,,$$

which is integrable over $\overline{N}$. Since $F_\sigma$is holomorphic in λ and for a fixed η is rapidly decreasing in ξ the integral (196) can be shifted to $\mathfrak{a}^* + i\eta$. Letting then $t \to +\infty$ we get the expression

$$\int_{\mathfrak{a}^*+i\eta} e^{i\lambda(H_0)} (\mathbf{c}(\lambda)\mathbf{c}(-\lambda))^{-1} F_\sigma(\lambda, \ell M) \bigg(\int_{\overline{N}} e^{-(i\lambda+\rho)(H(\overline{n}))}\, d\overline{n}\, d\lambda \bigg)$$

$$= \int_{\mathfrak{a}^*+i\eta} e^{i\lambda(H_0)} F_\sigma(\lambda, \ell M) (\mathbf{c}(-\lambda))^{-1}\, d\lambda\,.$$

Here we can shift the integration back and take $\eta = 0$. Using the definition of $\overline{J}$, Q_k and relation (116) the theorem follows.

To summarize we now compare the flat wave equation in $\mathbf{R}^n$ with the multitemporal wave equation by the following table.

Remarkable Analogies

The Wave Equation in $\mathbf{R}^n$	**Multitemporal Wave Equation in G/K**
$Lu = \frac{\partial^2 u}{\partial t^2}$, $u(x,0) = f_0(x)$, $u_t(x,0) = f_1(x)$	$Du = \partial(\Gamma(D))u$, $\partial(p_i)u(x,0) = f_i(x)$
Radon Transform Solution $u(x,t) = \int_{\mathbf{S}^{n-1}} (Sf)(w, \langle x,w\rangle + t)\, dw$ $Sf = c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)$	**Horocycle Transform Solutions** $u(x,H) = e^{\rho(H)}$. $\int_B (S_\sigma f)(b, \exp(A(x,b) + \sigma H))e^{2\rho(A(x,b))}\, db$ $(S_\sigma f)(\xi) = \sum_{j=1}^{w} (\Lambda_{j,\sigma}\widehat{f}_j)(\xi)$.
Fourier Transform Solution $u(x,t) = (f_0 * T_t')(x,t) + (f_1 * T_t)(x,t)$ $\frac{\sin\lvert\lambda\rvert t}{\lvert\lambda\rvert} = \int_{\mathbf{R}^n} e^{-i\langle x,\lambda\rangle} dT_t(x)$	**Fourier Transform Solution** $u(x,H) = \sum_{j=1}^{w} (f_j \times S_H^j)(x)$ $\sum_\sigma q^j(i\sigma\lambda)e^{i\sigma\lambda(H)} =$ $\int_X e^{(-i\lambda+\rho)(A(x,b))} dS_H^j(x)$
Mean value solution, n odd $u(x,t) = c_n \left[\frac{\partial}{\partial t}(I_t f_0)(x) + (I_t f_1)(x)\right]$ $(I_t f)(x) = \left(\frac{\partial}{\partial(r^2)}\right)^{\frac{n-3}{2}} \left(r^{n-2}(M^r f)(x)\right)$	**Mean value solution, G complex** $u(x,H) =$ $w\sum_{j=1}^{w} \partial(h^j)_H \left[\delta^{\frac{1}{2}}(\exp H)\left(M^{\exp H} f_j\right)(x)\right]$
$u(x,t)$ incoming if $u(x,t) = 0$ in $\lvert x\rvert < -t \qquad t < 0$ For $\mathbf{R}^{2n+1}$: $u(x,t)$ is incoming if and only if $(\partial\widehat{f}_0 + \widehat{f}_1)(w,s) = 0$ for $s < 0$, all w	$u(x,H)$ σ-incoming if $u(x,H) = 0$ for $d(0,x) < \lvert H\rvert$, $\quad \sigma H \in \mathfrak{a}^-$ For $\mathbf{H}^3$: $u(x,H)$ is e-incoming if and only if $[H_0(e^\rho\widehat{f}_0) + c_n e^\rho \widehat{f}_1](b,H) = 0$ for all b, $H \in \mathfrak{a}^-$.