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Preface

The origins of this monograph lie, firstly, in the pioneering contributions from H. Weyl, J. von Neumann, M.H. Stone, E.C. Titchmarsh, K. Kodaira to the theory of linear differential operators in Hilbert function spaces and, secondly, in the significant contributions made by the Ukrainian (former Soviet Union) mathematicians M.G. Krein, M.A. Naimark and I.M. Glazman to the study of boundary value problems for linear, ordinary quasi-differential equations on any real interval.

The results of Glazman in his seminal memoir of 1950, influenced by both Krein and Naimark, led to the now-named GKN theorem within the general theory of quasi-differential operators (which include and generalize the classical linear ordinary differential operators) in Hilbert function spaces. The significant contribution from Glazman, mirrored in part by the work (also in 1950) of Kodaira, led to the then new formulation of boundary conditions required to construct self-adjoint differential operators, representing the boundary value problem. The original GKN theorem is stated for real-valued, thereby necessarily of even order, quasi-differential expressions; the theorem gives an elegant, necessary and sufficient condition for Lagrange symmetric differential expressions to generate self-adjoint operators in the appropriate Hilbert space of functions on the prescribed real interval.

The Glazman idea is to represent the homogeneous boundary conditions in terms of the skew-symmetric, sesquilinear form associated with the quasi-differential expression and the corresponding Green's formula; the quasi-differential expressions are now known to define a real symplectic space, and the boundary conditions to correspond to Lagrangian subspaces of this symplectic space, as recently recognized and realized by the current authors. The properties of these real symplectic spaces, and their geometry and symplectic linear algebra, have long been advanced by mathematicians and physicists in a number of different applications; in particular in Lagrangian analytical dynamics and quantum theory.

The original GKN theory was confined to the real-valued, quasi-differential expressions of arbitrary even order. However complex-valued quasi-differential expressions, of arbitrary (positive) integer order, had been studied earlier by Halperin and Shin, and later by Everitt and Zettl. In the years following the untimely death of Glazman in 1968 these complex expressions have been extensively studied, with particular reference to the Lagrange symmetric (formally self-adjoint) expressions. This formulation of the GKN Theorem has consequently been extended to these complex quasi-differential expressions of arbitrary integer order; however this extension has required the introduction and study of linear complex symplectic geometries and the algebra of their Lagrangian subspaces, as defined and described in these pages. The consequences of this study are to be seen in the contents of this monograph.

Two special comments are called for in respect of these complex symplectic spaces:

1. The complex spaces have a much richer structure and range of properties in comparison with the real spaces. Real symplectic spaces exist in even dimensions only, and moreover there is a unique real space (up to symplectic isomorphism) in each such even dimension. Every real symplectic space can be complexified to a complex symplectic space of the same even (complex) dimension. However there exist even-order complex symplectic spaces that are not the complexification of any real space, and there exist different complex symplectic spaces of each odd integer order.

2. The complex, Lagrange symmetric, quasi-differential expressions, of arbitrary positive integer order n, also have additional structures in comparison with the corresponding real expressions. The most significant property, in this respect, is that the complex expressions lead to minimal, closed symmetric operators, defined in the appropriate Hilbert function space, which can have unequal deficiency indices; these indices are now re-interpreted as algebraic invariants of the corresponding complex symplectic space [see Section III, Theorem 1]. Of course, such a minimal symmetric operator has self-adjoint extensions if and only if the two deficiency indices are of the same value, say a non-negative integer d, which is less than or equal to the order n. In fact, an informal paraphrase of our new version of the GKN Theorem asserts (for a precise statement see Section II, Theorem 1):

Each such self-adjoint operator is specified explicitly by a Lagrangian d-space within the corresponding boundary complex symplectic $2d$-space, and conversely each Lagrangian d-space corresponds to exactly one such self-adjoint operator.

However in our detailed analysis of the kinds of boundary conditions that can occur we partition the basic interval into left and right sub-intervals, on each of which the restricted differential operator may have unequal deficiency indices. Thus the full range for the deficiency indices, equal or not, plays an important role in the theory (compare Section V, Proposition 1 with the Weyl-Kodaira formulas and the Deficiency Index Conjecture 2).

It can be argued that complex symplectic spaces have richer structures in order to support the extensive properties of complex quasi-differential expressions; vice versa there is a case to state that these complex differential expressions force the structure of the complex symplectic spaces to exist in order to support their properties.

The two main and significant consequences of writing this research monograph are:

1. There is now a complete and connected account of the geometric and algebraic structure of real and complex symplectic spaces and their Lagrangian subspaces, for all integer orders, with special attention to the algebraic properties of direct sum decompositions such as are relevant for the study of boundary conditions, especially with regard to properties of separation or coupling at the boundary endpoints.

2. There is a complete account of the canonical form of all possible symmetric boundary conditions (with respect to separation or coupling at the endpoints) for the extended GKN theory of Lagrange symmetric, linear, quasi- differential expressions (real or complex) of all integer orders on arbitrary real intervals.

In addition to this main text there are two substantial appendices. The first deals with the canonical form of classical ordinary differential expressions when these are considered as quasi-differential expressions and then settles certain technical questions concerning adjoint operators; the second treats the problems of the complexification of real symplectic spaces, and the analysis of self-adjoint operators which are non-real yet arise from real differential expressions.

In all these areas the authors have made significant and extensive new contributions, in addition to re-organizing established theories into a satisfying synthesis with the results within this monograph. As an illustration of our approach and of some of the new results, we offer here two very specific and explicit findings:

(i) The balanced intersection principle (Section III, Theorem 3 for precise details) provides an algebraic criterion for describing and classifying the divers kinds of self-adjoint boundary conditions for quasi-differential expressions of arbitrary integer order, and for all boundary value problems whether regular on compact intervals or singular on general intervals. In particular the coupling grade is defined for each Lagrangian d-space (and hence for the corresponding self-adjoint operator) and from this we deduce the minimal number of coupled boundary conditions necessary in the specification of the operator domain. For a regular problem of arbitrary positive order, on a compact interval, there is always the same number of separated boundary conditions at the left endpoint as at the right endpoint of the interval (assuming that minimal coupling is employed).

For singular problems this is not necessarily the case; however there is an arithmetic formula relating the number of separated boundary conditions at each of the two endpoints with the invariants of the left and right endpoint complex sympletic spaces. For example, consider a Lagrange symmetric real quasi-differential expression of order four on the closed half-line $[0, \infty)$. We find [see Table 3 and Example 3 of Section V] that the common deficiency index d can take the values 2,3 or 4, which generalizes the limit-point and limit-circle classifications that Weyl defined for second-order differential expressions. As an indication of the explicit nature of our calculations and tabulations, we mention that it is then possible to have three (independent) boundary conditions, when $d = 3$, to define a self-adjoint operator, with one separated at the left end and two coupling the ends, but it is impossible to define a self-adjoint operator by one separated condition at the left end and two separated at the right end.

(ii) Lagrange symmetric quasi-differential expressions that are real can determine self-adjoint operators that are real (definable by real boundary conditions) or else complex operators that are non-real. An investigation of this phenomenon is conducted in Appendix B, where the complexification of real symplectic spaces, and the associated concept of self-conjugate Lagrangian subspace, are described in great detail.

We provide an affirmative answer [Appendix B, Theorem 3] to a long-standing open question concerning the existence of real differential expressions of even order ≥ 4, for which there are non-real self-adjoint differential operators specified by strictly separated boundary conditions, *i.e.* complex Lagrangian subspaces which are not self-conjugate and which have coupling grade zero; in fact, we prove the existence of such Lagrangian subspaces of every possible prescribed coupling grade. This is somewhat surprising because it is well known that for order $n = 2$ strictly separated boundary conditions can produce only real operators (that is, any such given complex boundary conditions can always be replaced by real boundary conditions). Our analyses and examples are entirely explicit for regular problems on compact intervals; moreover corresponding explicit results also hold in the general singular case.

In undertaking this project we have reviewed the theory of both differential and quasi-differential expressions and have assembled all relevant information in the opening sections of this monograph to provide a convenient source of reference. In particular we have put together details of the connections between classical differential expressions and the extended class of quasi-differential expressions.

