Contents

Preface

The complete title of this book should be "Nonlocal bifurcations, normal forms, and elements of hyperbolic theory".

We present the modern theory of normal forms for local families of vector fields and diffeomorphisms. This presentation contains a complete list of integrable normal forms in the finitely smooth classification. This classification is the most suitable one for applications to nonlocal theory.

Hyperbolic and partial hyperbolic theory is the tool for the description of the invariant sets that occur under nonlocal bifurcations.

We restrict ourselves to the study of the bifurcations that occur on the boundary of the set of Morse–Smale systems and are generated by the loss of hyperbolicity of singular points and periodic orbits. Even this moderate goal leads to a variety of effects, which are only partly investigated up to now. A very rich domain of study is related to the homoclinic tangency of stable and unstable manifolds of singular points and periodic orbits. This subject, discussed in [**PT**], is beyond the scope of this book.

The most celebrated nonlocal bifurcations are those of the homoclinic orbit of a saddle. Bifurcations of this kind in space, both those that generate one periodic orbit and an arbitrary number of Smale horseshoes, are described in many books. The first occur at the boundary of the Morse–Smale set; the others are distant from this boundary. The general case of bifurcations of homoclinic orbits of saddles in spaces of arbitrary dimension was not described in detail in previous monographs on the same subject. All these results are presented below.

Less celebrated are bifurcations of homoclinic orbits of saddlenode singular points. One homoclinic curve generates one periodic orbit. But without increasing the rate of degeneracy, several homoclinic curves of the same point may occur. Their bifurcation gives rise to a nontrivial hyperbolic set: an Ω-explosion takes place.

The most complicated are the bifurcations of homoclinic surfaces of a saddle-node periodic orbit. It is not difficult to imagine a semistable cycle in a 2-torus filled by homoclinic orbits of this cycle. A homoclinic surface of this type may occur in a space of arbitrary dimension. But much more complicated homoclinic surfaces may occur as well. There may be a Klein bottle instead of a torus. Several homoclinic surfaces of the same periodic orbit may occur simultaneously. Their bifurcations lead to a new class of dynamical systems, whose investigation is now at the very beginning. The homoclinic surfaces may be topologically complicated. The so-called twisted homoclinic surfaces may occur in a higher-dimensional space (see Figure 1.15 below). The corresponding bifurcation gives rise to a hyperbolic attractor of solenoidal type.

Some new results related to bifurcations of homoclinic surfaces described above are presented in this book. Namely,

• smooth homoclinic tori and Klein bottles preserved under small perturbations;

• an invariant set with a random dynamical system on it occurring under the bifurcation of several homoclinic surfaces;

• a strange attractor generated under the bifurcation of the twisted homoclinic surface (Shilnikov and Turaev, 1995; the first complete proof was given by the first author during the work on this book).

The theory of normal forms drastically simplifies the study of nonlocal bifurcations. It provides integrable normal forms not only for the unperturbed equation near the equilibrium point, but for the perturbation as well. Therefore the map of the cross-sections transversal to invariant manifolds of the singular point along the orbits of the vector field may be explicitly calculated. Thus simple formulas replace the delicate estimates of the previous investigations.

There are two types of applications of these normal forms to nonlocal bifurcations.

The first is related to planar bifurcations. Here there are two directions of study as well: families with few and with many parameters.

Families with a small number of parameters may be investigated in full detail. The classical results of Andronov and his school are related to the nonlocal bifurcations in planar one-parameter families. The theory of normal forms transforms the proofs of these results into simple exercises. The general study of two- and three-parameter families in the plane was suggested by Arnold in 1985. At the beginning of the nineties Kotova collected the "zoo" of all polycycles that may occur in generic two- and three-parameter families. The cyclicity of these polycycles was investigated in a number of papers by Dumortier, Grozovskiĭ, Kotova–Stanzo, Morsalami, Mourtada, Roussarie, Rousseau, Sotomayor and others. The concluding paper was recently published by Trifonov.

The study of many-parameter families in the plane is mainly related to the so-called Hilbert–Arnold problem: to prove that the polycycle that occurs in a generic finite-parameter family generates no more than a finite number of limit cycles, and this number depends on the number of parameters. This problem is solved for polycycles with elementary singular points as vertexes, the so-called elementary polycycles. These results are included in two collections [**I2, IYa3**], where the other sources are quoted. Recently, Kaloshin obtained an explicit estimate of the number $E(k)$ of limit cycles generated by elementary polycycles in typical k-parameter families.

The second application of normal forms to nonlocal bifurcations is the study of spatial bifurcations. A systematic study is carried out for one-parameter families. This is the subject of this book. It seems that two-parameter spatial families may be studied in detail as well. The complete study of nonlocal bifurcations in three-parameter spatial and four-parameter planar families seems to be too complicated to be ever obtained.

The theory of normal forms for local families is presented in this book from the very beginning. We describe the homotopy method, which is the most convenient tool for the local smooth theory. The results we present are in a sense complete: we give the list of smooth integrable normal forms for the simplest families; smooth classification of more complicated families has functional moduli.

Elements of the hyperbolic theory are also presented from the very beginning and lead up to the newest results. The presentation of the nonlinear Smale horseshoe map deals with classical material. On the other hand, we study parallel results for partially hyperbolic maps. Roughly speaking, these maps have stable, unstable, and central subbundles of the tangent bundle. In general, stable and unstable foliations for such maps do exist, while the center-stable and center-unstable do not. The maps we study are subject to special geometric assumptions that guarantee the existence of all the invariant foliations mentioned above. This gives rise to a new kind of dynamics: random dynamical systems appear to be subsystems of smooth ones. The investigation of these systems is beyond the scope of this book. Here we only prove their birth under the bifurcations of several homoclinic surfaces of a saddlenode periodic orbit.

We describe all the bifurcations from a uniform point of view. Namely, all the proofs are obtained by studying the interaction of the theory of normal forms and hyperbolic theory. The main subject is the global Poincaré map represented as a product of singular and regular maps. The singular map is a map of cross-sections along the orbits passing near a singular point. This map is not everywhere defined and produces an unbounded distortion. It contracts in some directions and expands in others. The regular map is once more a map of cross-sections, this time along the orbits distant from a singular point. It is well defined, and bounded together with its derivatives.

The theory of normal forms gives explicit formulas for singular maps. The genericity assumptions guarantee that the regular map does not mix the contracting and expanding directions of the singular one. For the product of these maps hyperbolicity wins: the unbounded distortion of the singular maps overcomes the influence of the smooth regular map. Thus the global Poincaré map becomes the subject of hyperbolic theory, which provides the description of the invariant sets of this map.

Our uniform approach is illustrated in Figures 4.9, 4.12, 5.12, 7.7, 7.12, 7.18.

The idea of this book arose when the survey "Bifurcation theory" by Afraimovich, Arnold, Ilyashenko, and Shilnikov, 1986, was written. In the process, Arnold said that the survey should reflect the development of bifurcation theory for at least the twenty five years to come. The present book is a partial response to this challenge. Bifurcations, like torches, shed light on the transition from simple dynamical systems to complicated ones. Complicated systems occurring under bifurcations of homoclinic surfaces of nonhyperbolic periodic orbits partially described in Chapters 5 and 6 are the subject for promising future study.

The present book develops the third chapter of the survey [**AAIS**] (description of the bifurcations themselves) and the end of the second chapter of the same survey, where normal forms for local families were listed for the first time. In 1988 Professor Zhang Zhifen from Beijin organized a seminar on nonlocal bifurcations, where the proofs missing in the survey were reconstructed. The second author was an active participant of this seminar. In 1991–93 he had a scholarship in Moscow. During this period the first draft of the book was written. The present version is the result of several rewritings produced by both authors.

