

Preface

Volumes 5, 7, and 8 of this series form a trilogy treating the Generic Case of the classification proof. An overview of the general strategy for this set of volumes, along with a brief history of the original treatment of these results, is provided in the preface and Chapter 1 of Volume 5. We shall not repeat that here; rather, we refer the reader to Volume 5.

By the end of Volume 7, we arrived at the existence in our \mathcal{K} -proper simple group G , for a suitably chosen prime p , of one of the following:

- (a) (Alternating case) a subgroup $G_0 \leq G$ such that $G_0 \cong A_n$ for some $n \geq 13$, $p = 2$, and $\Gamma_{D,1}(G)$ normalizes G_0 for any root 4-subgroup D of G_0 . Thus for any involution $d \in G_0$ which is the product of two disjoint transpositions, $C_G(d)$ normalizes G_0 ; or
- (b) (Lie-type case) an element x of order p whose centralizer $C_G(x)$ has a p -generic quasisimple component $K = {}^d\mathcal{L}(q) \in \mathcal{C}hev(r)$, where p, r are a pair of distinct primes with q a power of r , such that either $p = 2$ and r is odd, or $r = 2$ and p divides $q^2 - 1$.

In Chapter 16 of this volume, we shall prove that in case (a), $G = G_0$, so that G is a \mathcal{K} -group, as desired. However, almost all our attention in this volume will be on case (b), where in order to catch up with case (a), we construct a subgroup $G_0 \in \mathcal{C}hev$ such that $\Gamma_{D,1}(G) := \langle N_G(Q) \mid 1 \neq Q \leq D \rangle$ normalizes G_0 for a suitable subgroup $D \cong Z_p \times Z_p$ of G .

The results of Volume 7 provide a lot more information in case (b). Thus, we know that $C_G(K)$ has a cyclic or quaternion Sylow p -subgroup, and if $p > 2$, then K itself contains a copy of $Z_p \times Z_p \times Z_p$. Significantly, also, a family of centralizers “neighboring” $C_G(x)$ also have semisimple p -layers. Considerable additional information is known about these centralizers. All this is spelled out precisely in Theorem 1.2 in Chapter 12 of this volume. [Note: The numbering of chapters in this volume continues that of Volumes 5 and 7. In particular, Chapter 12 is the first chapter of this volume.] Roughly speaking, using the terminology of the initial treatment of the Classification Theorem, we are faced with “standard form problems” for components in $\mathcal{C}hev$ whose centralizers have p -rank 1.

In this volume we complete the proof of

Theorem \mathcal{C}_7^* . *Let G be a \mathcal{K} -proper simple group. Assume that $\gamma(G) \neq \emptyset$ and that G does not possess a p -Thin Configuration for any prime $p \in \gamma(G)$. Then $G \cong G_0$ for some $G_0 \in \mathcal{K}^{(7)*}$.*

(See p. 1 and p. 4 for the definitions of the set $\mathcal{K}^{(7)*}$ of “generic” known simple groups and the set $\gamma(G)$ of primes associated to G .) In particular, in conjunction with the 2-Uniqueness Theorems in Volume 4 and the main theorems of Volume 6,

we have completed our treatment of simple groups of odd type, in the sense that we have proved the following theorem.

Theorem O. *Let G be a \mathcal{K} -proper simple group. Assume that either $m_2(G) \leq 2$ or $\mathcal{L}_2^o(G) \cap (\mathcal{T}_2 \cup \mathcal{G}_2) \neq \emptyset$. Then either G is an alternating group or $G \in \mathcal{C}hev(r)$ for some odd prime r or $G \cong M_{11}, M_{12}, J_1, Mc, Ly$, or $O'N$.*

Put another way, we have the following theorem.

Theorem. *Let G be a minimal counterexample to the Classification Theorem. Then G is of even type, i.e., the following conditions hold:*

- (a) $m_2(G) \geq 3$;
- (b) $O_{2'}(C_G(t)) = 1$ for all $t \in \mathcal{I}_2(G)$; and
- (c) $\mathcal{L}_2^o(G) \subseteq \mathcal{C}_2$.

Theorem \mathcal{C}_7^* also includes the following result when G is of even type.

Theorem GE. *Let G be a \mathcal{K} -proper simple group of even type. Assume that $\gamma(G)$ contains at least one odd prime and that G does not possess a p -Thin Configuration for any prime $p \in \gamma(G)$. Then $G \in \mathcal{C}hev(2)$.*

Historically, the portion of Theorem O covered in Volumes 5, 7, and 8 was principally the work of M. Aschbacher, J. H. Walter, D. Gorenstein, J. G. Thompson, and M. E. Harris. In particular, Chapter 14 in this volume relies at many points on arguments originally given by Aschbacher in his Classical Involution Paper [A9]. His work has been a continual source of inspiration for us.

Theorem GE was treated originally by the first two authors [GL1] in conjunction with a paper of Gilman and Griess [GiGr1]. Our work here follows a similar outline to this earlier work and at numerous points benefits from arguments found therein. However, although the earlier work relied almost exclusively on the Gilman-Griess Theorem (see Section 2 of Chapter 13), we employ a variety of recognition techniques, as discussed also in Sections 1 and 3 of Chapter 13. In particular, the theory of Curtis-Tits and Phan systems, developed in the last several decades, plays a very important role. Also useful is an identification method pioneered by W. J. Wong [Wo1] in the early 1970s for the recognition of classical groups in odd characteristic, and later extended by Finkelstein and the third author to many cases of classical groups in characteristic 2, e.g. [FinS1].

