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Preface

Model categories, first introduced by Quillen in [**Qui67**], form the foundation of homotopy theory. The basic problem that model categories solve is the following. Given a category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in the quotient category. The morphisms between two objects in the quotient category may not even be a set. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient category from X to Y are simply homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y.

Because this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important. However, so far their utility has been mostly confined to areas historically associated with algebraic topology, such as homological algebra, algebraic K-theory, and algebraic topology itself. The author is certain that this list will be expanded to cover other areas of mathematics in the near future. For example, Voevodksy's work [**Voe97**] is certain to make model categories part of every algebraic geometer's toolkit.

These examples should make it clear that model categories are really fundamental. However, there is no systematic study of model categories in the literature. Nowhere can the author find a definition of the category of model categories, for example. Yet one of the main lessons of twentieth century mathematics is that to study a structure, one must also study the maps that preserve that structure.

In addition, there is no excellent source for information about model categories. The standard reference [**Qui67**] is difficult to read, because there is no index and because the definitions are not ideal (they were changed later in [**Qui69**]). There is also [**BK72**, Part II], which is very good at what it does, but whose emphasis is only on simplicial sets. More recently, there is the expository paper [**DS95**], which is highly recommended as an introduction. But there is no mention of simplicial sets in that paper, and it does not go very far into the theory.

The time seems to be right for a more careful study of model categories from the ground up. Both of the books [**DHK**] and [**Hir97**], unfinished as the author writes this, will do this from different perspectives. The book [**DHK**] overlaps considerably with this one, but concentrates more on homotopy colimits and less on the relationship between a model category and its homotopy category. The book [**Hir97**] is concerned with localization of model categories, but also contains a significant amount of general theory. There is also the book [**GJ97**], which concentrates on simplicial examples. All three of these books are highly recommended to the reader.

This book is also an exposition of model categories from the ground up. In particular, this book should be accessible to graduate students. There are very few prerequisites to reading it, beyond a basic familiarity with categories and functors, and some familiarity with at least one of the central examples of chain complexes, simplicial sets, or topological spaces. We also require some familiarity with the basics of set theory, especially ordinal and cardinal numbers. Later in the book we do require more of the reader; in Chapter 7 we use the theory of homotopy limits of diagrams of simplicial sets, developed in [**BK72**]. However, the reader who gets that far will be well equipped to understand [**BK72**] in any case.

This book is the author's attempt to understand the theory of model categories well enough to answer one question. That question is: when is the homotopy category of a model category a stable homotopy category in the sense of [**HPS97**]? We do not in the end answer this question in quite as much generality as one would like, though we come fairly close to doing so in Chapter 7. As I tried to answer this question, it became clear that the theory necessary to do so was not in place. After a long period of resistance, I decided it was important to develop the necessary theory, and that the logical and most useful place to do so was in a book that would assume almost nothing of the reader. A book is the logical place because the theory I develop requires a foundation that is simply not in the literature. I think this foundation is beautiful and important, and therefore deserves to be made accessible to the general mathematician.

We now provide an overview of the book. See also the introductions to the individual chapters. The first chapter of this book is devoted to the basic definitions and results about model categories. In highfalutin language, the main goal of this chapter is to define the 2-category of model categories and show that the homotopy category is part of a pseudo-2-functor from model categories to categories. This is a fancy way, fully explained in Section 1.4, to say that not only can one take the homotopy category of a model category, one can also take the total derived adjunction of a Quillen adjunction, and the total derived natural transformation of a natural transformation between Quillen adjunctions. Doing so preserves compositions for the most part, but not exactly. This is the reason for the word "pseudo". In order to reach this goal, we have to adopt a different definition of model category from that of [**DHK**], but the difference is minor. The definition of [**DHK**], on the other hand, is considerably different from the original definition of [**Qui67**], and even from its refinement in [**Qui69**].

After the theoretical material of the first chapter, the reader is entitled to some examples. We consider the important examples of chain complexes over a ring, topological spaces, and chain complexes of comodules over a commutative Hopf algebra in the second chapter, while the third is devoted to the central example of simplicial sets. Proving that a particular category has a model structure is always difficult. There is, however, a standard method, introduced by Quillen [**Qui67**] but formalized in [**DHK**]. This method is an elaboration of the small object argument and is known as the theory of cofibrantly generated model categories. After examining this theory in Section 2.1, we consider the category of modules over a Frobenius ring, where projective and injective modules coincide. This is perhaps the simplest nontrivial example of a model category, as every object is both cofibrant and fibrant. Nevertheless, the material in this section seems not to have appeared before, except in Georgian [**Pir86**]. Then we consider chain complexes of modules over an arbitrary ring. Our treatment differs somewhat from the standard one in that we do not assume our chain complexes are bounded below. We then move on to topological spaces. Here our treatment is the standard one, except that we offer more details than are commonly provided. The model category of chain complexes of comodules over a commutative Hopf algebra, on the other hand, has not been considered before. It is relevant to the recent work in modular representation theory of Benson, Carlson, Rickard and others (see, for example [**BCR96**]), as well as to the study of stable homotopy over the Steenrod algebra [**Pal97**]. The approach to simplicial sets given in the third chapter is substantially the same as that of [**GJ97**].