The first chapter contains all the necessary definitions beginning with very elementary ones. Its goal is to present the main results about nonlocal bifurcations. They are summarized in the main table at the end of the chapter.

Chapter 2 contains two sections that are in fact outside the main content of the book. The first discusses "prevalence": the concept of genericity different from the traditional "category" notion. The revision of the genericity assumptions throughout this book from the "prevalent" point of view is a challenging problem. The second large section is devoted to the Hausdorff dimension of attractors. The results of this section are applied only once, in Chapter 6. Yet the subject seems to be of independent interest, and was therefore included. The third section contains an elementary description of the Smale horseshoe, understandable for high school students. The rest of the chapter contains a summary of the hyperbolic and normal form theories used throughout the book.

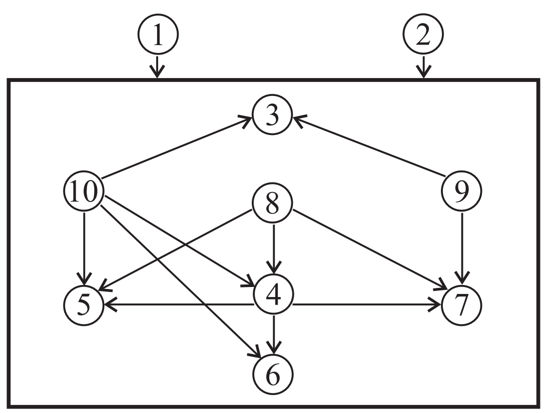
The next five chapters present nonlocal bifurcations. Chapter 3 is elementary and deals with one-parameter planar families of vector fields. The only exception is the end of the chapter, where simultaneous occurrence of several saddle connections in two-parameter families is studied.

Chapter 4 contains the main ideas of our subsequent study. It demonstrates the mechanism of the occurrence of hyperbolicity via the singular and transversality properties of the Poincaré map.

The next three chapters are the main ones. Chapter 5 studies bifurcations of smooth homoclinic surfaces of saddlenode cycles. It contains the new results mentioned above. Chapter 6 deals with nonsmooth homoclinic surfaces. It presents the first detailed exposition of the study of the strange attractor begun by Afraimovich and Shilnikov in the seventies and followed by Newhouse, Palis, Takens in the eighties. Chapter 7 describes bifurcations of the homoclinic orbit of a saddle in the space of arbitrary dimension.

The hyperbolic and partial hyperbolic theory is presented in Chapter 8. The last two chapters are devoted to normal forms for local families. These three chapters contain proofs of the results stated in the last two sections of Chapter 2.

The dependence of chapters is shown below.



The system of references is as follows. Theorems, lemmas, propositions, and formulas have double numbers $a.b$, where a is the number of the section, and b is the number of the statement. The references inside the same chapter use this numeration. The reference to item $a.b$ in Chapter A looks like $A.a.b$.

The authors are grateful to Professor Zhang Zhifen who organized their cooperation; to Alexander Ilyashenko for the preparation of the computer version of the figures; to Helen Ilyashenko and Sergei Yakovenko, who keyboarded part of the manuscript; to Alexander Bufetov, Anton Gorodetskii, Maria Saprykina and other participants of the Moscow seminar on differential equations for helpful comments and technical assistance. We are grateful as well to Dr. Sergei Gelfand and Dr. Alexei Sossinskii who helped a great deal at the final stage of the preparation of the manuscript. The first author is grateful to Cornell University, where the large part of the final draft was written, to UNAM, Mexico, and CIMAT, Guanajuato, where the first author presented a course on nonlocal bifurcations. The second author had a fruitful stay at IMPA, Rio de Janiero, and at the University of São Paulo. Both authors are grateful to all the friends and colleagues for the cordial and stimulating atmosphere during these stays.

Further, the first author is grateful to Valya Afraimovich, who taught him nonlocal bifurcations while they were working together over the survey [**AAIS**].

CHAPTER 1

Introduction

This book is devoted to nonlocal bifurcations that may occur in typical one-parameter families crossing the boundary of the set of Morse–Smale systems in the space of vector fields. This approach allows us to build up a broad picture covering most of the results on nonlocal bifurcations obtained from the sixties to the present day. At the same time, some yet unsolved problems naturally arise.

Multidimensional structurally stable systems generally exhibit such complex behavior that it is hopeless to try to give a complete description of bifurcations occurring on the boundary of the set of such systems. The systematic study of these problems was never carried out. On the other hand, Morse–Smale systems are rather simple and constitute a sufficiently rich class of objects: most of the nonlocal bifurcations studied up to now occur on the boundary of that class. We shall consider only generic points of this boundary. This means that we study *generic one-parameter families* of vector fields and bifurcations occurring in such families.

§1. Structural stability and Morse–Smale systems

In this section we introduce the concept of structural stability and some basic properties of Morse–Smale systems.

1.1. Structural stability. The idea behind the definition of the structural stability of a vector field is the following: the field is structurally stable if a sufficiently small perturbation does not change the dynamical properties of the field. To give a precise definition, we need to specify what is meant by a small perturbation of a vector field f.

DEFINITION 1.1. The C^r-*topology* on the space of C^r smooth functions defined in a bounded domain $\Omega \subset \mathbb{R}^n$ is given by the following convergence rule: a sequence of functions converges to a certain limit in this topology if all derivatives of the functions from the sequence up to order r converge to the corresponding derivatives of the limit function uniformly in Ω.

The C^r-topology on the space of C^r smooth vector fields $\chi^r(\Omega)$ or maps from Ω to $\mathbb{R}^m$ (denoted by $C^r(\Omega, \mathbb{R}^m)$) is defined in terms of C^r-proximity of each component.

The convergence rule defines neighborhoods in the natural way: we say that the field $\tilde{f}$ is a C^r-small perturbation of the field f if the two fields are close in the C^r-topology.

Now we introduce an equivalence relation expressing the fact that two dynamical systems generated by equivalent vector fields have the same geometric structure. Roughly speaking, we say that two dynamical systems are equivalent if there exists a map taking the phase space of one system into that of the other and this map sends trajectories into trajectories. This definition can be made precise in several different ways.

DEFINITION 1.2. Two vector fields are *smoothly equivalent* if there exists a diffeomorphism taking one field into the other. If we denote the vector fields by v and w, and the diffeomorphism by H, then the latter condition means that

$$\frac{\partial H}{\partial x} \cdot v = w \circ H. \tag{1.1}$$

The smooth classification has numerical moduli, the eigenvalues of linearizations of vector fields at singular points: for smoothly equivalent vector fields the eigenvalues coincide. Recall that the set of eigenvalues of a vector field at a singular point is the spectrum of the linear operator linearizing the field at that point. Indeed, if both vector fields have a singularity at the origin $0 \in \mathbb{R}^n$ and $H(0) = 0$, then

$$CAC^{-1} = B, \tag{1.2}$$

where

$$C = \frac{\partial H}{\partial X}(0), \qquad v(x) = Ax + \cdots, \qquad w(x) = Bx + \cdots, \tag{1.3}$$

and the dots denote terms of order $\geqslant 2$. Thus it is clear that no vector field can be smoothly equivalent to all close fields if the given field has at least one singular point.

DEFINITION 1.3. Two vector fields are *topologically equivalent* if there exists a homeomorphism between the phase spaces of these fields that conjugates their flows.

Denote by g_v^t (resp., by g_w^t) the flows of the vector fields v and w respectively. If H is the homeomorphism from Definition 3, then

$$g_v^t \circ H(x) = H \circ g_w^t(x) \tag{1.4}$$

for all values x and t such that both parts of the equality make sense. The topological classification also has numerical moduli, one of them being the period of a closed periodic orbit. Thus one can see that a vector field with an isolated periodic orbit cannot be topologically equivalent to all perturbations.

