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Preface

With three preliminary volumes under our belt, we at last turn directly to the proof of the Classification Theorem; that is, we begin the analysis of a minimal counterexample G to the theorem. Thus G is a $\mathcal{K}$-proper simple group. Specifically, the heart of this volume provides a collection of uniqueness and pre-uniqueness theorems for G, proved in Chapters 2 and 3.

Chapter 2 is primarily dedicated to the proof of a fundamental 2-uniqueness theorem of Michael Aschbacher, which we call Theorem ZD. Historically there is a sequence of basic papers leading up to Theorem ZD. First is a fundamental paper of Michio Suzuki [Su4] on 2-transitive permutation groups; the result of that paper is one of our Background Results [P1, Part II]. Building on that, Helmut Bender [Be3] established the Strongly Embedded Subgroup Theorem, which we call Theorem SE. Finally Aschbacher [A3] proved Theorem ZD and its corollaries, based on Bender's theorem. In both Bender's and Aschbacher's papers, the first major step is to establish that the group under investigation acts 2-transitively on a certain set. Aschbacher's arguments for this step are modelled closely on Bender's and so it seems natural to combine these proofs. This unified proof constitutes Sections 3–7 of Chapter 2, and follows the original arguments of Bender and Aschbacher closely. However, our treatment of Section 7 benefits considerably from unpublished notes of David Goldschmidt. After this point the proofs by Bender and Aschbacher diverge both in the originals and here. Indeed we follow the originals quite closely at most points, and we have made use of Peterfalvi's revision of this proof [P1] in Sections 8–11 of Chapter 2.

The remainder of Chapter 2 treats three corollaries of Theorem ZD. The classification of $\mathcal{K}$-proper simple groups with a 2-uniqueness subgroup, Theorem SU, combines a proof of Aschbacher's theorem [A3] on groups with a proper 2-generated core, but only for $\mathcal{K}$-proper simple groups, with ideas of Koichiro Harada [H1]. Finally the important Theorems SA and SF of Goldschmidt [Go5] and Holt [Ho1] are proved only for $\mathcal{K}$-proper simple groups of even type, which circumvents many of the difficulties in the original papers.

Our approach to Theorem ZD and the consequent Theorems SA and SZ in Chapter 2 is to formulate them not just for G but for an arbitrary finite group X. Thus for the major part of this chapter, we depart from our original plan regarding 2-uniqueness theorems, as announced in [I_1] and carried out in preprints preliminary to this chapter. That strategy, which remains workable, was to prove the results only for the $\mathcal{K}$-proper simple group G; in the case of Theorem SE, for example, making central use of a classification of $\mathcal{K}$-groups M whose set of involutions is permuted transitively by some subgroup of M of odd order. Indeed, such a group M can be proved either to be solvable or to have a unique composition factor of even order, which is isomorphic to $L_2(q)$ for some $q \equiv 3 \mod 4$. We have returned to an approach close to the original one of Bender and Aschbacher, for the sake of efficiency.

Chapter 3 is devoted to five "p-component pre-uniqueness theorems", Theorems PU_1–PU_5. The common theme is a maximal subgroup M which has a p-component K such that the centralizer W of $K/O_{p'}(K)$ has p-rank at least 2 and such that $C_G(y) \le M$ for every non-identity p-element of W. This type of situation will arise typically when K is maximal in some ordering on the set of p-components of centralizers of p-elements of G, and the p'-cores $O_{p'}(C_G(x))$, $x \in W$, have already been assembled, for example by the signalizer functor method or by an assumption that they are trivial. The generic conclusion is that M is a strongly p-embedded subgroup of G, which yields an immediate contradiction if $p = 2$. When p is odd and the $\mathcal{K}$-proper simple group G has even type, it will later be shown that again G does not exist. Thus the eventual import of Chapter 3 will be to establish in general that maximal p-components of centralizers of p-elements have centralizers of p-rank 1. (Of course there are counterexamples to this statement, for example when $p = 2$ and G is an alternating group.) Historically results of this type were established first for $p = 2$ by Powell and Thwaites, whose ideas are incorporated into Section 17 of Chapter 3. Shortly thereafter, Aschbacher proved his Component Theorem, a more definitive version for $p = 2$. Robert Gilman gave a somewhat different proof of Aschbacher's result. Many of the ideas of Aschbacher and Gilman also appear here, but some of their delicate analysis is replaced by a detailed consideration of $\mathcal{K}$-groups. The results proved here are new in the case that p is odd. We remark that in the proof of the first major pre-uniqueness theorem, Theorem PU_1, the case $p = 2$ is handled fairly quickly thanks to Aschbacher's criterion for a strongly embedded subgroup (Theorem ZD). Thus in Sections 7–15 of Chapter 3 the prime p is odd.