In respect of symmetric boundary problems for these differential expressions, and the associated self-adjoint differential operators, there is complete generality; regular problems on compact intervals and the more general singular problems are all treated in full detail. From the algebraic data all the classification results for boundary conditions then follow; thereby the infinite dimensional functional analysis is reduced to finite dimensional linear symplectic algebra.

The introduction of these algebraic and geometric methods has led to the discovery of new kinds of qualitative insight into the topology of the boundary value problem in terms of the Lagrange-Grassmannian manifold.

The axiomatic formulation of these mathematical structures leads immediately to applications for other types of boundary value problems such as the multi-interval or interfacial conditions of the multi-particle systems of quantum mechanics, or the general theory of linear elliptic partial differential equations; these applications depend on the extension of the ideas considered in this monograph to infinite dimensional, complex sympletic spaces.

In concluding this work we have to survive a disappointment. It had been our hope at the start of these labors that the algebra of complex sympletic spaces would throw new light on the "deficiency index conjecture" for complex quasi-differential operators. This has not been the case and the so-called range conjecture, formulated precisely in this work, remains unsolved. We have little doubt that the conjecture is true. While our analysis and classification of symmetric boundary conditions do not rest on the validity of this conjecture, we have occasionally used it (with appropriate warnings) to give insight and guidance into the search for new properties and interrelations among classes of such boundary value problems.

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W. Norrie Everitt

Department of Mathematics and Statistics, University of Birmingham, Birmingham, B15 2TT, England

Lawrence Markus

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455 USA
and
Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England

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SECTION I

Introduction: Fundamental Algebraic and Geometric Concepts Applied to the Theory of Self-Adjoint Boundary Value Problems

1. Survey of problems, methods, goals: organization of results

Boundary value problems for complex linear ordinary differential (or quasi-differential) equations

$$\mathcal{M}[y] = \lambda w y, \tag{1.1}$$

where $\mathcal{M}$ is a complex formal differential (or quasi-differential) operator or expression, defined on an assigned real interval $\mathcal{I} \subset \mathbb{R}$ with a prescribed positive weight function w, and $\lambda \in \mathbb{C}$ is a complex spectral parameter, (see details below), are customarily treated within the fundamental theory of unbounded linear operators, usually self-adjoint on some appropriate complex function Hilbert spaces (refer to the classical spectral theory of Weyl-Kodaira-Titchmarsh [**NA**], [**TI**] and to the general references [**AG**], [**CL**], [**DS**]). The purpose of this paper is to develop useful algebraic descriptions of these sets of self-adjoint operators (as specified by appropriate boundary conditions), and this goal is achieved through the methods of quasi-derivatives and complex symplectic geometry and algebra. This introductory Section I formulates basic notations, and reviews the relevant established concepts and theory; the technical new developments begin with Section II.

Here $\mathcal{M}$ is either a classical linear differential expression M or, more generally, a quasi-differential expression M_A (see Examples 1 and 2 below); $\mathcal{I}$ is a nondegenerate interval (open, closed, half-open or closed, finite or infinite—with endpoints denoted by$-\infty \le a < b \le +\infty$); $w \in \mathcal{L}^1_{\text{loc}}(\mathcal{I})$ satisfies $w(x) > 0$ a.e. for $x \in \mathcal{I}$; and the domain $\mathcal{D}(\mathcal{M})$ is to be specified within $AC_{\text{loc}}(\mathcal{I})$. As usual $\mathcal{L}^1_{\text{loc}}(\mathcal{I})$ denotes all complex-valued functions that are locally (i.e. on each compact subinterval) integrable on $\mathcal{I}$, and $AC_{\text{loc}}(\mathcal{I})$ consists of all complex-valued locally absolutely continuous functions on $\mathcal{I}$.

Example 1. $\mathcal{M} = M$, a classical differential expression of order $n \ge 2$ on $\mathcal{I}$. Here

$$M[y] = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1\, y' + p_0 y \tag{1.2}$$

with complex coefficients $p_j \in \mathcal{L}^1_{\text{loc}}(\mathcal{I})$, $j = 0, 1, 2, \ldots, n-1$, and further $p_n \in AC_{\text{loc}}(\mathcal{I})$ with $p_n(x) \ne 0$ for all $x \in \mathcal{I}$. The corresponding domain for M is

$$\mathcal{D}(M) :\equiv \{y : \mathcal{I} \to \mathbb{C} \mid y^{(r)} \in AC_{\text{loc}}(\mathcal{I}) \text{ for } r = 0, 1, \ldots, n-1\}, \tag{1.3}$$

in terms of the ordinary derivatives $y^{(r)}$, so $y^{(n)}$ and also $M[y] \in \mathcal{L}^1_{\text{loc}}(\mathcal{I})$.

There is an important special case where M has smooth coefficients $p_j \in C^j(\mathcal{I})$, or even $p_j \in C^\infty(\mathcal{I})$, for $j = 0, 1, \ldots, n$ (where $C^k(\mathcal{I}),\ 0 \le k \le \infty$ consists of all complex-valued functions with k continuous derivatives on some open neighborhood of $\mathcal{I}$, as usual).

EXAMPLE 2. $\mathcal{M} = M_A$, a quasi-differential expression based on a Shin-Zettl matrix $A \in Z_n(\mathcal{I})$ of order $n \ge 2$ (see definitions in Section I.2 below). Here

$$M_A[y] = i^n y_A^{[n]}, \tag{1.4}$$

with the domain $\mathcal{D}(M_A)$, which we usually write as $\mathcal{D}(A)$,

$$\mathcal{D}(A) :\equiv \{y : \mathcal{I} \to \mathbb{C} \mid y_A^{[r]} \in AC_{\mathrm{loc}}(\mathcal{I}) \text{ for } r = 0, 1, \ldots, n-1\}, \tag{1.5}$$

so $y_A^{[n]}$ and $M_A[y]$ belong to $\mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$. The factor i^n (where $i^2 = -1$) is required in (1.4) to insure that M_A is formally self-adjoint—under appropriate conditions (1.8) and (2.13).

These quasi-derivatives $y_A^{[r]}$, for $r = 0, 1, \ldots, n$, are defined relative to the matrix $A \in Z_n(\mathcal{I})$, as explained in Section I.2 below, and in full detail in [**EV**], [**EZ**]. The theory of classical differential expressions (with complex smooth coefficients) is treated in detail in [**DS,** Ch. XII, XIII]; real, even-order quasi-differential expressions in [**AG,** Appendix 2], [**NA,** Ch. V]; general complex quasi-differential expressions of arbitrary finite order in [**EZ**].

In Appendix A, at the end of this paper, it is demonstrated that each classical differential expression M (as in Example 1, even with non-smooth coefficients) can be re-written in the format M_A as a quasi-differential expression; and, of course, then $\mathcal{D}(M) = \mathcal{D}(M_A)$. However, it will also be clear that there exist quasi-differential expressions (even formally-self-adjoint – see definitions below) that are not equal to any classical differential expression M. Thus the theory of quasi-differential expressions is a proper generalization of the classical theory, and we shall concentrate our attentions on such M_A.

In any case we observe that the differential expression $w^{-1}\mathcal{M}$ defines or generates a linear operator T, once the domain $\mathcal{D}(T)$ is suitably specified,

$$Tf = w^{-1}\mathcal{M}[f] \quad \text{for } f \in \mathcal{D}(T). \tag{1.6}$$

For motivation we note that f is an eigenfunction of T just in case f satisfies the equation (1.1) – and, of course, both f and $w^{-1}\mathcal{M}[f]$ must lie in the relevant Hilbert space, which we choose as $\mathcal{L}^2(\mathcal{I}; w)$ (reducing to $\mathcal{L}^2(\mathcal{I})$ in the familiar case $w(x) \equiv 1$ where the measure $w(x)dx$ reduces to the Lebesgue measure on $\mathcal{I}$). The notation $\mathcal{L}^2(\mathcal{I}; w)$ refers to the complex Hilbert space of all measurable functions $f : \mathcal{I} \to \mathbb{C}$ (or equivalence classes agreeing a.e.) with norm given by

$$\|f\|^2 = \int_{\mathcal{I}} |f|^2 w dx < \infty,$$

and with the corresponding scalar or inner product

$$(f, g) = \int_{\mathcal{I}} f\bar{g}\, w dx, \tag{1.7}$$

for $f, g \in \mathcal{L}^2(\mathcal{I}; w)$ (where $\bar{g}$ is the complex conjugate of g).

