Preface

With this book we begin Part V of our program, the analysis of the special even case. Three theorems belong to that Part: $\mathcal{C}_4$, $\mathcal{C}_5$, and $\mathcal{C}_6$. All three assume that the $\mathcal{K}$-proper simple group G being analyzed is of even type.

Theorem $\mathcal{C}_4$, to be treated in a later book, concerns the case $e(G) \leq 3$. Thanks to the monumental Quasithin Theorem of M. Aschbacher and S. D. Smith [**ASm1**] [**ASm2**], which we now include in our Background Results, it is really the case $e(G) = 3$.

This book contains a complete proof of Theorem $\mathcal{C}_5$, which covers the "bicharacteristic" subcase of the $e(G) \geq 4$ problem. The outcome of this theorem is that G is isomorphic to one of the six sporadic groups for which $e(G) \geq 4$, or one of six groups of Lie type which exhibit both characteristic 2-like and characteristic 3-like properties.

Finally, in Chapter 7, we begin the proof of Theorem $\mathcal{C}_6$ and its generalization Theorem $\mathcal{C}_6^*$, which cover the "p-intermediate" case. This case will be shown in the end to be satisfied by no $\mathcal{K}$-proper simple group G. In this case, for some odd prime p and $x \in G$ of order p such that $m_p(C_G(x)) \geq 4$, $C_G(x)/O_{p'}(C_G(x))$ has a component K which is neither in the "generic" set $\mathcal{G}_p$ nor in the "characteristic p-like" set $\mathcal{C}_p$, and satisfies a condition close to p-terminality. In particular, K might have cyclic Sylow p-subgroups (with a couple of exceptions). In the preceding book in this series, we had promised complete proofs of Theorems $\mathcal{C}_6$ and $\mathcal{C}_6^*$ in this book, but because of space considerations, we postpone the completion of those theorems to the next volume.

This volume has had a long gestation period, a zeroth draft of Theorem $\mathcal{C}_5$ having been written by the second and third authors in the early 1990s, suggesting the concept of groups of bicharacteristic type as a unifying approach to a local characterization of many of the large sporadic simple groups. The prehistory of this approach dates to John Thompson's second paper [**Tho70**] on simple N-groups, where he characterized the groups $PSp_4(3)$ and $G_2(3)$ by their 2-local and 3-local structures. Indeed, $PSp_4(3)$ is the primordial bicharacteristic group, with incarnations as $U_4(2)$, $\Omega_5(3)$ and $\Omega_6^-(2)$, as well as $W(E_6)'$ and the group of the 27 lines in a cubic surface; and it is a progenitor of the Fischer groups, which are prominent in this volume.

Not long after Thompson's work, K. Klinger and G. Mason [**KMa1**] proved the following theorem.

THEOREM. *Let G be a finite group of characteristic 2-type with $F^*(G) = E(G)$. Let p be an odd prime and suppose that G is also of characteristic p-type. Then every 2-local subgroup of G has p-rank at most 2.*

A finite group G is called quasithin if the 2-local p-rank of G is at most 2 for every odd prime p. Thus, an immediate corollary of the Klinger-Mason Theorem is the following statement.

COROLLARY. *Let G be a finite group of characteristic 2-type with $F^*(G)$ simple. Then either G is quasithin or there exists an odd prime p such that G has 2-local p-rank ≥ 3 and G is not of characteristic p-type.*

In the late 1960s, Gorenstein and J. H. Walter pioneered a strategy for the classification of finite simple groups which are not of characteristic 2-type. Their strategy was implemented by a large team of group theorists, mostly during the 1970s. Later, Gorenstein and Lyons [**GL1**] developed a strategy for the classification of finite simple groups (CFSG) which are of characteristic 2-type but for which there exists an odd prime p such that G has 2-local p-rank ≥ 4 and G is not of characteristic p-type. The above corollary shows that, in conjunction with a classification of simple groups of characteristic 2-type that are either quasithin or satisfy $e(G) = 3$, this would complete the CFSG. And indeed this strategy was implemented in the first proof of CFSG, with the $e(G) = 3$ theorem of Aschbacher and the Quasithin Theorem of Aschbacher and Smith.

From Thompson's original N-Group argument through Klinger-Mason up to the Trichotomy Theorem of Gorenstein-Lyons [**GL1**], a significant subcase of the analysis was the so-called "$GF(2)$-type" case, i.e., the case when a 2-central involution z of G has the property that $F^*(C_G(z))$ is a 2-group of symplectic type, in the sense of Philip Hall. As an independent problem (sometimes called the O_2 extraspecial problem), this generated a large body of research, starting with the work of Z. Janko and his students and culminating in classification theorems by F. G. Timmesfeld, S. D. Smith and others in the mid-1970s. This entailed characterization theorems for many of the sporadic simple groups, as well as most simple groups of Lie type defined over the field $\mathbf{F}_2$.