DEFINITION 1.4. Two vector fields are *orbitally topologically equivalent* if there exists a homeomorphism between the phase spaces that takes phase trajectories (curves) of the first field into those of the second, preserving the natural orientation on the curves.

Finally we come to the principal definition.

DEFINITION 1.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A vector field $f \in \chi^r(\Omega)$ is *structurally stable* if there exists an $\varepsilon > 0$ such that all ε-small perturbations in the sense of the C^1-topology are orbitally topologically equivalent to f.

1.2. Nonwandering points. What is meant when we say that a certain point in the phase space of a vector field has a nontrivial dynamic behavior? Naturally, points which return in time to an arbitrarily small neighborhood of their original position exhibit nontrivial dynamic behavior. Examples of such points are

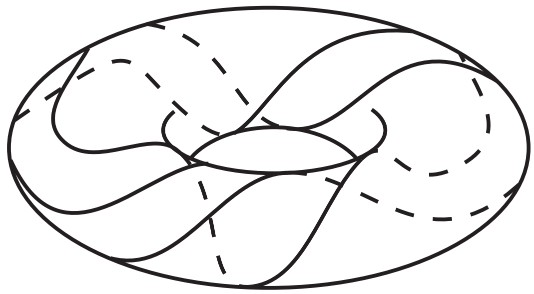


FIGURE 1.1. Irrational flow on a 2-torus.

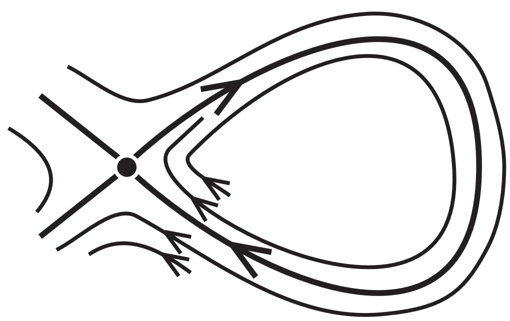


FIGURE 1.2. The homoclinic orbit of a saddle.

singular points or periodic orbits. There also may be more complex recurrence (see Figure 1.1).

On the other hand, a point may never return to its original position, though arbitrarily close points exhibit recurrent motion, as shown in Figure 1.2. This argument motivates the following definition.

DEFINITION 1.6. A point of the phase space is called *nonwandering* for the flow if any small neighborhood of that point, when moved by the flow, intersects its original position in any distant future.

In other words, the point p is nonwandering if for any neighborhood $U \ni p$ and for any $T < +\infty$ there exists a moment $t > T$ such that

$$g^t(U) \cap U \neq \varnothing. \tag{1.5}$$

An orbit of a nonwandering point is called a nonwandering orbit.

1.3. Hyperbolic singular points and cycles. In this section we describe singular points and periodic orbits which are *locally* structurally stable, i.e., there exists a small neighborhood of the point (resp., the cycle) such that the restriction of the field to that neighborhood is structurally stable.

DEFINITION 1.7. A singular point of a vector field is *hyperbolic* if it has no eigenvalues on the imaginary axis.

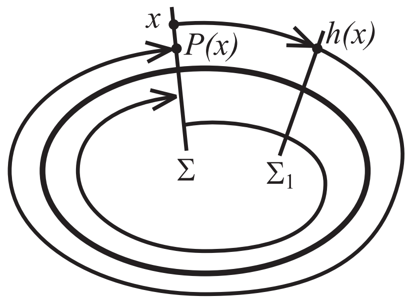


FIGURE 1.3. The Poincaré map of a cycle.

To give a similar definition for a periodic orbit, we recall the basic construction of the Poincaré return map. Take a hypersurface Σ transversal to the vector field at a certain point of the periodic orbit (cycle). For all points sufficiently close to the intersection of this surface with the cycle, the first return map is defined as taking a point x on Σ to the first intersection point of the positive semiorbit emanating from x with the surface Σ, as shown in Figure 1.3. We denote this map by P. It has numerous names: *Poincaré map, first return map, monodromy map*, etc. Note that the point of intersection of Σ with the cycle itself is a fixed point of the map P.

DEFINITION 1.8. Two C^r smooth maps F, G are *C^k-equivalent* if there exists a C^k smooth diffeomorphism h conjugating them: $h \circ F = G \circ h$. If $k = 0$, then such equivalence is called *topological equivalence*.

DEFINITION 1.9. The *multipliers* of a C^1 smooth map at a fixed point are the eigenvalues of the linearization of this map at that point. The multipliers of a periodic orbit (cycle) are the multipliers of its monodromy map at the fixed point corresponding to the cycle itself.

REMARK. Let Σ and Σ_1 be two transversal sections to the same cycle, and P and P_1 the two corresponding Poincaré maps. Let $h\colon \Sigma \to \Sigma_1$ be the map projecting one surface onto the other along flow curves, as shown in Figure 1.3. Then $P_1 \circ h = h \circ P$ (which means that the maps P and P_1 are C^r-equivalent if the field is of class C^r). From this simple observation we immediately conclude that the matrices of the linearization of P and P_1 are similar, hence have the same spectrum. In other words, the multipliers of a cycle are defined independently of the choice of the transversal section.

DEFINITION 1.10. A fixed point of a diffeomorphism is *hyperbolic* if it has no multipliers on the unit circle in the complex plane. A cycle is hyperbolic if its monodromy has a hyperbolic fixed point.

1.4. Topological classification of flows near hyperbolic singular points and cycles. For any finite-dimensional phase space, there is only a finite number of topological orbital normal forms for a vector field near a hyperbolic singular point. The same is true for hyperbolic fixed points of diffeomorphisms.

THEOREM 1.1 (Grobman–Hartman). *A C^1 smooth vector field (C^1-diffeomorphism) near a hyperbolic singular (resp., fixed) point is topologically equivalent to its linearization at that point.*

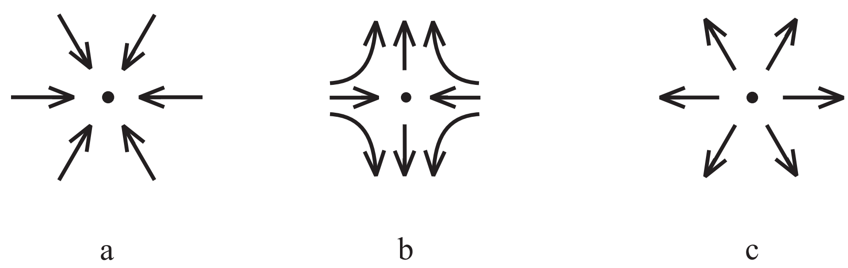


FIGURE 1.4. Topological classification of hyperbolic singularities in the plane.

FIGURE 1.5. Hyperbolic linear maps that preserve or reverse the orientation.

This result immediately implies that the problem of topological classification of hyperbolic singular points and cycles is reduced to the same problem for linear fields (maps). The last step is elementary. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, and denote by n_- (resp., n_+) the number of its eigenvalues in the left (resp., right) open half-plane so that $n = n_- + n_+$ due to hyperbolicity. Then the differential equation

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{1.6}$$

is topologically equivalent to the *standard saddle*

$$\begin{aligned} \dot{y} &= y, \qquad && y \in \mathbb{R}^{n_+}, \\ \dot{z} &= -z, \qquad && z \in \mathbb{R}^{n_-}. \end{aligned} \tag{1.7}$$

This concludes the topological classification of vector fields near a hyperbolic singularity. One may see from Figure 1.4 that the topological equivalence is a very robust relation. However, this equivalence clearly distinguishes between sinks, saddles and sources.