The structure and embedding of p-component uniqueness subgroups when $p = 2$ and K has 2-rank one is somewhat exceptional. In particular the main assertion of Theorem PU_4, that terminal components are standard, is not valid in this situation. In other words, K could commute elementwise with a conjugate. Historically Aschbacher and Richard Foote were able to show that there could be only one such conjugate, and we prove a similar result in Theorem PU_5. Thanks are due to Professor Foote, who suggested years ago that such a result ought to be simple to prove.

As noted above, with the exception of Theorem ZD and its corollaries Theorems SE and SZ, all the results are proved for a $\mathcal{K}$-proper simple group G. The proof thus can and does rely heavily on the theory of almost simple $\mathcal{K}$-groups established in $[\mathrm{I}_A]$. Those $\mathcal{K}$-group properties essential for Chapters 2 and 3 are collected in Chapter 4 of this volume and either extend or follow directly from the theory presented in our preceding volume $[\mathrm{I}_A]$. A much briefer Chapter 1 similarly extends our second volume $[\mathrm{I}_G]$ with some "general" (as opposed to $\mathcal{K}$-group theoretic) results pertinent to our task. Notable here is some theory of permutation groups underlying the proof of Theorem ZD.

In references (even within this volume), we shall specify the four chapters of this volume as II_1, II_2, II_3 and II_4, respectively.

We are grateful to Michael O'Nan for his assistance with the proof of Theorem 3.2 of Chapter 1; to Sergey Shpectorov for his enthusiastic support and constructive comments during the preparation of Chapter 2; to Michael Aschbacher and Helmut Bender for their support and helpful comments and suggestions; and to Inna

Korchagina for rooting out errors in the previous volume. Our deep appreciation goes to John G. Thompson, Peter Sin and Chat-Yin Ho at the University of Florida both for the many corrections and improvements which they have suggested and for the moral support which they have provided.

We also gratefully acknowledge the support of the National Science Foundation grant #DMS 97-01253; and the second author thanks the Ohio State University for its generosity and warm hospitality during his visit in the winter of 1998, during the preparation of this manuscript.

Our joy in preparing this exposition of the beautiful theory of finite simple groups has been tempered for all these years by the haunting memory of Danny Gorenstein. His conception of this project is still a sure guide for us, and we miss his companionship, insight and energy as much as ever.

During the preparation of this volume the world lost another master of finite group theory: Michio Suzuki, who took ill in the winter of 1997-1998 and died on May 31, 1998. Chapter 2 revolves around the classification of finite groups with a strongly embedded subgroup, which as we have noted was pioneered by Suzuki in the 1950's and completed by Bender over ten years later. This theorem has been a cornerstone of the theory of finite simple groups; virtually every general classification theorem in the subject has relied on it—except the Odd Order Theorem, of course. While Suzuki's fundamental contributions are to be found over many parts of the group-theoretic landscape, this particular one is surely the most far-reaching and typifies his spirited originality and powerful insight. Out of deep respect and affection we have dedicated this book to his memory.

October, 1998

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