Accordingly by the study of the boundary value problem (1.1), we mean the description of all self-adjoint operators T on $\mathcal{D}(T) \subset \mathcal{L}^2(\mathcal{I}; w)$, as generated by $w^{-1}\mathcal{M}$; that is, with adjoint operator $T^* = T$ on the common domain $\mathcal{D}(T^*) = \mathcal{D}(T)$ in $\mathcal{L}^2(\mathcal{I}; w)$. With this objective we next formulate a very general condition defining the self-adjoint property for such a formal expression $\mathcal{M}$ on $\mathcal{I}$ – a pre-condition for the existence of such self-adjoint operators T, see [**FR**].

Definition 1. A formal differential expression $\mathcal{M}$ on $\mathcal{I}$ (either M or M_A as above) is formally self-adjoint or Lagrange symmetric in case:

$$\int_{\mathcal{I}} \{\mathcal{M}[f]\bar{g} - f\overline{\mathcal{M}[g]}\} dx = 0 \tag{1.8}$$

for all $f, g \in \mathcal{D}_0(\mathcal{M})$, where

$$\mathcal{D}_0(\mathcal{M}) = \{y \in \mathcal{D}(\mathcal{M}) \mid \text{compact support } (\operatorname{supp} y) \text{ lies interior to } \mathcal{I}\}. \tag{1.9}$$

Remarks. If $\mathcal{M} = M$ is a classical differential expression with smooth coefficients, then M is formally self-adjoint if and only if M coincides with its Lagrange adjoint M^+:

$$M[y] = M^+[y] :\equiv (-1)^n(\bar{p}_n y)^{(n)} + (-1)^{n-1}(\bar{p}_{n-1} y)^{(n-1)} + \cdots + \bar{p}_0 y. \tag{1.10}$$

(It is sufficient to verify $M = M^+$ for all $y \in C^\infty(\mathcal{I})$).

However, for general M (with non-smooth coefficients) we are able to test for Lagrange symmetry only by replacing M by an equivalent quasi-differential expression M_A (see Appendix A), but even this does not lead to an explicit decision procedure.

For formally self-adjoint differential expressions $\mathcal{M}$ on $\mathcal{I}$, the analysis of the boundary value problem (1.1) rests on two mathematical approaches, namely, (1) Functional analysis of linear operators - which *is not* the subject of this investigation; and (2) Algebraic methods involving boundary conditions - which *is* the focus of our new concepts and results.

Accordingly, as a reference for the functional analysis background, we next tabulate a listing of well-known steps [**DS,** Ch. XII.4] relating to the extension of a symmetric operator to self-adjoint operators, within the context of the action of $w^{-1}\mathcal{M}$ on the Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$. Afterwards we comment on the significance of this material for our algebraic developments.

Among the operators generated by $w^{-1}\mathcal{M}$ on $\mathcal{L}^2(\mathcal{I}; w)$ we recognize a maximal operator T_1, on the largest feasible domain $\mathcal{D}(T_1)$, namely,

$$\begin{aligned} &T_1 f = w^{-1}\mathcal{M}[f] \qquad \text{for } f \in \mathcal{D}(T_1) \\ &\mathcal{D}(T_1) :\equiv \{f \in \mathcal{D}(\mathcal{M}) \mid f \quad \text{and } w^{-1}\mathcal{M}[f] \text{ in } \mathcal{L}^2(\mathcal{I}; w)\}. \end{aligned} \tag{1.11}$$

It can be shown [**DS**], [**NA,** Ch. V] that T_1 is a closed operator, and, of course, $\mathcal{D}(T_1)$ is a linear manifold (but not a closed subspace) in $\mathcal{L}^2(\mathcal{I}; w)$.

Now use the Lagrange-Green Formula to define the boundary form $[f : g]_{\mathcal{M}}$ for $f, g \in \mathcal{D}(T_1)$.

DEFINITION 2. The boundary form for $f, g \in \mathcal{D}(T_1)$,

$$[f:g]_{\mathcal{M}} :\equiv \int_{\mathcal{I}} \{w^{-1}\mathcal{M}[f]\bar{g} - fw^{-1}\overline{\mathcal{M}[g]}\}w dx = \int_{\mathcal{I}} (\mathcal{M}[f]\bar{g} - f\overline{\mathcal{M}[g]})dx, \tag{1.12}$$

is a semibilinear (or conjugate bilinear) form on

$$\mathcal{D}(T_1) \times \mathcal{D}(T_1) \to \mathbb{C}. \tag{1.13}$$

Clearly the boundary form is skew-Hermitian

$$[f:g]_{\mathcal{M}} = -\overline{[g:f]}_{\mathcal{M}}, \quad \text{for } f, g \in \mathcal{D}(T_1), \tag{1.14}$$

and this property will play a central role in all our constructions.

DEFINITION 3. Let $\mathcal{M}$ be a formal differential expression on $\mathcal{I}$ (either M or M_A as above) with domain $\mathcal{D}(\mathcal{M})$ and maximal operator T_1 on $\mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$. Define

(1.15)
$$\mathcal{D}_0(T_1) :\equiv \{f \in \mathcal{D}(T_1) \mid \operatorname{supp} f \text{ lies in a compact set interior to } \mathcal{I}\},$$
$$\mathcal{D}_0(T_1) = \mathcal{D}(T_1) \cap \mathcal{D}_0(\mathcal{M}).$$

REMARK. Assume that $\mathcal{M}$ is formally self-adjoint (Lagrange symmetric) on $\mathcal{I}$, so then

$$[f:g]_{\mathcal{M}} = 0, \qquad \text{for all } f, g \in \mathcal{D}_0(T_1). \tag{1.16}$$

This implies that $w^{-1}\mathcal{M}$ generates a symmetric operator on the domain $\mathcal{D}_0(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$. In order to employ the general theory of symmetric operators on a complex Hilbert space (say, as in the von Neumann-Stone theory [**NA,** Ch. IV]), it is necessary to demonstrate that $\mathcal{D}_0(T_1)$ is a dense linear manifold in $\mathcal{L}^2(\mathcal{I}; w)$. This result is unexpectedly difficult to obtain, especially in the case of quasi-differential expressions M_A (where it is difficult even to verify that $\mathcal{D}_0(M_A)$ is nontrivial – i.e. non-zero) but the proof will be sketched later in Section I.2, and presented in full detail in the Density Theorem 1 of our Appendix A below.

In this situation, when $w^{-1}\mathcal{M}$ generates a symmetric operator on the dense domain $\mathcal{D}_0(T_1)$, we define the minimal operator T_0, as generated by $w^{-1}\mathcal{M}$ on $\mathcal{L}^2(\mathcal{I}; w)$, by

$$\begin{aligned} &T_0 \subseteq T_1 \\ &T_0 f = w^{-1}\mathcal{M}[f] \quad \text{for } f \in \mathcal{D}(T_0), \\ &\mathcal{D}(T_0) = \{f \in \mathcal{D}(T_1) \mid [f : \mathcal{D}(T_1)]_{\mathcal{M}} = 0\}. \end{aligned} \tag{1.17}$$

It is known that T_0, on its domain $\mathcal{D}(T_0) \subseteq \mathcal{D}(T_1)$, can also be defined as the unique minimal closed operator which is generated by $w^{-1}\mathcal{M}$ on $\mathcal{D}_0(T_1)$, see [**DS,** Ch. XII. 4] for the classical operators M (where this closed operator is denoted by $\bar{T}_0$) and [**NA,** Ch. V] for certain quasi-differential operators M_A. The proof of this result for the general case is presented in Theorem 2 of our Appendix A. But since this property of T_0 is not required in our theory, we shall here use the definition of T_0 on $\mathcal{D}(T_0)$ according to (1.17) – and this is entirely sufficient for our investigations.

