Preface

In the first volume of this series [$\mathbf{I}_2$, Sections 14–16], we formulated seven theorems, Theorems $\mathcal{C}_i$, $1 \leq i \leq 7$, to handle our sevenfold case division for the proof of the classification theorem for the finite simple groups. The seven rows of the "Classification Grid" [$\mathbf{I}_2$, Section 3] gave a coarse outline of the proofs of Theorems $\mathcal{C}_1$–$\mathcal{C}_7$ as we foresaw them. Together with various supporting Uniqueness theorems collected in the "Uniqueness Grid" [$\mathbf{I}_2$, Section 2], and together with a specified list of Background Results, Theorems $\mathcal{C}_1$–$\mathcal{C}_7$ imply the classification theorem. Several of the necessary Uniqueness theorems, but not all, were proved in Book 4. These included all the so-called 2-Uniqueness theorems — those necessary to complete the characterization of target groups of odd type. (Roughly speaking, the simple groups of odd type are the alternating groups, and the groups of Lie type of odd characteristic, with a small finite number of additions and subtractions.) It is important to note that the completion of the so-called Odd Uniqueness theorems, which are those Uniqueness theorems needed for target groups of even type, is a major step still to be covered in future volumes.

Theorem $\mathcal{C}_1$ is the Odd Order Theorem of Feit-Thompson, which is one of our assumed Background Results. Theorems $\mathcal{C}_2$ and $\mathcal{C}_3$, on the Special Odd case, were proved in Book 6. In Books 5, 7, and 8, we proved Theorem $\mathcal{C}_7$ (the theorem for the Generic case), modulo a new Theorem $\mathcal{C}_6^*$, which in turn is a mild strengthening of Theorem $\mathcal{C}_6$. In particular, in conjunction with the 2-Uniqueness theorems established in Book 4 (the archetype of which is the Bender-Suzuki theorem on groups containing a strongly embedded subgroup), Books 5–8 established that a minimal counterexample G to the classification theorem is of even type, indeed "restricted" even type. In fact, with a boost from Theorem $\mathcal{C}_6^*$, proved in Chapter 10 of the current volume, and Odd Uniqueness theorems, they would establish more, as we shall see presently.

In Book 9, we began the special even type analysis by proving Theorem $\mathcal{C}_5$, yielding among other targets some large sporadic groups, including the Monster. Also in that book we began the proofs of Theorems $\mathcal{C}_6$ and a variant $\mathcal{C}_6^*$. In this book we first complete the proof of Theorems $\mathcal{C}_6$ and $\mathcal{C}_6^*$, and a consequence Theorem $\mathcal{C}_6^{**}$, in Chapter 10. Theorem $\mathcal{C}_6^{**}$ is stated as follows (Chapter 10, Theorem 1.5):

THEOREM. *If G is of weak $\mathcal{LT}_p$-type, p odd, then G contains a strong p-uniqueness subgroup of component type.*

(See the Introductory section of Chapter 10 for the definition of weak $\mathcal{LT}_p$-type.)

Together with Theorems $\mathcal{C}_5$ [$\mathbf{V}_1$, p. 1] and $\mathcal{C}_7^*$ [$\mathbf{III}_1$, p. 5] from Books 5, 7, 8, and 9, these theorems yield the following milestone:

THEOREM. *Let G be a $\mathcal{K}$-proper simple group of restricted even type. If p is an odd prime such that $m_{2,p}(G) \geq 4$, then either $G \in \mathcal{K}$ or G has a strong p-uniqueness subgroup.*

Modulo Odd Uniqueness theorems, then, a minimal counterexample to the classification theorem satisfies $e(G) \leq 3$.

