Preface

The purpose of this book is to develop the diagrammatic theory of knotted surfaces in 4-dimensional space in analogy with the classical theory of knotted and linked circles in 3-space. This goal may sound unachievable to some readers, how can we perceive phenomena that occur in 4-space? A related issue is perception in 3-dimensional space which we discuss briefly.

For sighted people, the 3-dimensional world is projected upon a 2-dimensional surface, the retina, at any particular moment. Spatial relations are determined by a pair of two-dimensional images. The sense of touch is patently two-dimensional since the world comes into contact with us by means of our skin — a 2-dimensional surface. Sound is perceived as vibrations along a 2-dimensional membrane. The sense of taste is discrete with four states that are either excited or not. Only the olfactory sense has a degree of multi-dimensionality; it is difficult to classify and relate various smells. Each odor seems to be an independent quantity. But the sense of smell may also be discrete with an enormously large set of states.

Since we usually perceive the world by a series of 2-dimensional impression, how do we come up with a 3-dimensional model of it? One can argue that the relative position of the wrist, elbow, and shoulder allow for a 3-dimensional world. So even though the world projects to us on our skin, retinas, or eardrums, we see the world as 3-dimensional. Additional perceptual clues come from moving the eyes to see around a corner, moving the hands to feel a different facet, or turning the ear towards a sound.

So in order to develop the diagrammatic theory of knotted surfaces in 4-dimensions, we will project the surfaces into 3-dimensions, and we will move the surfaces around to see their different facets. It is not unreasonable that we will develop some 4-dimensional intuition in the process.

In classical knot theory, invariants (Alexander, Conway, Jones, HOMPTFLY, Kauffman polynomials) are computed diagrammatically. Category theoretical interpretations of knot diagrams play a key role in the study of quantum invariants. The braid form of a classical knot, which is both algebraic and diagrammatic, is used not only to define new invariants but also as geometric machinery for the study of knots. Most of these concepts and computations can be generalized to 4-dimensions via diagrams. Thus we will develop the theory of knotted surfaces and thereby provide the machinery for algebraic and geometric computations.

Here is the outline of the book.

Chapter 1 develops the notion of a knotted surface diagram. A diagram consists of a generic surface in 3-space together with crossing information indicated along the double point curves. Such a diagram can be projected further onto a plane to create a *chart* — a planar graph with labeled edges. The chart can be used to reconstruct the surface and to construct two other models. A *movie* consists of the diagram with a fixed height function in 3-space. In such a movie we can consistently fix the height functions in the stills to create a fully combinatorial description of the surface. The combinatorial description is called a *sentence*; this is a sequence of words that are related by some grammatical rules. We give examples of each of these descriptions, and discuss some other diagrammatical methods.

Chapter 2 contains the theory of Reidemeister moves. For each description of the knotted surfaces there is a finite set of moves such that equivalent knottings are related by a finite sequence of moves taken from this set. We give examples of applications of the moves in the various context. Chapter 2 closes with the classical argument that a coffee cup and a doughnut represent isoptopic surfaces in 3-space.

Chapter 3 reviews the theory of surface braids that has been extensively developed by S. Kamada. We discuss generalizations of Alexander, Markov and Artin theorems. In particular, for a generalized Artin theorem, we give a finite list of moves to surfaces braids such that equivalent surface braids are related by a finite application of moves taken from this list, and we give examples of the Alexander isotopy. We close the chapter with a discussion on a homotopy theory interpretation of the surface braids.

Chapter 4 contains material that contrasts the knotted surface case with the classical theory. We show that not all generic surfaces lift to embeddings. Triple point smoothing and applications thereof are given. We define signs and colors of triple and branch points, and relate them to the normal Euler number. Cancellation of cusps and branch points on the projections are discussed. Some of the work in this chapter is joint work with Vera Carrara.

Chapter 5 contains methods for computing the fundamental group and a presentation matrix for the Alexander module of the knotted surface. We give explicit computations for several examples. The chapter closes with a description of the Seifert algorithm for knotted surfaces. Such an algorithm was used by Giller [**Gi**] in the case that the projection had no triple points. We developed the full algorithm in [**CS2**] and constructed Heegaard diagrams using our algorithm; Kamada wrote a version of the algorithm in the surface braid case. We use Kamada's approach to give a Heegaard diagram of the Seifert solid in the case the surface is given in braid form.

Chapter 6 is a review of the algebraic and categorical aspects of knotted surfaces. We present solutions to the equations that are generalizations of the Yang-Baxter equation. Our solutions are based on diagrammatic methods and provide a good testimony to the power of these methods. The definition of a braided monoidal 2-category with duals (as given in [**KV2**], [**BN**] and [**BL**]) is sketched. We indicate that embedded surfaces in 3-space form a monoidal 2-category with duals while surfaces embedded in 4-space form a braided monoidal 2-category with duals. The chapter closes with the result of Baez and Langford that states that embedded surfaces form a free braided monoidal category with duals on one self dual object generator. This result forms the backbone of the future search for invariants that are analogous to the Jones polynomial.

Some of the exercises are labeled research problems. That means that we do not know the outcome of the research. If the reader finds a solution before we do, then that is great!

Throughout this book, the term *the classical case* refers to the theories of knotted and linked circles in 3-dimensional space or planar diagrams thereof. All manifolds and maps are smooth, and 4-space has the standard smooth structure.

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