We recall the definition of the adjoint operator T^* on $\mathcal{D}(T^*)$, for any operator T on a dense domain $\mathcal{D}(T) \subseteq \mathcal{L}^2(\mathcal{I}; w)$. Namely, $f \in \mathcal{D}(T^*)$ in $\mathcal{L}^2(\mathcal{I}; w)$ just in case there exists some $F \in \mathcal{L}^2(\mathcal{I}; w)$ such that the scalar products satisfy

$$(f, Tg) = (F, g), \qquad \text{for all } g \in \mathcal{D}(T).$$

Since $\mathcal{D}(T)$ is dense in $\mathcal{L}^2(\mathcal{I}; w)$, this function F is unique and we define $T^*f = F$, so we can then write

$$(f, Tg) = (T^*f, g), \qquad \text{for all } f \in \mathcal{D}(T^*), \quad g \in \mathcal{D}(T).$$

Furthermore, T is self-adjoint in case

$$T = T^* \quad \text{ on } \mathcal{D}(T) = \mathcal{D}(T^*).$$

It is then evident, from (1.12) and (1.17), that $\mathcal{D}(T_0) \subseteq \mathcal{D}(T_1^*)$ and T_0 is the restriction of T_1^* to $\mathcal{D}(T_0)$. However, it is known – first for special cases (see [**EZ**], [**NA**]) – and now for the general case (see Theorem 2 in Appendix A) that the adjoint operators T_0^* and T_1^* satisfy

$$T_0^* = T_1 \text{ on } \mathcal{D}(T_0^*) = \mathcal{D}(T_1), \quad \text{ and } T_1^* = T_0 \text{ on } \mathcal{D}(T_1^*) = \mathcal{D}(T_0), \tag{1.18}$$

so T_0 is self-adjoint on $\mathcal{L}^2(\mathcal{I}; w)$ if and only if $T_0 = T_1$. If $T_0 \neq T_1$, then we are led to the classical problem [**DS,** Ch. XII.4] of finding all (if any) self-adjoint (hence closed) operators T on domains $\mathcal{D}(T)$, which are extensions of T_0 on $\mathcal{D}(T_0)$ (and, of course, restrictions of T_1 on $\mathcal{D}(T_1)$). That is,

$$T_0 \subseteq T \subseteq T_1 \quad \text{on } \mathcal{D}(T_0) \subseteq \mathcal{D}(T) \subseteq \mathcal{D}(T_1), \tag{1.19}$$

respectively. Furthermore, each such self-adjoint operator T, or more specifically $\mathcal{D}(T)$, is determined by appropriate (generalized) boundary conditions at the ends of $\mathcal{I}$ (see Theorem 1 in Section II).

The existence of self-adjoint extensions T of T_0 is determined by the structure of the linear manifold $\mathcal{D}(T_1)$, which itself becomes a complete Hilbert space under the T_1-graph norm [**DS,** Ch. XII.4.2]. Moreover, there is then an orthogonal direct sum decomposition of the Hilbert space $\mathcal{D}(T_1)$ into closed (in T_1-graph norm) Hilbert subspaces,

$$\mathcal{D}(T_1) = \mathcal{D}(T_0) \oplus \mathcal{D}^- \oplus \mathcal{D}^+, \tag{1.20}$$

where the deficiency spaces $\mathcal{D}^+$ and $\mathcal{D}^-$ of T_0 are defined by

$$\begin{aligned} \mathcal{D}^{\pm} &= \operatorname{span}\{f \in \mathcal{D}(T_0^*) \mid T_0^* f = \pm i f\} \\ &= \operatorname{span}\{f \in \mathcal{D}(T_1) \mid T_1 f = \pm i f\}, \\ &= \operatorname{span}\{y \in \mathcal{D}(\mathcal{M}) \cap \mathcal{L}^2(\mathcal{I}; w) \mid \mathcal{M}[y] = \pm i w y\}, \end{aligned} \tag{1.21}$$

and their (complex) dimensions are the deficiency indices

$$d^{\pm} = \dim \mathcal{D}^{\pm}. \tag{1.22}$$

The theory of von Neumann [**NA,** Ch. IV] asserts that there exist self-adjoint extensions T of T_0 if and only if $d^+ = d^-$, and we shall *make this assumption in our boundary value theory* (but not necessarily in some related algebraic topics), where we denote the deficiency index by

$$d = d^+ = d^-, \text{written } d = d^{\pm}. \tag{1.23}$$

Here d is an integer $0 \leq d \leq n$ – since a homogeneous linear differential equation of order n (see (1.21)) has just n independent solutions.

REMARKS. It is easy to see that when $\mathcal{M}$ is real, (say, classical expression $M = \bar{M}$, or else M_A with $A = \bar{A}$, and n even) then $d^+ = d^-$. If $\mathcal{I}$ is compact (the regular case), then (even without reality assumptions) $d^+ = d^- = n$. However, in the general singular case, where $\mathcal{I}$ is noncompact, then we must make the explicit assumption $d = d^\pm$ (although on subintervals of $\mathcal{I}$ we encounter problems for which there are unequal deficiency indices).

However it should be noted that these deficiency indices can be very difficult to compute [**EI**]. For instance, it is unknown whether there exists a quasi-differential expression M_A of order $n = 5$, formally self-adjoint on $\mathcal{I} = [a, b)$, with $d^+ = 2$, $d^- = 4$.

The deficiency index $d = d^\pm = 0$ if and only if $T_0 = T_1$, in which case T_0 is the sole self-adjoint operator generated by $w^{-1}\mathcal{M}$ on $\mathcal{L}^2(\mathcal{I}; w)$. In any other case where $d > 0$ the structure of the $2d$-space $\mathcal{S}$ (since $\dim \mathcal{S} = \dim \mathcal{D}^- \oplus \mathcal{D}^+ = 2d$), for

$$\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0), \tag{1.24}$$

will later be used to determine the boundary conditions for the domains $\mathcal{D}(T)$ of self-adjoint operators T, according to our Theorem 1 in Section II, using the complex symplectic geometry on $\mathcal{S}$.

REMARK. While we shall assume that $d = d^\pm$ in our studies of self-adjoint boundary value problems, so that $\mathcal{S}$ is a complex symplectic $2d$-space (see Section II), in our general investigations of the structure of abstract symplectic spaces S in Section III, we shall consider the cases where $d^- \neq d^+$. Note that we systematically use the notation S for abstract symplectic spaces defined algebraically, and $\mathcal{S}$ for the corresponding spaces of functions, as in (1.24).

This approach involving deficiency spaces is familiar in the renowned spectral theory of Weyl-Kodaira-Titchmarsh (see [**NA**], [**TI**] and also general references in [**AG**], [**CL**], [**DS,** Ch. XII–XIII]) for classical differential expressions M (with smooth coefficients), and in such analyses this method results in generalized Fourier integral transforms and expansion theorems for each $f \in \mathcal{L}^2(\mathcal{I}; w)$–and, when $\mathcal{I}$ is compact, Fourier series type expansions in a basis of eigenfunctions of T. For further details on convergence, with special classes of functions, say $f \in \mathcal{D}(T)$, additional hard analysis is required.

We shall not contribute to this well-known theory of functional analysis, except to clarify later some of the conditions and hypotheses in the case $\mathcal{M} = M_A$ of quasi-differential expressions, especially in Appendix A.

But algebraic methods are also used to describe and classify the set of all such self-adjoint operators T, as generated by such an expression $w^{-1}\mathcal{M}$ in $\mathcal{L}^2(\mathcal{I}; w)$. Our contributions in this paper introduce new algebraic methods of complex symplectic geometry to resolve such problems. We describe this totality of self-adjoint operators as a single global finite dimensional geometric structure. The concepts and notations of quasi-derivatives make the introduction of symplectic geometry especially simple and elegant, when used for classifying the various self-adjoint operators in terms of their defining boundary conditions. Furthermore these methods illuminate many classical results such as the Weyl limit circle-point cases for order $n = 2$, and, in addition, they suggest interesting generalizations to higher orders.