The topological classification of hyperbolic linear maps is almost the same, the only difference being caused by the preservation or reversal of orientation (see Figure 1.5).

Denote by S the standard mirror symmetry of the space $\mathbb{R}^m$ in the hyperplane $x_1 = 0$: $S(x_1, x_2, \dots, x_m) = (-x_1, x_2, \dots, x_m)$. Suppose that a nonsingular linear operator $B\colon \mathbb{R}^n \to \mathbb{R}^n$ has n_- eigenvalues strictly inside the unit circle and n_+ eigenvalues strictly outside, again $n_- + n_+ = n$. Then the topological type of B depends on its restrictions to its contracting (stable) and expanding (unstable)

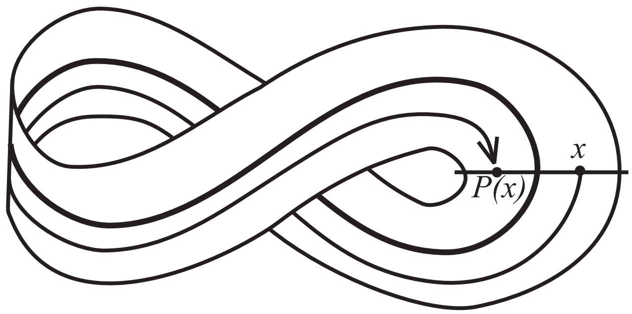


FIGURE 1.6. Poincaré map of a cycle on the Möbius band.

planes. Namely, different types occur for orientation-preserving and orientation-reversing restrictions. Let

$$B(y, z) = (y', z'), \qquad y \in \mathbb{R}^{n_+},\ z \in \mathbb{R}^n. \tag{1.8}$$

The map B is topologically equivalent to one of the following four normal forms (the formulas give expressions for (y', z')):

$$(2y, z/2), \quad (2Sy, z/2), \quad (2y, Sz/2), \quad (2Sy, Sz/2). \tag{1.9}$$

Note that the monodromy map of a cycle in an orientable phase space is orientation-preserving. Therefore, two of the cases from the above list are ruled out for Poincaré maps on orientable manifolds. On nonorientable manifolds all four cases may occur, as the simplest example of the Möbius band shows (see Figure 1.6).

We conclude the topological classification of flows near a hyperbolic cycle by the following remark: two vector fields on a neighborhood of a cycle are topologically orbitally equivalent if and only if their monodromies are topologically equivalent.

1.5. Hadamard–Perron theorem. The Grobman–Hartman theorem implies that a neighborhood of a hyperbolic singular point (a hyperbolic cycle) has continuous "stable and unstable invariant manifolds".

DEFINITION 1.11. A manifold is said to be *invariant* for a vector field if, together with any point of the manifold, it contains the entire phase curve of the field passing through this point.

EXAMPLE. Let A be the same linear operator as in 1.4. Then the equation

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{1.10}$$

has two invariant planes, $\mathbb{R}^{n_-}$ and $\mathbb{R}^{n_+}$; the phase curves lying on these planes exponentially tend to the origin as the time t tends to $+\infty$ or $-\infty$, respectively.

The Grobman–Hartman theorem implies that the differential equation

$$\dot{x} = Ax + \cdots, \qquad x \in \mathbb{R}^n, \tag{1.11}$$

with hyperbolic singular point 0 has two continuous invariant manifolds of dimensions $n_\pm$ exhibiting the above stability property for trajectories in direct or reverse time. The Hadamard–Perron theorem below asserts that these invariant manifolds,

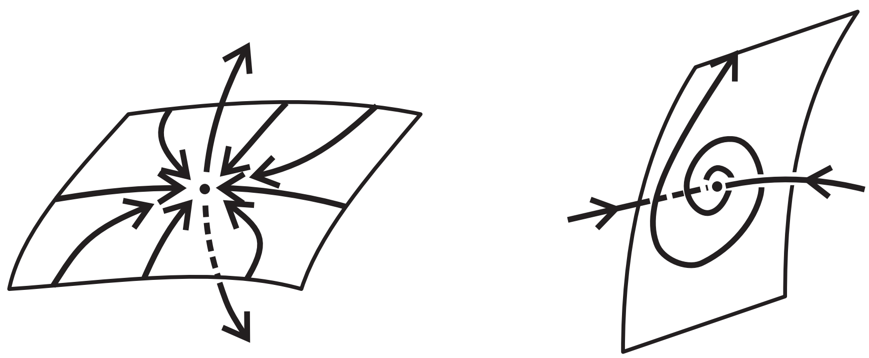


FIGURE 1.7. Trajectories on stable and unstable manifolds.

whose existence is guaranteed by the Grobman–Hartman theorem, are in fact as smooth as the vector field is.

THEOREM 1.2 (Hadamard–Perron). *Assume that the right-hand side of equation* (1.11) *is of class* C^k *with* $k \leqslant \infty$ *or* $k = \omega$ (C^ω *stands for the class of analytic functions*), *and the singularity at the origin is hyperbolic. Suppose that the operator* A *has* n_- *eigenvalues to the left of the imaginary axis and* n_+ *of them to the right, and denote by* $T^\pm$ *the corresponding linear invariant subspaces for* A.

Then equation (1.11) *has two* C^k *smooth invariant manifolds* W^u (*unstable*) *and* W^s (*stable*) *tangent at the origin to* T^+ *and* T^-, *respectively. All orbits starting on* W^s *exponentially approach the origin as time increases to* $+\infty$; *trajectories on* W^u *exponentially converge to the singular point in the reverse time.*

Examples illustrating the behaviors of phase curves are given in Figure 1.7. In the planar case, trajectories different from the singular point and belonging to the invariant manifolds are called *separatrices*.

A similar assertion holds for diffeomorphisms near hyperbolic fixed points and for vector fields near hyperbolic cycles.

If the phase space is a compact manifold, then the flow of any vector field is defined globally, for all values of time. In particular, all orbits constituting the stable (unstable) manifold of a hyperbolic singular point (a cycle) may be infinitely continued. Saturation of locally defined stable and unstable manifolds by phase trajectories is a pair of immersed submanifolds of the phase space; their location may be rather complicated. In order to describe their mutual position, we need the concept of transversality.

1.6. Transversality.

DEFINITION 1.12. Two submanifolds of a smooth manifold intersect each other transversally if the following alternative holds:

(i) their intersection is empty, or
(ii) at each intersection point the tangent spaces to those manifolds together span the tangent space to the ambient manifold.

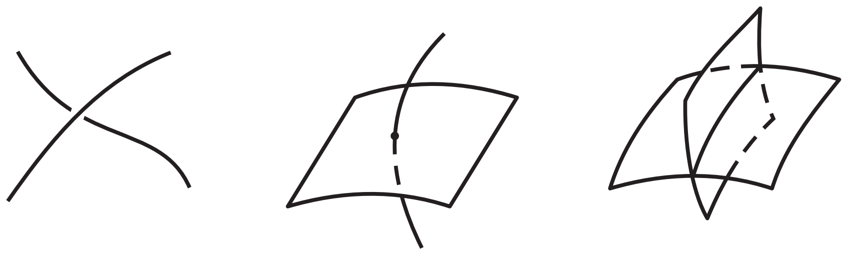


FIGURE 1.8. Transversally intersecting submanifolds in $\mathbb{R}^3$.

