Contents

Preface

The last twenty years have seen an especially active interaction between mathematics and physics. This interaction has given birth to a number of remarkable new areas of mathematics and has provided powerful new tools in various fields of theoretical physics. This book is devoted to one of those new areas, which deals with mathematical structures of conformal field theory and their q-deformations. It arose from the course of lectures on the classical and quantum Knizhnik-Zamolodchikov equations, given by one of the authors of this book (I. B. F.) in the Spring of 1992 at Yale University, and inspired by his recent (at that time) joint paper with Nicolai Reshetikhin. The instructor was lucky to find two enthusiastic graduate students and future co-authors, who improved and extended the exposition and later gave related courses at Harvard (P. I. E.) and MIT (A. A. K.).

By that time, all three of us had already been severely afflicted with the "q-disease", a dangerous mathematical illness whose earliest victim was Euler, but which was first diagnosed by Richard Askey. Mathematicians working in practically every field, be it algebra, geometry, analysis, differential equations – you name it – are vulnerable to its addictive charm. The first symptom of the q-disease is that one day you realize that most of the results obtained or acquired during your mathematical life admit a q-deformation. The second stage is indicated by the idea that the q-case is much more interesting than the classical one. It was at that stage that we started writing the second part of the book, with the intention to q-deform all structures of conformal field theory. This turned out to be a difficult task, which has taken five years, and is still not completed. Luckily, during these years the area grew up quite significantly, and we were able to use some of the more recent results and refer the reader to active new developments and promising research problems.

When writing this book, we followed one of the rules we learned from Israel Gelfand: for every new theory, choose the simplest non-trivial example and write down everything explicitly for this example. Therefore, all general constructions and theorems are accompanied by explicit calculations for the Lie algebra $\mathfrak{sl}_2$.

We also wanted to put this new area of mathematics into the general perspective of development of representation theory. With this intention we wrote the first and the last lecture, trying to help the reader navigate in the mist of modern representation theory and mathematical physics. The bare scheme of the development of the theory is captured by the diagram at the end of Section 1.6 in the Introduction. Familiar to many mathematicians working in the area, it is closer to alchemical formulas than to mathematics. Nevertheless, it was instrumental in the discovery of some of the structures studied in this book, and it might still be useful again.

This book is written for people who are familiar with the representation theory of simple Lie algebras, but requires no knowledge of physics. It can be used for teaching an advanced one-semester graduate course, though the instructor and the students should work really hard, as it was in the Spring of 1992 at Yale. The number of lectures exactly corresponds to the number of weeks in one semester at Yale University. Each lecture is assumed to take two and a half hours and can be split into two or three weekly classes depending on the strength and enthusiasm of the participants. Our experience shows that the golden mean is always the best solution. Also the first and the last lectures should not be taken as seriously (for some people as lightly) as the rest of the book, giving the students and the instructor an opportunity to relax.

This theory was created by the efforts of many people, and some results were circulating as "folklore" for a number of years. We tried to give the references in the first lecture and especially in the introduction to each of the consecutive lectures, but we would like to apologize in advance for unintentional omissions.

One of the main purposes of this book was to simplify, extend and provide all the necessary details of the results in the original article [**FR**]. We hope that in our present exposition we have corrected more misprints and inaccuracies in that paper than we have added new ones.

The credit for the eventual existence of this book rightfully belongs to our editor, Sergei Gelfand, who achieved a seemingly impossible goal of persuading the authors – after a five year delay – to bring this book to a conclusion.

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