Various preliminary studies in this direction have been based on the famous theorem of *Glazman-Krein-Naimark* (GKN), and its extensions by Everitt and Zettl [**EZ**], but our results are more concise and present a more general, global viewpoint of the theory – see Theorem 1 in Section II.

The organization of this paper is as follows. In the introductory Section I.1 we review certain relevant aspects of functional analysis for classical differential, and quasi-differential, expressions. The basic definitions and concepts for quasi-derivatives, and their motivations for the introduction of complex symplectic geometry and algebra, are provided in Section I.2 of this Introduction. Only definitions and accompanying discussions are presented in Section I, but theorems and their proofs are omitted or merely quoted with appropriate references. Our objective here is to offer a coherent exposition, although still a sketch, of already known mathematical theorems and results, as a background for the developments in later sections.

In Section II our new version of the GKN Theorem, including the extension by Everitt and Zettl [**EZ**], [**NA**], makes precise the interrelation between the symplectic boundary form and the self-adjoint operators T generated by $w^{-1}M_A$. In this result, Theorem 1, the simplifications produced by the use of quasi-derivatives and symplectic geometry are of the greatest interest–and this achieves the first goal of our paper.

Our second, and main, goal is the examination and description of the set $\{T\}$ of all such self-adjoint operators on $\mathcal{L}^2(\mathcal{I}; w)$, as generated by $w^{-1}M_A$, for a given formally self-adjoint quasi-differential expression M_A on $\mathcal{I}$. Here the set $\{T\}$ is put into a bijective correspondence with the set $\{L\}$ of all Lagrangian d-spaces of the complex symplectic $2d$-space (that is, $\dim \mathcal{S} = 2d$, see (1.24)),

$$\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0),$$

where $0 \leq d \leq n$ is the deficiency index of T_0 (see (1.22) and (1.23)). That is, the boundary conditions defining such a domain $\mathcal{D}(T)$ are specified by the corresponding Lagrangian d-space L. A further decomposition

$$\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+, \tag{1.25}$$

corresponding to the boundary conditions at the left and right endpoints of $\mathcal{I}$, enables us to obtain new results, even for classical boundary value problems, e.g. Section III, Theorems 3, 4, 5, 6, and especially 7. As a quite elementary illustration of the consequences of our theory, it will be made clear that for a regular boundary value problem (for $\mathcal{I}$ compact) each $\mathcal{D}(T)$ has an equal number of *separated* boundary conditions at each of the two endpoints, when using a *minimally coupled basis* for L—and then further generalizations hold for the singular cases, for instance when $\mathcal{I}$ is non-compact.

In Section III the geometry of Lagrangian subspaces of a complex symplectic space is treated in some detail, and these results are applied to regular boundary value problems in Section IV and to singular problems in Section V.

Appendix A gives explicit constructions for quasi-differential operators and a required Density Theorem; Appendix B presents details on the complexification of real symplectic spaces, and a "real version" of the GKN-Theorem.

Our overall goal is to provide a geometric classification of such sets of self-adjoint operators, as generated by a linear quasi-differential expression of order $n \geq 2$, with particular attention to the cases of separated versus nonseparated (or coupled) boundary conditions. Hence our methods will be almost entirely algebraic and geometric, compare [**CL,** Ch. 7, 10] and [**NA,** Ch. V], and we assume known the standard analytic results (with exceptions developed in Appendix A). It is this special emphasis on the algebraic problems, arising from the boundary separation at the endpoints of $\mathfrak{I}$, that leads to new concepts in complex symplectic geometry, and to new results in the boundary value theory of linear differential operators.

2. Review of basic definitions and standard results for quasi-differential expressions; and basic concepts of symplectic geometry and symplectic algebra

As described above (1.4), (1.5), a linear quasi-differential expression of order $n \geq 2$ (first order differential expressions can easily be treated separately, see [**EM 1**]),

$$w^{-1}M_A[y] = i^n w^{-1} y_A^{[n]} \tag{2.1}$$

is defined on the domain $\mathcal{D}(M_A)$, also denoted by $\mathcal{D}(A)$,

$$\mathcal{D}(A) = \{y : \mathfrak{I} \to \mathbb{C} \mid y_A^{[j]} \in AC_{\text{loc}}(\mathfrak{I}) \text{ for } j = 0, 1, \ldots, n-1\}, \tag{2.2}$$

where $\mathfrak{I}$ is a real interval (open, closed, half-open or closed, finite or infinite – with endpoints $-\infty \leq a < b \leq +\infty$). Also the given real weight function $w \in \mathcal{L}^1_{\text{loc}}(\mathfrak{I})$ satisfies $w(x) > 0$ a.e. for $x \in \mathfrak{I}$. The quasi-derivatives $y_A^{[j]}$ for $j = 0, 1, \ldots, n$ are defined below, as certain linear combinations of the ordinary derivatives $y^{(j)}$, in terms of a prescribed complex $n \times n$ matrix $A = A(x)$ for $x \in \mathfrak{I}$, of Shin-Zettl type [**EV**], [**EM**], [**EZ,** Section 1], and we write $A \in Z_n(\mathfrak{I})$. This means that $A(x) = (a_{rs}(x))$, say with row index $1 \leq r \leq n$ and column index $1 \leq s \leq n$, has complex entries $a_{rs} \in \mathcal{L}^1_{\text{loc}}(\mathfrak{I})$ which satisfy the two conditions defining the class of Shin-Zettl matrices $Z_n(\mathfrak{I})$, see [**EM**], [**EZ**], [**HA**], [**NA**], [**SH**], [**SI**], [**SN**], [**ZE**], as given next.

Definition 4. The matrix $A = (a_{rs}) \in Z_n(\mathfrak{I})$ in case $a_{rs} \in \mathcal{L}^1_{\text{loc}}(\mathfrak{I})$ for $1 \leq r, s \leq n$, and

$$\begin{aligned} &(\alpha_1) \quad a_{rs}(x) \neq 0, \quad \text{a.e. for } x \in \mathfrak{I},\ 1 \leq r \leq n-1,\ s = r+1 \\ &(\alpha_2) \quad a_{rs}(x) \equiv 0, \quad \text{a.e. for } x \in \mathfrak{I},\ 1 \leq r \leq n-2,\ s \geq r+2. \end{aligned} \tag{2.3}$$

Hence A is like a companion matrix in that the first super diagonal consists of non-vanishing elements (rather than merely 1's) and the higher super diagonals are all zero.

Fix a choice of the matrix $A \in Z_n(\mathfrak{I})$, and then the corresponding quasi-derivatives, $f_A^{[j]}$ for $j = 0, 1, \ldots, n$, are defined recursively for suitable complex functions $f : \mathfrak{I} \to \mathbb{C}$. Namely, define, for all $x \in \mathfrak{I}$,

$$\begin{aligned} f_A^{[0]}(x) &= f(x), \text{ and then} \\ f_A^{[1]}(x) &= a_{12}(x)^{-1}\{f_A^{[0]}(x)' - a_{11}(x) f_A^{[0]}(x)\} \\ &\ \vdots \\ f_A^{[n-1]}(x) &= a_{n-1,n}(x)^{-1}\{f_A^{[n-2]}(x)' - \sum_{s=1}^{n-1} a_{n-1,s}(x) f_A^{[s-1]}(x)\}, \end{aligned} \tag{2.4}$$

where the prime marks the ordinary derivative, and the reciprocals $a_{12}(x)^{-1}, \ldots, a_{n-1,n}(x)^{-1}$ exist by condition (α_1) of (2.3) (although these reciprocals may not belong to $\mathcal{L}^1_{\text{loc}}(\mathfrak{I})$). Then finally define

$$f_A^{[n]}(x) = f_A^{[n-1]}(x)' - \sum_{s=1}^{n} a_{ns}(x) f_A^{[s-1]}(x). \tag{2.5}$$