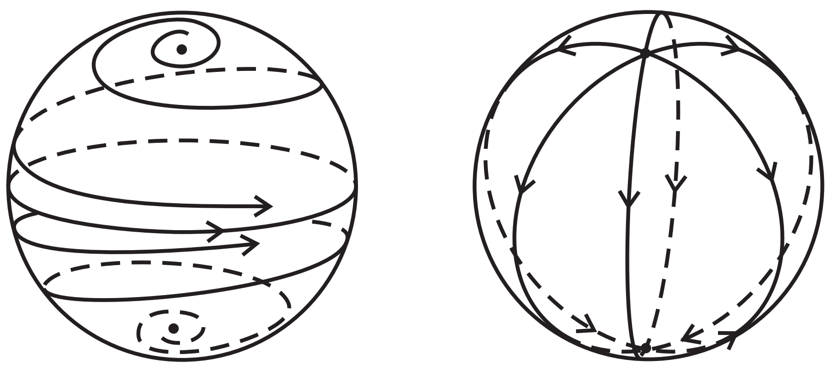


FIGURE 1.9. Examples of Morse–Smale systems on the 2-sphere.

REMARK. If the sum of dimensions of the two submanifolds is strictly smaller than the dimension of the ambient manifold (say, two curves in $\mathbb{R}^3$), then the transversality condition implies that their intersection is void.

Examples of transversally intersecting submanifolds are shown in Figure 1.8.

Now everything is prepared to introduce the class of Morse–Smale systems.

1.7. Morse–Smale systems.

DEFINITION 1.13. A C^1 smooth vector field on a manifold or a diffeomorphism of this manifold is called a *Morse–Smale system* if the following three conditions hold:

(i) the set of nonwandering points consists of a finite number of singular points and periodic orbits (for a diffeomorphism a fixed point is a periodic point of period 1);

(ii) all singular points and periodic orbits are hyperbolic;

(iii) stable and unstable manifolds of hyperbolic singular points and periodic orbits intersect transversally.

Figure 1.9 shows two examples of Morse–Smale vector fields on the 2-sphere.

The fundamental property of Morse–Smale systems is their stability with respect to C^1-small perturbations.

THEOREM 1.3. *Morse–Smale systems on a compact manifold are structurally stable.*

Below we mention some results about Morse–Smale systems on closed surfaces. Morse–Smale vector fields on the 2-sphere admit a simple description, being characterized by the following two properties:

(i*) all singular points and periodic orbits of the field are hyperbolic;

(ii*) the field has no saddle connections; in other words, there are no separatrices common to two singularities, including loops through the same saddle point.

The equivalence of this particular definition to the general one for the case of the 2-sphere is not a very trivial fact. It follows from the Poincaré–Bendixson theorem, which is heavily based on Jordan's lemma (each Jordan curve divides the 2-sphere into two pieces). On the 2-torus and other orientable surfaces, conditions (i*) and (ii*) are not sufficient for a system to be of Morse–Smale type: irrational winding on the torus provides an easy counterexample; see Figure 1.1.

Andronov and Pontryagin were the first to define the concept of structural stability. They proved that conditions (i*) and (ii*) are necessary and sufficient for fields on the 2-sphere. In the early sixties Peixoto proved the following two statements.

THEOREM 1.4. *A vector field on a compact two-dimensional surface is structurally stable if and only if it is Morse–Smale.*

THEOREM 1.5. *For any natural r, Morse–Smale C^r smooth vector fields constitute an open dense subset of the space $\chi^r(M)$ for any compact orientable surface M of any genus and for any nonorientable surface of low $(\leqslant 3)$ genus.*

Later Smale proved the converse statement for vector fields in the multidimensional case (in dynamical systems theory, "multidimensional" always means "of dimension greater than 2").

THEOREM 1.6. 1. *There exists a structurally stable system which is not Morse–Smale.*

2. *Structurally stable vector fields are not dense in $\chi^r(M)$ for $\dim M \geqslant 3$.*

Thus we see that the following challenging problem remains: *is it true that C^r-structurally stable vector fields on a nonorientable surface of high genus* (> 3) *are dense*? *The* (*positive*) *answer is known only for the case* $r = 1$.

Now we proceed with the description of the boundary of the set of Morse–Smale systems. For brevity we will refer to MS-sets and MS-boundaries.

1.8. Degeneracies occurring on the boundary of MS-sets. For simplicity (in order to avoid repetitious remarks) we restrict ourselves to the case of vector fields. On an MS-boundary the following phenomena may occur.

1. Systems with nonhyperbolic singular points.
2. Systems with nonhyperbolic cycles.
3. Systems with nontransversal intersections of stable and unstable invariant manifolds of singular points or cycles.
4. Systems with infinite number of nonwandering orbits.

We will study bifurcations occurring in generic one-parameter families crossing the MS-boundary. Genericity means that this crossing occurs at a generic point of the MS-boundary, therefore only one degeneration from the above list can materialize, and no additional equality-type obstructions arise (see below).

PROBLEM (Arnold). *Is it possible that a system with infinite nonwandering set may occur at the crossing of the boundary of an MS-set by a generic one-parameter family without the simultaneous occurrence of any system of type* 1–3 *above? In other words, may the explosion of the nonwandering set occur on the MS-boundary without the simultaneous occurrence of nonhyperbolic points or cycles or homoclinic tangency of stable and unstable manifolds of hyperbolic singular points or cycles?*

It is not known whether the first non-MS point on a generic curve passing through an MS-point towards MS-boundary in $\chi^r(\mathbb{R}^3)$ may be of type 4.

In all families studied below, the bifurcations are caused by degeneracies of types 1–3, and an infinite nonwandering set appears as a result of such bifurcations. Therefore we will study only phenomena relevant to the cases 1–3. The natural starting point is the investigation of *local* bifurcations.

§2. Equivalence and local bifurcations in generic one-parameter families

In this section we formulate the reduction principle of Shoshitaĭshvili. This principle gives the strongest possible assertion on the similarity of the germ of a vector field and its linear part. At the same time, it is the cornerstone for the foundation of local bifurcation theory.

2.1. Local and principal families. In local dynamics, the concept of *germ* is frequently used to refer to objects without specifying their domains explicitly. The precise definition follows.

DEFINITION 2.1. Two functions (vector fields, maps) defined in two neighborhoods of the same point are *equivalent* if they coincide on some smaller neighborhood of that point. The corresponding equivalence class is called the *germ* of the function (vector field, map). Any element of the equivalence class is a *representative* of the germ.

Evidently, all representatives of the same germ take the same value at the point p, so without ambiguity one may speak of the value of the germ at the reference point. In the same manner, derivatives of the germ at the reference point are well defined.

The notion of germ immediately finds an application. Let U be a subset of the Cartesian product $\mathbb{R}^n \times \mathbb{R}^p$ whose points are pairs (x, ε). A family of vector fields depending on the parameter $\varepsilon \in \mathbb{R}^p$ is a vector field defined in U and parallel to the phase space. In the coordinates (x, ε) this field gives rise to the equation

$$\begin{cases} \dot{x} = v(x, \varepsilon), \\ \dot{\varepsilon} = 0. \end{cases}$$

DEFINITION 2.2. A *local family* of vector fields is the germ of a family of vector fields considered as a single vector field on the Cartesian product of the phase space and the space of parameters at a certain point (x_0, ε_0). The latter is called the *reference point* of the local family, and ε_0 is the *initial value* of the parameters.

We denote the local family by $(v; x_0, \varepsilon_0)$. Sometimes we call it a *deformation* of the germ of the vector field $v_0 = v(\,\cdot\,, \varepsilon_0)$.