In barest outline, the strategies for the non-alternating generic case of Theorem O and for Theorem GE are identical. The main theorems of Volume 7 have produced, for some $p \in \gamma(G)$, an element $x \in \mathcal{I}_p(G)$ and a component K of $C_G(x)$ with $K \in \mathcal{C}hev(r)$ and with $m_p(C_G(K)) = 1$, having an acceptable subterminal (x, K) -pair (y, L) (defined on p. 7), where $D := \langle x, y \rangle \cong Z_p \times Z_p$ and L is a large component of $C_K(y)$. Moreover, we have some $u \in D - \langle x \rangle$ such that the pumpup L_u of L in $C_G(u)$ is quasisimple with $L < L_u$, and in general, K and L_u are defined over the same field. The strategy is to use one of the recognition theorems mentioned in the previous paragraph and described carefully in Chapter 13 to identify the group $G_0 := \langle \mathcal{N}(x, K, y, L) \rangle$ as a quasisimple group in $\mathcal{C}hev(r)$. Here, by definition, G_0 is the group generated by K and all of the neighboring components L_u , $u \in D^\#$ lying over the subcomponent L inside K . Because of our hypothesis that G has no proper p -uniqueness subgroup, G_0 will turn out to be G , and we will have completed the identification of G .

In practice, we treat the cases $p = 2$ and p odd separately. For the construction of G_0 , the $p = 2$ case is handled in Chapter 14, while the p odd case is treated in

Chapter 15. The principal difference is that, when $p = 2$, the existence of commuting involutions in the centers of suitable fundamental $SL_2(q)$ subgroups of K and of L_u permits these subgroups to be lined up properly for the verification of suitable Curtis-Tits or Phan relations. An analogue is not available when p is odd, but—following Gilman and Griess—we use a toral subgroup B of exponent p visible both in $C_G(x)$ and all $C_G(u)$, $u \in D^\#$, permitting the construction of a Weyl group W for G as a quotient of $N_G(B)$. Then G_0 may be identified with $\langle K, N \rangle$ for suitable $N \leq N_G(B)$ covering W . A version of this strategy of recognizing G_0 as $\langle K, N \rangle$ was first implemented by W. J. Wong for the characterization of symplectic and orthogonal groups in odd characteristic. Moreover, in the Classical Involution Paper, M. Aschbacher constructed a toral 2-subgroup of G and an associated Weyl-type group (which he calls the Thompson group), building on ideas of J. G. Thompson.

By the end of Chapter 15, we have constructed in case (b) a subgroup G_0 of G with $G_0 \in \mathcal{C}hev(r)$ and $\Gamma_{D,1}(G)$ normalizing G_0 . Finally, in Chapter 16, we prove that $G = G_0$ in both case (a) and case (b) by showing that, if not, then $N_G(G_0)$ is a proper strong p -uniqueness subgroup of G of component type, contrary to hypothesis. (Note that for $p = 2$, the simple groups with a proper strong 2-uniqueness subgroup are classified in Volume 4, using the arguments of Bender and Aschbacher.)

In this volume, we have found it of great convenience, and sometimes of necessity, to use some known theorems not previously included in our set of Background Results, viz., L. E. Dickson's classification of smallest-dimensional faithful modules for the symmetric groups [Di2]; J. McLaughlin's characterization of groups generated by transvections on a finite vector space of odd order [McL2]; R. Gramlich's exposition of the state of Phan theory [Grm2]; and theorems of R. Blok, C. Hoffman, and S. Shpectorov on amalgams of Curtis-Tits and Phan types [BHS]. *The full set of Background Results consists now of these four references and the Background Results listed in our first volume [I₁].* At an appropriate future moment, we shall add to these the Aschbacher-Smith tomes on quasihomomorphisms.

We continue the notational conventions established in Volume 2 of this series [I_G]. We refer to the chapters of this book as [III₁₂], [III₁₃], [III₁₄], [III₁₅], [III₁₆] and [III₁₇]. As in previous volumes, the last chapter [III₁₇] collects the necessary \mathcal{K} -group lemmas for the main chapters [III₁₄]–[III₁₆], and thus logically precedes them.

As early as 1999, the authors began a discussion with Curtis Bennett and Sergey Shpectorov concerning the possibility of a modern treatment of the theorems of Phan characterizing the finite unitary groups in odd characteristic. They took up this problem and expanded its scope, later recruiting Corneliu Hoffmann and Rieuwert Blok as collaborators. Also, in the early 2000s, Ralf Gramlich began his dissertation work under the supervision of Aryeh Cohen. Gramlich visited Rutgers during part of the 2001–02 academic year, which the third author spent at Rutgers. Gramlich became a leader in the development of Phan Theory and recruited many others to this project. We extend our warmest thanks to all of these mathematicians for their contributions to the development of this body of recognition theorems.

We also thank Bob Griess, Jon Hall, Len Scott, Gary Seitz, and Gernot Stroth for numerous helpful conversations and valuable suggestions. Special thanks once

again go to our wives, Lisa and Rose, for their patience during our many hours of mathematical distraction. And, lastly, we recall with deep regret the passing of our colleague and friend, Kay Magaard, with whom we had many hours of fruitful discussion of some the topics destined for future volumes of this series, along with many lighter moments of laughter and good fellowship. We will miss him often as the years go by.

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Richard Lyons and Ronald Solomon
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