In the 1980s, with the first proof of CFSG largely complete, Gorenstein and Lyons began a rethinking of the classification strategy. They observed that the endgame for a unifying methodology for the classification of most of the alternating groups and groups of Lie type was to classify simple groups G having an element x of prime order p such that $C_G(x)$ has p-rank ≥ 3 and $C_G(x)$ has a p-component K with $K/O_{p'}(K)$ either an alternating group (with $p = 2$) or a quasisimple group of Lie type of characteristic $r \neq p$. Of course this endgame fails to include the sporadic simple groups. Many of these are quasithin groups, but most of the remaining ones have elements of order p whose centralizers are either p-constrained or have components which are either of Lie type in characteristic p or sporadic.

This motivated the replacement of the dichotomy between characteristic 2-type and non-characteristic 2-type groups with the dichotomy between groups of even type and those of odd type. While all involution centralizers in groups of characteristic 2-type are 2-constrained, involution centralizers in groups of even type are allowed to have components K, but only with K of Lie type in characteristic 2 (plus or minus a small finite number of groups, mainly sporadic). It is still required that involution centralizers be core-free. A similar notion of p-type emerged for odd primes p, allowing centralizers of elements of order p with p-components K where $K/O_{p'}(K)$ is of Lie type in characteristic p (again plus or minus a handful of groups). Here $O_{p'}(C_G(x))$ is required only to have odd order for x of order p. From this point of view, the residual case arising in the endgame of their classification strategy was the Bicharacteristic Case, i.e., the case when G is a finite simple group of even type for which there exists an odd prime p such that G has 2-local p-rank ≥ 3 and G is of p-type. A draft treating the Bicharacteristic Case, with some additional simplifying hypotheses, most notably that G has 2-local p-rank at least 4, was completed by Gorenstein and Lyons in 1992, the outcome being the five groups Fi_{23}, Fi'_{24}, Co_1, F_2, and F_1. A strengthened version (but still only for 2-local p-rank ≥ 4) is Theorem $\mathcal{C}_5$, which is the principal content of the current volume. In addition to the five sporadic groups just mentioned, Fi_{22} appears, as do six groups of Lie type. Careful definitions and a precise statement appear in Chapter 1.

A word about recognition theorems: the six large sporadic groups in the conclusion of Theorem $\mathcal{C}_5$ are recognized by well-established characterizations by M. Aschbacher, R. Griess, U. Meierfrankenfeld, and Y. Segev. Our original Background Results included the statement that these groups were characterized by their full centralizer of involution patterns. However, that is an impractical approach, given the above-mentioned characterizations, especially since some such characterization theorems would still eventually be needed to obtain the six groups up to isomorphism. Thus, we change the Background Results as follows. We add two books by Aschbacher [**A2**], [**A19**], and two papers, one by Segev on the Baby Monster [**Seg1**] and one by Griess, Meierfrankenfeld and Segev, on the Monster [**GrMeSeg**]. (And we remove the result that these six groups are characterized up to isomorphism by their centralizer of involution patterns.) What is needed from these publications are centralizer of involution characterizations of the six large groups, plus a couple of results from Aschbacher's *Sporadic Groups* [**A2**] relating to Co_2, the binary Golay code, and the Todd module for M_{24}. *We have now supplemented the Background Results listed in our initial volume* [$\mathbf{I}_1$] *with these four references, the Quasithin Theorem of Aschbacher and Smith* [**ASm1**] [**ASm2**]*, and the four references mentioned in the Preface to the previous volume in this series* [**GLS8**].

The six large sporadic groups are in some sense on the border between large-rank generic and non-generic groups. When we round them up in Theorem $\mathcal{C}_5$, we also capture six groups of Lie type which can be considered to lie on that border as well: $\Omega_7(3)$, $P\Omega_8^{\pm}(3)$, $U_7(2)$, ${}^2D_5(2)$, and ${}^2E_6(2)$. Characterizing the first three of these – the orthogonal groups over $\mathbf{F}_3$ – was one of the last details to be attended to in the original proof of the classification, and was accomplished by Aschbacher [**A23**]. We have relied heavily on that paper to get from a standard component isomorphic to $L_4^{\pm}(3)$ or $2U_4(3)$ to an endgame configuration. For the final recognition of these and the last three groups of Lie type over $\mathbf{F}_2$, we use tools similar to those used in our previous volume [**GLS8**], namely, the Curtis-Tits theorem as well as methods of Wong and Finkelstein-Solomon [**GLS8**, pp. 18–32].

As usual, we continue the notational conventions established in the second volume of this series [$\mathbf{I}_G$]. We refer to the first seven chapters of this book as [$\mathbf{V}_1$], ..., [$\mathbf{V}_7$], and the eighth chapter as [$\mathbf{V}_K$]. As usual, the chapter [$\mathbf{V}_K$] consists of supporting $\mathcal{K}$-group lemmas for the main chapters [$\mathbf{V}_3$]–[$\mathbf{V}_7$], and thus logically precedes them.

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