DEFINITION 2.3. Two local families of vector fields $(v; x_0, \varepsilon_0)$ and $(w; y_0, \eta_0)$ are (*topologically orbitally*) *equivalent* if there exists a germ of a homeomorphism H such that:

(i) the germ H has the reference point at (x_0, ε_0) and takes the value (y_0, η_0) at this point;

(ii) there exists a representative of the germ H which is fibered over the parameter spaces; this means that the representative has the form

$$H\colon (x, \varepsilon) \mapsto (y, \eta) = (H_1(x, \varepsilon), H_2(\varepsilon)); \tag{2.1}$$

(iii) for every ε, the map $H_1(\,\cdot\,, \varepsilon)$ is a homeomorphism taking phase curves of the field $v(\,\cdot\,, \varepsilon)$ to those of $w(\,\cdot\,, \eta)$, $\eta = H_2(\varepsilon)$ and preserving their orientation.

REMARK. We do not require that $H(x_0, \varepsilon) = y_0$ for $\varepsilon \neq \varepsilon_0$.

DEFINITION 2.4. We say that two germs of vector fields are *weakly* (*orbitally topologically*) *equivalent* if there exist a germ of a map H satisfying conditions (i)–(iii) above, except for the continuity assumption, which is relaxed in the following way: we do not require that H be continuously dependent on ε. More precisely, in (2.1) the map $H_1(\,\cdot\,, \varepsilon)$ must be continuous for every ε, but the dependence on ε may be discontinuous.

Sometimes it is natural to consider local families of vector fields depending on the same set of parameters. For such families we may avoid reparametrization by introducing the following definition.

DEFINITION 2.5. Two local families are *strongly equivalent* if they are equivalent and the corresponding homeomorphism H preserves the parameter:

$$H_2 = \mathrm{id}\,. \tag{2.2}$$

DEFINITION 2.6. The local family $(u; x_0, \mu_0)$ is *induced* from the local family $(v; x_0, \varepsilon_0)$ if there exists a germ of a continuous map from the parameter space of the first family to the parameter space of the second family, $\mu \mapsto \varepsilon = \phi(\mu)$, $\phi(\mu_0) = \varepsilon_0$, such that

$$u(x, \mu) = v(x, \phi(\mu)). \tag{2.3}$$

Now we describe local families that are in a sense maximal: they contain all possible deformations up to topological equivalence.

DEFINITION 2.7. A local family $(v; x_0, \varepsilon_0)$ is a *topologically orbitally versal* (in short, *versal*) deformation of the germ of the field $v_0 = v(\,\cdot\,, \varepsilon_0)$ if any other local family containing the same germ v_0 is strongly equivalent to a family induced from (v, x_0, ε_0).

If in Definition 2.7 we replace strong equivalence by weak equivalence, then the notion of a *weakly versal deformation* arises.

Now we define *principal families* as topological normal forms for unfoldings of a field having degenerate singular point. Fix some type D of degeneracy (say, a class of germs satisfying certain equality-type conditions imposed on lower order terms).

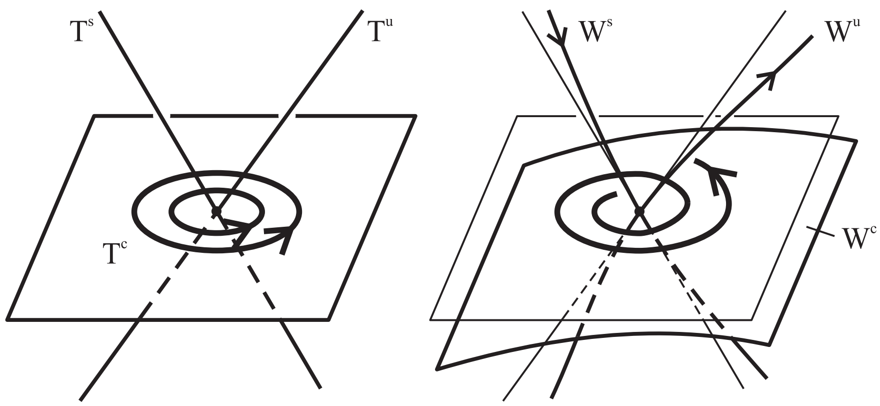


FIGURE 1.10. Stable, unstable and center manifolds.

DEFINITION 2.8. A *tuple of principal families for the degeneracy D* in codimension ν is a finite collection of ν-parameter local families with the following characteristic property: any generic ν-parameter family containing the degeneracy D for the initial value of the parameters is orbitally topologically equivalent to one of the families constituting the tuple.

Versal deformations and principal families contain, in very concentrated form, complete information about bifurcations occurring in local families unfolding degenerate singularities.

2.2. Center manifold theorems. Consider a linear operator $A\colon \mathbb{R}^n \to \mathbb{R}^n$. The linear space $\mathbb{R}^n$ is split into the direct sum of three A-invariant subspaces,

$$\mathbb{R}^n = T^s \oplus T^u \oplus T^c \tag{2.4}$$

(^{s} and ^{u} stand for stable and unstable respectively, ^{c} for center) in the following way: the restriction $A|_{T^s}$ possesses a spectrum contained in the open left half-plane, the spectrum of the restriction $A|_{T^u}$ lies to the right of the imaginary axis, and all eigenvalues of the restriction $A|_{T^c}$ have zero real parts. It turns out that a similar property holds for nonlinear vector fields. The next theorem is the strongest general result of that sort.

THEOREM 2.1 (the center manifold theorem for flows). *Let $v(x)$ be the germ at the origin of a C^{r+1} smooth vector field on $\mathbb{R}^n$ with $r < \infty$. Denote by A its linearization, so that $v(x) = Ax + \cdots$. Let T^s, T^u, and T^c be the invariant subspaces of the operator A defined in* (2.4).

Then the differential equation $\dot{x} = v(x)$ has three invariant manifolds W^s, W^u, and W^c of smoothness C^{r+1}, C^{r+1}, and C^r respectively, having the tangent spaces T^s, T^u, and T^c at the origin. The restriction of the flow on W^s (*resp., on W^u*) *is exponentially contracting* (*resp., contracting in reverse time*) *exactly as in the assertion of Hadamard–Perron theorem*; *the behavior of the flow on W^c is determined by both linear and nonlinear terms of v.*

Theorem 2.1 is represented by Figure 1.10.

REMARK. Under the same set of assumptions, the equation $\dot{x} = v(x)$ possesses two other invariant manifolds, the center-stable manifold $W^{sc} \supseteq W^s \cup W^c$ and the center-unstable manifold $W^{uc} \supseteq W^u \cup W^c$ tangent to $T^s \oplus T^c$ and $T^u \oplus T^c$ respectively and of the class C^r.

The manifolds W^s and W^u are referred to as the stable and unstable manifold as in the hyperbolic case; if the germ is C^∞ or C^ω smooth, then they are also of the same smoothness class. Unlike them, the manifolds W^c, W^{sc}, and W^{us} even for a C^∞ or C^ω smooth field are only of *finite differentiability*: for any $k \in \mathbb{N}$ there exists a neighborhood of the origin such that the intersection of the above invariant manifolds with this neighborhood is C^k smooth. With k increasing, the size of this neighborhood decreases and it shrinks to a singular point, so there can be no C^∞ smooth invariant manifold.

DEFINITION 2.9. The manifold W^c is called the *center manifold*. The plane $T^s \oplus T^u$ is the *plane of hyperbolic variables*.

2.3. Reduction principle. The dynamics of the restrictions of a flow to stable and unstable invariant manifolds was already described. It turns out that in general the topology of a flow is determined by its linear part and the restriction of this flow to the center manifold.