In Chapter 11, we consider what cases remain for the classification theorem. Now, Theorems $\mathcal{C}_5$, $\mathcal{C}_6$, and $\mathcal{C}_7$ are distinguished from one another by their hypotheses on the p-local structure of G for some single odd prime p (let us call it the "critical" prime). In these different theorems, different hypotheses are placed on the isomorphism types of components of $C_G(x)/O_{p'}(C_G(x))$ for certain elements $x \in G$ of order p (the set of all such isomorphism types is called $\mathcal{L}_p^o(G)$). In all cases the 2-local p-rank $m_{2,p}(G)$ of G is assumed to be at least 4, and indeed some 2-local subgroup of G containing almost a whole Sylow 2-subgroup of G is assumed to have p-rank at least 4. On the other hand, certain generalizations ($\mathcal{C}_6^*$ and $\mathcal{C}_7^*$) of Theorems $\mathcal{C}_6$ and $\mathcal{C}_7$ are also proved in Books 5, 7, and 8 and Chapter 10 of the current volume. These generalizations weaken the assumption $m_{2,p}(G) \geq 4$ to $m_p(G) \geq 4$ for the critical prime p. Hence a minimal counterexample G to the classification theorem must have, at this point, additional properties beyond $e(G) \leq 3$. Taking advantage of this, we formulate in Chapter 11 a new Theorem $\mathcal{C}_4^*$ to serve in the place of Theorem $\mathcal{C}_4$. We then see that the proof of Theorem $\mathcal{C}_4^*$ divides into three main cases. First, there is the "classical" Quasithin case $e(G) \leq 2$. Next is the "Bicharacteristic" case, in which $e(G) = 3$ but $m_p(G) > 3 = m_{2,p}(G)$ for some odd prime p, and every group in $\mathcal{L}_p^o(G)$ is in the explicit set $\mathcal{C}_p$ of "$\mathcal{C}hev(p)$-like" quasisimple $\mathcal{K}$-groups [$\mathbf{I}_2$, 12.1]. Last is the so-called "Case B," in which $e(G) = 3$ and $m_p(G) = 3$ for every odd prime p such that $m_{2,p}(G) = 3$.

The classical Quasithin case has been handled by a celebrated theorem of Aschbacher and Smith [**ASm1**], [**ASm2**]. One version of their result was designed to be quotable here:

THEOREM (Aschbacher, Smith). *Let G be a $\mathcal{K}$-proper simple group of even type with $e(G) \leq 2$. Then $G \in \mathcal{K}$.*

In Chapters 12–16 of the current volume, we handle the Bicharacteristic case, thereby proving:

THEOREM $\mathcal{C}_4^*$ (Case A). *Let G be a $\mathcal{K}$-proper simple group of restricted even type. Suppose that p is an odd prime such that $m_{2,p}(G) = 3 < m_p(G)$.*

Suppose also that $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$. *Then* $p = 3$ *and one of the following holds:*

(a) $G \cong A_{12}$; *or*
(b) $G \cong Co_2$, Co_3, Suz, F_3, *or* F_5; *or*
(c) $G \cong U_5(2)$, $U_6(2)$, $D_4(2)$, $Sp_8(2)$, *or* $F_4(2)$.

Note that in this theorem, as was also the case in Theorem $\mathcal{C}_5$, there is no alternative conclusion that a strong p-uniqueness subgroup exists. This contrasts with Theorems $\mathcal{C}_6^*$ and $\mathcal{C}_7^*$, and also with the future Theorem $\mathcal{C}_4^*$ (Case B).

(Note also that in Definition 1.1 of Chapter 1 of Book 5 [$\mathbf{III}_1$, p. 1], we have defined "strong p-uniqueness subgroup of G of component type" for any prime p which, if odd, satisfies $m_p(G) \geq 4$. The definition does not require $m_{2,p}(G) \geq 4$.)

In summary, the theorems to be proved in future volumes are Case B of Theorem $\mathcal{C}_4^*$ and Odd Uniqueness theorems. The change to Theorem $\mathcal{C}_4^*$ has consequences for the proofs of both of these. While we originally conceived of a mainly 2-local proof of Theorem $\mathcal{C}_4$, we now plan to use odd local analysis as well, following the fundamental groundbreaking papers of Aschbacher [**A13**], [**A24**]. In particular, Sections 15 and 22 of [$\mathbf{I}_2$] no longer fit our plans and can be replaced by Chapter 11 of this volume, which contains more precise details on Theorem $\mathcal{C}_4^*$. Furthermore, our Theorem $\mathcal{C}_4^*$ will also necessitate a strengthened version of the Odd Uniqueness theorems — Theorem and Corollary $U(\sigma)$, which in turn depend on Theorem $\mathcal{M}(S)$.

Just as we have found it necessary in other volumes of this series, we again need to expand the Background Results, this time in connection with the recognition of the sporadic group Co_2. The expansion is made precise in (9B) of Chapter 16.

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We have recently passed January 1, 2023, the centenary of the birth of Danny Gorenstein. It is fitting that we extend a special tribute to his inspiration and guidance as we continue to work towards the fulfillment of his vision.

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