In the fourth chapter we consider model categories that have an internal tensor product making them into closed monoidal categories. Almost all the standard model categories are like this: chain complexes of abelian groups have the tensor product, for example. Of course, one must require the tensor product and the model structure to be compatible in an appropriate sense. The resulting monoidal model categories play the same role in the theory of model categories that ordinary rings do in algebra, so that one can consider modules and algebras over them. A module over the monoidal model category of simplicial sets, for example, is the same thing as a simplicial model category. Of course, the homotopy category of a monoidal model category is a closed monoidal category in a natural way, and similarly for modules and algebras. The material in this chapter is all fairly straightforward, but has not appeared in print before. It may also be in [**DHK**], when that book appears.

The fifth and sixth chapters form the technical heart of the book. In the fifth chapter, we show that the homotopy category of any model category has the same good properties as the homotopy category of a simplicial model category. In our highfalutin language, the homotopy pseudo-2-functor lifts to a pseudo-2-functor from model categories to closed Ho **SSet**-modules, where Ho **SSet** is the homotopy category of simplicial sets. This follows from the idea of framings developed in [**DK80**]. This chapter thus has a lot of overlap with [**DHK**], where framings are also considered. However, the emphasis in [**DHK**] seems to be on using framings to develop the theory of homotopy colimits and homotopy limits, whereas we are more interested in making Ho **SSet** act naturally on the homotopy category. There is a nagging question left unsolved in this chapter, however. We find that the homotopy category of a monoidal model category is naturally a closed algebra over Ho **SSet**, but we are unable to prove that it is a *central* closed algebra.

In the sixth chapter we consider the homotopy category of a pointed model category. As was originally pointed out by Quillen [**Qui67**], the apparently minor condition that the initial and terminal objects coincide in a model category has profound implications in the homotopy category. One gets a suspension and loop functor and cofiber and fiber sequences. In the light of the fifth chapter, however, we realize we get an entire closed Ho $\mathbf{SSet}_*$-action, of which the suspension and loop functors are merely specializations. Here Ho $\mathbf{SSet}_*$ is the homotopy category of pointed simplicial sets. We prove that the cofiber and fiber sequences are compatible with this action in an appropriate sense, as well as reproving the standard facts about cofiber and fiber sequences. We then get a notion of pre-triangulated categories, which are closed Ho $\mathbf{SSet}_*$-modules with cofiber and fiber sequences satisfying many axioms.

The seventh chapter is devoted to the stable situation. We define a pre-triangulated category to be *triangulated* if the suspension functor is an equivalence of categories. This is definitely not the same as the usual definition of triangulated categories, but it is closer than one might think at first glance. We also argue that it is a better definition. Every triangulated category that arises in nature is the homotopy category of a model category, so will be triangulated in our stronger sense. We also consider generators in the homotopy category of a pointed model category. These generators are extremely important in the theory of stable homotopy categories developed in [**HPS97**]. Our results are not completely satisfying, but they do go a long way towards answering our original question: when is the homotopy category of a model category a stable homotopy category?

Finally, we close the book with a brief chapter containing some unsolved or partially solved problems the author would like to know more about.

Note that bold-faced page numbers in the index are used to indicate pages containing the definition of the entry. Ordinary page numbers indicate a textual reference.

I would like to acknowledge the help of several people in the course of writing this book. I went from knowing very little about model categories to writing this book in the course of about two years. This would not have been possible without the patient help of Phil Hirschhorn, Dan Kan, Charles Rezk, Brooke Shipley, and Jeff Smith, experts in model categories all. I wish to thank John Palmieri for countless conversations about the material in this book. Thanks are also due Gaunce Lewis for help with compactly generated topological spaces, and Mark Johnson for comments on early drafts of this book. And I wish to thank my family, Karen, Grace, and Patrick, for the emotional support so necessary in the frustrating enterprise of writing a book.