We shall later encounter the quasi-differential equation

$$y_A^{[n]} = w\varphi, \quad \text{for an arbitrary "control function" } \varphi \in \mathcal{L}^\infty_{\text{loc}}(\mathfrak{I}), \tag{2.6}$$

which is merely a notational scheme asserting that the column n-vector $\underset{\sim}{y}_A = \operatorname{col}(y_A^{[0]}, y_A^{[1]}, \ldots, y_A^{[n-1]})$ is an absolutely continuous solution of the vector-matrix ordinary differential equation (see [**EV,** Sections 3, 4, 5, 6])

$$\underset{\sim}{y}'_A = A\underset{\sim}{y}_A + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w \end{pmatrix} \varphi. \tag{2.7}$$

Since $A \in Z_n(\mathfrak{I})$, the coefficients of (2.7) are all locally integrable on $\mathfrak{I}$, and hence the classical existence, uniqueness and regularity theorems hold, see [**NA,** Ch. V]. Thus there must exist non-zero functions $y = y_A^{[0]}$ for which $y_A^{[j]} \in AC_{\text{loc}}(\mathfrak{I})$ for $j = 0, 1, \ldots, n-1$, and so $y \in \mathcal{D}(A)$ according to the Definition (2.2). In all cases $y_A^{[n]} \in \mathcal{L}^1_{\text{loc}}(\mathfrak{I})$ for $y \in \mathcal{D}(A)$. Furthermore, by means of Naimark's "patching lemma" [**NA,** Ch. V 17.3, Lemma 2], or else by control theory as in Corollary 1, Appendix A of this paper – with full details in [**EM**] – it can be proved that the sets (see (1.9) and (1.15))

$$\mathcal{D}_0(A) = \{f \in \mathcal{D}(A) \mid \text{compact support } (\operatorname{supp} f) \text{ lies interior to } \mathfrak{I}\} \tag{2.8}$$

and

$$\mathcal{D}_0(T_1) = \mathcal{D}(T_1) \cap \mathcal{D}_0(A) \tag{2.9}$$

are each infinite dimensional linear manifolds in $\mathcal{L}^2(\mathfrak{I}; w)$, and moreover that $\mathcal{D}_0(T_1)$ is dense in the complex Hilbert space $\mathcal{L}^2(\mathfrak{I}; w)$ under appropriate conditions–see (1.7), (1.8) and Appendix A. [Warning: Because $w(x)^{-1}$ and $a_{12}(x)^{-1}$, etc., can be unbounded, it is possible some C^∞-functions with compact supports can be excluded from $\mathcal{D}_0(T_1)$].

It is of special interest to emphasize the case where the prescribed matrix $A \in Z_n(\mathfrak{I})$ has the companion format [**CL,** Ch. 3]:

$$\begin{aligned} a_{r,r+1}(x) &\equiv 1, &&\text{for } 1 \le r \le n-1 \\ a_{rs}(x) &\equiv 0, &&\text{for } 1 \le r \le n-1,\ s \ne r+1. \end{aligned} \tag{2.10}$$

In this special case M_A will reduce to an ordinary differential expression M; in particular, $y_A^{[j]} = y^{(j)}$, so the quasi- and ordinary derivatives are equal for $j = 0, 1, \ldots, n-1$, when $y \in \mathcal{D}(A) = \mathcal{D}(M)$, and moreover

$$M_A[y] = i^n y_A^{[n]} = i^n \left\{ y^{(n)} - \sum_{s=1}^{n} a_{ns} y^{(s-1)} \right\}. \tag{2.11}$$

Hence, in this case, M_A is merely an ordinary differential expression M, see (1.2), (1.3), with $p_n(x) \equiv i^n$ on $\mathfrak{I}$; and conversely every such differential expression can be written in the form of a quasi-differential expression M_A. If p_n is constant, then normalize so $p_n = i^n$, which merely introduces a corresponding multiplier in the spectral parameter λ in (1.1). Furthermore, when M is formally self-adjoint, (see (1.8), (1.9), with clarifications later in Appendix A) we can additionally require that $A = A^+$; that is, A is Lagrange symmetric (see (2.13), (2.14) below) – a property not generally satisfied by the companion-type matrices.

Recall from Section I.1, in particular (1.11) and (1.17), the maximal and minimal operators T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$, respectively, as generated by the quasi-differential expression $w^{-1}M_A$ on $\mathcal{L}^2(\mathfrak{I}; w)$. With this in mind the Lagrange-Green identity (1.12) which defines the boundary form $[:]_A$, see (1.13), is denoted by

$$f, g \to [f : g]_A :\equiv \int_{\mathfrak{I}} (M_A[f]\bar{g} - f\overline{M_A[g]})dx, \quad \mathcal{D}(T_1) \times \mathcal{D}(T_1) \to \mathbb{C}. \tag{2.12}$$

In order to impose the requirement that M_A be formally self-adjoint, so $w^{-1}M_A$ generates the symmetric operator T_0 on $\mathcal{D}(T_0) \subset \mathcal{L}^2(\mathfrak{I}; w)$, we shall usually demand that the matrix $A \in Z_n(\mathfrak{I})$ be Lagrange symmetric [**EZ**]. That is, in addition to the conditions (α_1) and (α_2) of (2.3), we shall require the condition (α_3) given next,

$$(\alpha_3) \qquad A = A^+. \tag{2.13}$$

Here the Lagrange adjoint A^+ of $A \in Z_n(\mathfrak{I})$ is defined by

$$A^+ :\equiv -\Lambda_n^{-1} A^* \Lambda_n, \tag{2.14}$$

where $A^* = \bar{A}^t$ (the conjugate transpose of A, as usual), and $\Lambda_n = (\ell_{rs})$ is a certain fixed constant $n \times n$ matrix with $-1, +1, -1, +1 \ldots$ down the counter-diagonal and zeros elsewhere, that is,

$$\begin{aligned} \ell_{rs} &= (-1)^r, &&\text{for } r + s = n + 1, \text{ and} \\ \ell_{rs} &= 0, &&\text{otherwise.} \end{aligned} \tag{2.15}$$

Then easy computations [**EZ**], [**NA**], using the simplifying formulas

$$\Lambda_n^{-1} = \Lambda_n^t = (-1)^{n-1}\Lambda_n, \tag{2.16}$$

show that the boundary form has a particularly elegant and universal configuration when $A = A^+$.

Namely, for $A = A^+$ the Lagrange-Green identity can be written [**EV**], for all $f, g \in \mathcal{D}(A)$ and each compact interval $[\alpha, \beta]$ interior to $\mathfrak{I}$,

$$\int_\alpha^\beta \{M_A[f]\bar{g} - f\overline{M_A[g]}\}dx = [\![f, g]\!]_A(\beta) - [\![f, g]\!]_A(\alpha), \tag{2.17}$$

where we introduce the notation (simplifying the expression for the boundary form (1.12) and (2.12))

$$[\![f,g]\!]_A(x) :\equiv i^n \sum_{r=0}^{n-1} (-1)^r f_A^{[n-1-r]}(x)\overline{g_A^{[r]}(x)}, \quad \text{for } x \in \mathfrak{I}. \tag{2.18}$$

Thus when $A = A^+$ we observe that $[\![f,g]\!]_A(x) \equiv 0$ for all x in the complement of $(\operatorname{supp} f \cap \operatorname{supp} g)$ in $\mathfrak{I}$. Hence for $A = A^+$ we conclude that M_A is formally self-adjoint. Moreover the boundary form $[:]_A$ will then be defined by limits, see (1.12) and (2.12), as $[\alpha,\beta]$ expands to exhaust the interval $\mathfrak{I}$ which has endpoints $-\infty \le a < b \le +\infty$,

$$[f:g]_A = \lim_{\substack{\alpha\to a\\ \beta\to b}} \{[\![f,g]\!]_A(\beta) - [\![f,g]\!]_A(\alpha)\}, \quad \text{for } f,g \in \mathcal{D}(T_1). \tag{2.19}$$

Of course, for $f,g \in \mathcal{D}(T_1)$ the individual limits also exist and are finite

$$\lim_{\alpha\to a} [\![f,g]\!](\alpha) = [\![f,g]\!](a), \quad \lim_{\beta\to b} [\![f,g]\!](\beta) = [\![f,g]\!](b). \tag{2.20}$$

Thus the condition (α_3), $A = A^+$, guarantees that M_A is formally self-adjoint (Lagrange symmetric) and that $w^{-1}M_A$ generates maximal and minimal closed operators, T_1 and T_0 respectively, on $\mathcal{L}^2(\mathfrak{I}; w)$, as given above.