Consider a differential equation $\dot{x} = v(x)$ with C^r smooth right-hand side v, $r \geqslant 2$, having a singularity at the origin with the linearization A as in 2.2. Let $\mathbb{R}^n = T^s \oplus T^u \oplus T^c$ be the invariant splitting for A, and let W^s, W^u, and W^c be the corresponding invariant manifolds for the equation.

THEOREM 2.2. *In a sufficiently small neighborhood of the origin, the equation $\dot{x} = v(x)$ is topologically equivalent to the standard saddle suspension over its restriction to the center manifold*:

$$\begin{cases} \dot{x} = w(x), \\ \dot{y} = -y, \\ \dot{z} = z, \end{cases} \qquad x \in W^c,\ y \in T^s,\ z \in T^u. \tag{2.5}$$

This theorem has numerous applications. In fact, the entire modern theory of local stability of vector fields is based on this theorem, as well as the topological classification of germs of vector fields.

On the other hand, the theorem may be easily modified to cover the case of local families depending on parameters: to that end, we consider the extended system

$$\begin{cases} \dot{x} = v(x, \varepsilon), \\ \dot{\varepsilon} = 0, \end{cases} \qquad x \in \mathbb{R}^n,\ \varepsilon \in \mathbb{R}^k. \tag{2.6}$$

At the point $(0, 0)$ the system (2.6) has a center manifold of dimension $k + \dim T^c$. The precise formulation of the reduction principle for local families follows.

DEFINITION 2.10. Let $u, s \geqslant 0$ be the dimensions of the stable and unstable manifolds. The *standard saddle suspension* over the local family

$$\dot{x} = w(x, \varepsilon) \tag{2.7}$$

is the family

$$\begin{cases} \dot{x} = w(x,\varepsilon), \\ \dot{y} = -y, \qquad\qquad y \in \mathbb{R}^s,\ z \in \mathbb{R}^u. \\ \dot{z} = z, \end{cases} \tag{2.8}$$

DEFINITION 2.11. The center manifold of the local family $(v; 0, 0)$ satisfying $v(0, 0) = 0$ is the center manifold at the origin for the system (2.6).

SHOSHITAĬSHVILI REDUCTION PRINCIPLE. *An arbitrary local family of vector fields $(v; 0, 0)$ with $v(0) = 0$ is topologically equivalent to the saddle suspension over the restriction of the family to its center manifold.*

COROLLARY. *Let $(w; 0, 0)$ be the restriction of the family $(v; 0, 0)$ to the center manifold of the latter. If the former family is a versal deformation of the germ $w(\cdot, 0)$, then the family $(v; 0, 0)$ is a versal deformation of the germ $v(\cdot, 0)$.*

2.4. Local bifurcations of singular points in generic one-parameter families. In generic local one-parameter families of vector fields only two possible types of nonhyperbolic singularities may occur: exactly one zero eigenvalue (the *saddlenode case*) or exactly one pair of purely imaginary conjugate complex numbers (the *Andronov–Hopf case*). The restriction of the original local family to its center manifold will be called the *reduced family*. The Shoshitaĭshvili reduction principle asserts that without loss of generality only reduced families may be considered. In Table 1 below, we list all generic one-parameter reduced families. This table is organized as follows.

TABLE 1. Principal families in codimension 1.

Type of singularity	c	Typical germ	Genericity conditions	Principal families	Bifurcational diagram, phase portrait
One zero eigenvalue: the saddlenode case	1	$\dot{x} = ax^2$	$a \neq 0$	$\dot{x} = x^2 \pm \varepsilon$ (2.9$^\pm$)	Figure 1.11
One pair of imaginary eigenvalues: the Andronov–Hopf case	2	$\dot{z} = z(i\omega + A\rho)$	$\operatorname{Re} A \neq 0$	$\dot{z} = z(i + \varepsilon \pm \rho)$ (2.10$^\pm$)	Figure 1.12
Saddlenode maps (tangent to the identity)	1	$x \mapsto x + ax^2$	$a \neq 0$	$x \mapsto x + x^2 \pm \varepsilon$ (2.11$^\pm$)	Figure 1.13

In the first column we specify the type of degeneracy. The number c in the second column is the dimension of the center manifold. The third column gives the normalized jet of the degenerate vector field, and in column 4 the degenericity assumption (which rules out cases of deeper degeneracy) is given. Finally, we write principal families and make references to the figures showing the corresponding bifurcation diagrams.

THEOREM 2.3. *The set of all germs of vector fields at a nonhyperbolic singular point splits into the union of two open subsets and a complementary subset of codimension greater than one in the space of all germs of vector fields at the singular point. The first open subset corresponds to the saddlenode germs, the second consists of Andronov–Hopf germs. In both cases a generic germ from one of these two*

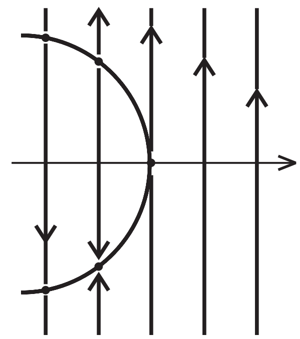


FIGURE 1.11. The bifurcation of a saddlenode singular point.

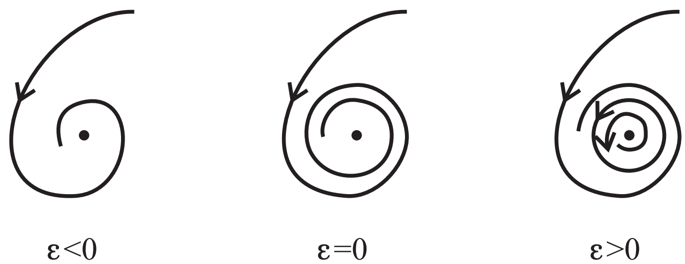


FIGURE 1.12. The Andronov–Hopf bifurcation for vector fields.

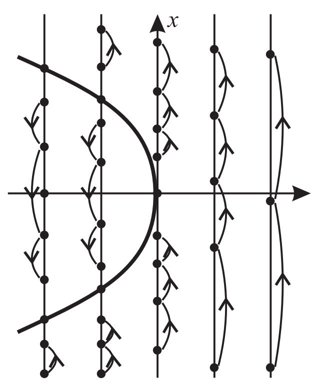


FIGURE 1.13. The bifurcation of a saddlenode fixed point.

subsets is orbitally topologically equivalent to its normalized jet given in Table 1. *Generic one-parameter deformations of such germs are stable equivalent* (*that is, become equivalent after an appropriate saddle suspension*).

The latter assertion means that the principal families shown in this table are stable versal deformations (i.e., become versal deformations after an appropriate saddle suspension).

The theorem is classical; the proof may be found, say, in [**A**].

The last line in the table corresponds to a so-called saddlenode local family of maps on the line. It is very simple in itself (see Figure 1.13). On the other hand, it is a key to nonlocal bifurcations of homoclinic surfaces of saddlenode cycles. These bifurcations are studied in Chapters 5 and 6.

We now describe the properties of the second principal family.

For the variable $\rho = z\overline{z}$, equations ($2.10^{\pm}$) (in Table 1) yield the so-called quotient system

$$\dot{\rho} = \rho(\varepsilon \pm \rho).$$

For the sign $+$, the equation has no nonzero singular points for $\varepsilon > 0$ and a unique nonzero singularity for $\varepsilon < 0$. The principal system ($2.10^{\pm}$) is itself invariant under rotations of the z-plane, and the nonzero singularity of the quotient system corresponds to the limit cycle of the principal family. Stability of this singular point for the quotient system coincides with that of the limit cycle of the principal system. This justifies Figure 1.12.

REMARKS. 1. The family (2.9^{+}) may be induced from the family (2.9^{-}) by reversing the parameter $\varepsilon \mapsto -\varepsilon$.