Moreover, this assumption (α_3) for $A \in Z_n(\mathfrak{I})$ can be made without loss of generality. This follows from the known result, see the paper of Everitt and Race [**ER**]: if M_B is formally self-adjoint for some $B \in Z_n(\mathfrak{I})$, then there exists a matrix $A \in Z_n(\mathfrak{I})$ such that $A = A^+$ and

$$M_B = M_A \quad \text{with } \mathcal{D}(B) = \mathcal{D}(A)$$

(or possibly, $M_B = -M_A$ when n is odd – see Appendix A). In this case we replace the Shin-Zettl matrix B, by the (non-unique) matrix A satisfying (α_1), (α_2), and (α_3). Henceforth, unless otherwise stated, we shall generally make the hypotheses that $A \in Z_n(\mathfrak{I})$ satisfies (2.3) (α_1) and (α_2), and (2.13) (α_3) in dealing with formally self-adjoint expressions M_A.

It is now understandable how the notation for quasi-derivatives can exploit, through transparently parallel formulas, the analogy between the quasi-differential expression $M_A[y] = i^n y_A^{[n]}$ (for A satisfying (α_1), (α_2), (α_3)) and the classical self-adjoint operator $M = i^n y^{(n)}$.

We illustrate these concepts and notations by the famous Sturm-Liouville formal operator with order $n = 2$. This example will serve to introduce the algebraic methods leading to symplectic geometry, but the precise definitions for complex symplectic geometry will be presented only following the example.

Example 3. Consider the classical Sturm-Liouville formal operator of order $n = 2$,

$$M[y] = -(py')' + qy, \tag{2.21}$$

where p and q are smooth real-valued functions, and $p(x) \neq 0$ for all x on $\mathfrak{I} = [0,1]$. We consider boundary value problems of the form

$$M[y] = \lambda y, \quad (\text{spectral parameter } \lambda \in \mathbb{C}),$$

with weight function $w(x) \equiv 1$ on $\mathfrak{I}$, and we consider various boundary conditions.

The Lagrange-Green identity becomes (as in (1.12))

$$\int_0^1 \{M[f]\bar{g} - f\overline{M[g]}\}dx = -[pf'\bar{g} - fp\bar{g}']_0^1 = [f : g], \tag{2.22}$$

for all suitably smooth complex functions f, g on $[0,1]$. Clearly this boundary form $[:]$ vanishes for all f and g that vanish in a neighborhood of the endpoints $0, 1$. Hence M is formally self-adjoint on $\mathfrak{I}$.

Then the familiar algebraic formula to specify self-adjoint boundary conditions is:

$$p(1)f'(1)\bar{g}(1) - f(1)p(1)\bar{g}'(1) = p(0)f'(0)\bar{g}(0) - f(0)p(0)\bar{g}'(0) \tag{2.23}$$

for f, g in some dense linear manifold $\mathcal{D}(T) \subset \mathcal{L}^2([0,1])$.

For instance, consider the boundary conditions indicated by the functionals,

$$f(0) = 0,\ f(1) = 0 \qquad \text{(same for } g\text{)} - \tag{2.24}$$

as separated boundary conditions; or else,

$$f(0) = f(1), \quad p(0)f'(0) = p(1)f'(1) \quad \text{(same for } g\text{)} - \tag{2.25}$$

as coupled or mixed boundary conditions; and each of (2.24) or (2.25) can be shown to specify a domain $\mathcal{D}(T)$ of a corresponding self-adjoint operator T, as generated by M on $\mathcal{D}(T_1)$, see (1.11).

Now let us examine this same Sturm-Liouville expression M, and the corresponding boundary value problems, from the viewpoint of quasi-differential expressions and the resulting symplectic geometry.

Take the 2×2 matrix $A = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix} \in Z_2(\mathfrak{I})$, involving the coefficient functions p and q on $\mathfrak{I}$. (See Appendix A for quite general rules for choosing such matrices $A \in Z_n(\mathfrak{I})$ satisfying the axioms (α_1), (α_2), (α_3)). Then the corresponding quasi-derivatives for $f \in \mathcal{D}(M_A)$ are $f_A^{[0]} = f$, $f_A^{[1]} = pf'$, $f_A^{[2]} = (pf')' - qf$.

Thus

$$M_A[y] = i^2 y_A^{[2]} = -(py')' + qy = M[y], \tag{2.26}$$

and this provides an algebraic simplification for $M[y]$ in (2.21). Furthermore the boundary form now becomes, as in (2.17), (2.18), (2.19) above,

$$\begin{aligned} [f : g]_A &= \left[-\sum_{r=0}^{1} (-1)^r f_A^{[1-r]}(x)\overline{g_A^{[r]}(x)} \right]_{x=0}^1 \\ &= \left[-f_A^{[1]}(x)\overline{g_A^{[0]}(x)} + f_A^{[0]}(x)\overline{g_A^{[1]}(x)} \right]_{x=0}^1 . \end{aligned} \tag{2.27}$$

So, see (2.22) above,

$$[f : g]_A = [-pf'\bar{g} + fp\bar{g}']_{x=0}^1 = [f : g].$$

For each such complex function f write the "boundary components" as a constant (row) 4-vector $\hat{f}$ in $\mathbb{C}^4$,

$$\hat{f} :\equiv (f_A^{[0]}(0), f_A^{[1]}(0), f_A^{[0]}(1), f_A^{[1]}(1)). \tag{2.28}$$

Then we can compute the value of the boundary form $[f : g]_A$ in terms of the corresponding 4-vectors $\hat{f}$ and $\hat{g}$; namely,

$$(2.29)\quad [f : g]_A = (f_A^{[0]}(0), f_A^{[1]}(0), f_A^{[0]}(1), f_A^{[1]}(1)) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{g_A^{[0]}}(0) \\ \overline{g_A^{[1]}}(0) \\ \overline{g_A^{[0]}}(1) \\ \overline{g_A^{[1]}}(1) \end{pmatrix}$$

or

$$[f : g]_A = \hat{f}\ H\ \hat{g}^*,$$

involving the nonsingular, skew-Hermitian 4×4 matrix

$$H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

But such a semibilinear (or conjugate bilinear) form on $\mathbb{C}^4$, as specified by the matrix H, defines a complex symplectic structure on $\mathbb{C}^4$, which here plays the role of $\mathcal{S}$ in (1.24), see (2.32) below.

Now re-interpret the self-adjoint boundary conditions (2.24), (2.25) in the notation of quasi-derivatives - namely,

$$(2.30)\qquad f_A^{[0]}(0) = 0,\ f_A^{[0]}(1) = 0 \qquad \text{– separated boundary conditions,}$$

or else

$$(2.31)\qquad f_A^{[0]}(0) = f_A^{[0]}(1),\ f_A^{[1]}(0) = f_A^{[1]}(1), \qquad \text{– coupled boundary conditions.}$$

We note that each of these two pairs of conditions (or linear functionals) defines a 2-dimensional subspace L in $\mathbb{C}^4$, whereon $\hat{f}H\hat{g}^* = 0$ for all $\hat{f}$ and $\hat{g}$ in L. Thus, in each case, L is a 2-dimensional subspace of the symplectic space $\mathbb{C}^4$ whereon the symplectic form vanishes - and this property defines Lagrangian 2-planes (see (2.33) below). But each such Lagrangian 2-plane L in the symplectic space $\mathbb{C}^4$, relative to the symplectic form $[:]_A$, determines a linear manifold $\mathcal{D}(T) = \{f \in \mathcal{D}(T_1) \mid \hat{f} \in L\} \subset \mathcal{L}^2([0,1])$, and, as we shall show later, a corresponding self-adjoint operator T on $\mathcal{D}(T)$, as generated by the Sturm-Liouville expression M_A of (2.26).

It is through this use of the boundary form $[:]_A$, vanishing on $\mathcal{D}(T)$, that the self-adjoint boundary conditions are imposed at the endpoints 0 and 1 for the Sturm-Liouville operator (2.21) or (2.26). Other self-adjoint boundary conditions can easily be found, corresponding to other Lagrangian 2-planes in $\mathbb{C}^4$, and this illustrates the general GKN-Theorem which has been re-interpreted as our Theorem 1 in Section II. This theorem asserts the one-to-one correspondence between the domains $\mathcal{D}(T)$, of self-adjoint operators T, as generated by a given quasi-differential expression $w^{-1}M_A$, and the Lagrangian d-spaces in a complex symplectic $2d$-space $\mathcal{S}$ – presuming that the quasi-differential expression $w^{-1}M_A$ is formally self-adjoint on a prescribed interval $\mathcal{I}$, with the deficiency index d, as before (1.16), (1.23), (1.24).

Finally this example illustrates how the notation of quasi-derivatives is eminently suited for characterizing the self-adjoint boundary conditions by replacing confusing algebraic manipulations, required for the classical differential expressions M (where complications arise because the boundary form explicitly involves the coefficients of M), with the algebra of symplectic geometry.

In order to fix the algebraic notation for symplectic geometry we next present the fundamental definitions, which will be expanded upon later in Section III, see the texts of Abraham and Marsden [**AM**], Markus [**MA**], McDuff and Salamon [**MS**], and the paper of Robbin [**RO**]. It will be our custom to refer to the abstract or purely algebraic structure of complex symplectic spaces by the symbol S, whereas the particular symplectic spaces that arise as complex-valued function spaces (as in the GKN-Theory) will be denoted by the related, but distinct, notation S.

Definition 5. A complex linear space S, together with a complex-valued function on the product space $S \times S$,

$$X, Y \to [X : Y], \qquad S \times S \to \mathbb{C} \tag{2.32}$$

is a *pre-symplectic space* in case:

(σ_1) (sesquilinear, semibilinear, or conjugate bilinear property)

$$\begin{aligned} &[Z : X + Y] = [Z : X] + [Z : Y] \\ &[X + Y : Z] = [X : Z] + [Y : Z] \\ &[\mu X : Y] = \mu[X : Y], \quad [X : \mu Y] = \bar{\mu}[X : Y] \end{aligned}$$

for all X, Y, Z in S, and $\mu \in \mathbb{C}$; and

(σ_2) (skew-Hermitian or alternating property)

$$[X : Y] = -\overline{[Y : X]} \qquad \text{for all } X, Y \text{ in } S$$

(or equally well, $i[X : Y] = \overline{i[Y : X]}$ is Hermitian symmetric).

Thus properties (σ_1), (σ_2) together assert that $i[X : Y]$ is a Hermitian symmetric semibilinear form on S, and conversely this means that S is a pre-symplectic space (with the skew-Hermitian semibilinear form [:]).

If in addition to properties (σ_1) and (σ_2) we further require

(σ_3) (nondegeneracy property)

$$[X : Y] = 0, \text{ for all } Y \in S, \text{ implies that } X = 0,$$

then S, together with the nondegenerate, skew-Hermitian, semibilinear form [:], is a *symplectic space*. (We often refer to this symplectic form as "bilinear" for short).

We consider complex symplectic *linear spaces* only, and their symplectic homomorphisms ($\mathbb{C}$-linear maps preserving the "symplectic product" [:]). The usual concepts of subspace, homomorphism, isomorphism, etc. all apply to complex symplectic spaces, and also to pre-symplectic spaces. The trivial case where $\dim S = 0$ is allowed as a symplectic space.

The concept of a real symplectic (linear) space is classical, [**AM**] [**MA**] and [**MH**], but this will be reviewed–together with methods of complexification, in Appendix B.

NOTE. We refer to the study of finite dimensional (real or complex) symplectic spaces, especially with the corresponding methodology of linear algebra, as *Symplectic Geometry*, or equally well, *Symplectic Algebra*. Further detailed discussions of Symplectic Geometry and Symplectic Algebra are found in Section III.

DEFINITION 6. Let S, together with the skew-Hermitian semibilinear form $[\,:\,]$, be a pre-symplectic space. A linear subspace $L \subset S$ is called *Lagrangian* in case:

$$[X : Y] = 0 \qquad \text{for all } X, Y \text{ in } L. \tag{2.33}$$

In a pre-symplectic space S, with the semibilinear form $[\,:\,]$, we note that

$$[X : X] = -\overline{[X : X]} \qquad \text{so } \operatorname{Re}[X : X] = 0.$$

Also

$$[\mu X : \mu X] = \mu\bar{\mu}[X : X] = |\mu|^2[X : X]$$

for each vector $X \in S$, and scalar $\mu \in \mathbb{C}$. Hence each vector $X \in S$ (moreover each complex 1-dimensional space μX, for $\mu \in \mathbb{C}$) is of exactly one of the following three types:

(2.34)

(1) positive, $\operatorname{Im}[X : X] > 0$

(2) negative, $\operatorname{Im}[X : X] < 0$

(3) neutral, $\operatorname{Im}[X : X] = 0$, so $[X : X] = 0$.

Note that a Lagrangian subspace $L \subset S$ consists of neutral vectors, with the additional property that $[X : Y] = 0$ for all $X, Y \in L$.

REMARKS. The concepts and nomenclature of symplectic geometry, symplectic algebra, and Lagrangian subspaces, arise in classical Hamiltonian mechanics [**AM**], [**MA**] and [**RO**], and hence much of the linear algebra occurring in our analysis is rather familiar. But many unusual features are caused through the introduction of complex coordinates. Further, the existence of distinguished subspaces, corresponding to the boundary conditions at the endpoints a, b of $\mathcal{I}$, lead to new and difficult problems of complex symplectic geometry. These matters will be explored in greater detail in Section III.

We further comment that the terminology for Lagrangian subspaces is usually restricted in the literature to n-dimensional subspaces of a real $2n$-dimensional symplectic space, and our Lagrangian spaces are often called isotropic [**AM**]. Sometimes, especially in the theory of self-adjoint operators, a Lagrangian space is referred to as "symmetric", see [**DS**], [**EZ,** Theorem 1.1 and Section 3].

EXAMPLE 4.

1. Real symplectic $2n$-space complexified to $\mathbb{C}^{2n}$. Take $\mathbb{R}^{2n}$ with a basis $X^1, X^2, \cdots, X^n$, $Y^1, \cdots Y^n$ and define the "symplectic products"

$$[X^j : X^k] = 0,\ [Y^j : Y^k] = 0,\ [X^j : Y^k] = \delta^{jk} \quad \text{(Kronecker-}\delta)$$

for $1 \le j, k \le n$. Then $\mathbb{R}^{2n}$ is a real symplectic space (see Section III, and Appendix B) with Lagrangian subspaces spanned by $\{X^1, \dots, X^n\}$, and

$\{Y^1, \dots, Y^n\}$. Now complexify $\mathbb{R}^{2n}$ to $\mathbb{C}^{2n}$ to obtain the corresponding complex symplectic form on the complex symplectic space $\mathbb{C}^{2n}$. As a trivial illustration consider a basis X, Y for the complex symplectic space $\mathbb{C}^2$, with $[X : X] = [Y : Y] = 0$, $[X : Y] = 1$; and note the Lagrangian 1-space spanned by the vector $X + Y$.

2. Consider the complex linear space $\mathbb{C}^3$ with the basis X, Y, Z and the corresponding symplectic form defined by

$$[X : X] = [Y : Y] = 0, [Z : Z] = i$$
$$[X : Y] = 1, [X : Z] = [Y : Z] = 0.$$

On the other hand, a different (non-isomorphic) symplectic structure can be defined on the linear space $\mathbb{C}^3$ by simply changing to $[Z : Z] = -i$. Yet neither of these complex symplectic 3-spaces contains a Lagrangian 2-plane.

3. It will be shown later in Section II that $\mathcal{D}(T_0)$ is an infinite dimension Lagrangian subspace of the pre-symplectic space $\mathcal{D}(T_1)$.

The GKN-theory of self-adjoint extensions of quasi-differential operators necessarily requires the consideration of complex pre-symplectic spaces of infinite dimensions, and also of finite dimensional complex symplectic spaces of both even and odd dimensions.