2. The families (2.10^{+}) and (2.10^{-}) are transformed into each other by the time reversal $t \mapsto -t$, the symmetry $z \mapsto \overline{z}$ and the parameter reversal. Still we distinguish between these two cases because the loss of stability occurs according to two different scenarios, called *soft* and *hard* stability loss. In the family (2.10^{-}) for $\varepsilon \leqslant 0$, the singular point at the origin is asymptotically stable; for $\varepsilon > 0$ it becomes unstable. Yet a small neighborhood of the singularity remains attracting for small positive ε: trajectories starting on the boundary of this neighborhood enter the neighborhood and stay in it forever, converging to the cycle of radius of order $\sqrt{\varepsilon}$ rather than to the singularity. This phenomenon is referred to as the soft stability loss in the physical slang.

In the family (2.10^{+}) for $\varepsilon < 0$ the singularity is stable, yet its basin of attraction shrinks to zero as $\varepsilon \to 0-$, and for $\varepsilon \geqslant 0$ the origin becomes unstable. So all trajectories (except for the singular point itself) must leave the neighborhood of the origin after sufficient time for any positive ε. This situation is called hard stability loss: when the parameter ε passes through the zero value, the system must jump to another stable regime. This new regime may be constant, periodic or more complicated, but in any case it must be far away from the original equilibrium point.

2.5. Local bifurcations of cycles in generic one-parameter families. Local bifurcations of cycles may be described in the same way as bifurcations of singular points. But this description is much more complicated and essentially lies beyond the scope of this book.

There are three possible types of nonhyperbolic cycles which may occur in generic one-parameter cycles: *saddlenode, flip*, and *Andronov–Hopf fixed point*; see 4.1 below. In all three cases the corresponding cycles may have homoclinic orbits, but only in the first case does the vector field belong to the boundary of the Morse–Smale set. This effect is briefly explained in §4 and in more detail in Chapter 2. Here we describe the local bifurcation of saddlenode cycles.

DEFINITION 2.12. A nonhyperbolic cycle of a vector field is of *saddlenode* type if exactly one of its multipliers is equal to +1, while the others are hyperbolic (not on the unit circle).

Now we need an analog of the local theory described in 2.1–2.4 for the case of maps rather than vector fields. This analog may be obtained by making the following replacements in all definitions and formulations:

1. germs of vector fields at singular points $\mapsto$ germs of maps at fixed points;
2. eigenvalues outside the imaginary axis $\mapsto$ eigenvalues outside the unit circle;
3. saddle suspensions of vector fields $\mapsto$ maps of the form

$$(x, y, z) \mapsto (x', y', z'), \qquad x' = w(x, \varepsilon), \quad y' = Ay, \quad z' = Bz,$$
$$y \in \mathbb{R}^s,\ z \in \mathbb{R}^u,\ \|A\| < 1,\ \|B^{-1}\| < 1.$$

In particular, the Shoshitaĭshvili reduction principle also holds for the case of maps and allows one to consider bifurcations occurring in a saddlenode family of maps in dimension 1. The last line in Table 1 gives the topological principal family.

REMARK 3. The families ($2.11^{\pm}$) from Table 1 may be transformed into each other by parameter reversal.

§3. Homoclinic trajectories of nonhyperbolic singular points

A *homoclinic trajectory* of a singular point is the trajectory which tends to this point both in direct and in reverse time. Only two types of nonhyperbolic singularities (see above) may appear in generic one-parameter families.

1. Saddlenodes (with exactly one zero eigenvalue).
2. Andronov–Hopf points (with only one pair of nonzero purely imaginary eigenvalues).

All other eigenvalues have nonzero real parts (and are thus hyperbolic). In the first case the center manifold is one-dimensional; in the second the dimension of the center manifold is two. No other degeneracies are allowed for singularities occurring in generic one-parameter families. Therefore the germ of the restriction of the vector field to the center manifold can be topologically normalized according to Table 1, §2. Hence a saddlenode generically occurring in such a family is topologically equivalent to the vector field

$$\dot{x} = x^2, \quad \dot{y} = -y, \quad \dot{z} = z, \qquad (x, y, z) \in (\mathbb{R}^1 \times \mathbb{R}^u \times \mathbb{R}^s, 0). \tag{3.1}$$

In the Andronov–Hopf case, the topological normal form is

$$\begin{gathered} \dot{z} = iz \pm z^2\overline{z}, \quad \dot{y} = -y, \quad \dot{p} = p, \\ z \in (\mathbb{C}, 0) \simeq (\mathbb{R}^2, 0), \quad (y, p) \in (\mathbb{R}^u \times \mathbb{R}^s, 0). \end{gathered} \tag{3.2$^{\pm}$}$$

DEFINITION 3.1. A *positive* (*negative*) *semitrajectory* of a point is the part of its trajectory corresponding to nonnegative (resp., nonpositive) time values. The *stable* (*unstable*) *set* of a nonhyperbolic singular point of a vector field is the union of all positive (resp., negative) semitrajectories tending to this point (resp., in reverse time).

The stable and unstable sets of a nonhyperbolic singularity are denoted by S^s and S^u; if we want to stress the fact that a point rather than a cycle is considered, then we use the notation S_0^s, S_0^u. Actually we deal with germs of those sets rather

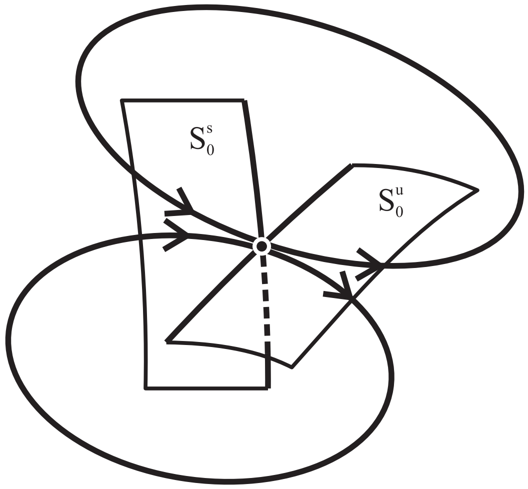


FIGURE 1.14. Homoclinic orbits of a saddlenode singular point.

than with the sets themselves. (A germ of a set at a point a is an equivalence class of sets with the following equivalence relation: two sets are equivalent if they coincide in some neighborhood of a.)

In the saddlenode case the (germs of) stable and unstable sets are homeomorphic to the (germ of the) closed half-space of dimension $s+1$ and $u+1$ respectively. Let n be the dimension of the total phase space. Then

$$n = s + u + 1,$$

since $\dim W^c = 1$. Therefore

$$\dim S_0^s + \dim S_0^u = n + 1.$$

These two manifolds with boundary may intersect in a one-dimensional manifold, which is either a single connected component or the union of several, perhaps an infinite number of, connected components (phase curves); see Figure 1.14.

For the Andronov–Hopf case ($3.2^{\pm}$), we distinguish between two subcases. If the sign is $+$, then the germs of the sets S_0^s and S_0^u are homeomorphic to T^s and $T^u \times W^c$ respectively, and $\dim W^c = 2$. Therefore

$$\dim S^s + \dim S^u = n.$$

When transversal, these two manifolds are disjoint (since the intersection must be at least one-dimensional, being invariant under the flow) except for one point of intersection at the origin. The other case (3.2^-) is treated similarly. Recall that the existence of an Andronov–Hopf point is a degeneracy itself. In generic one-parameter families no other degeneracies occur. Thus we conclude that no homoclinic trajectories are possible for an Andronov–Hopf point in codimension 1. So we need to study only homoclinic trajectories of saddlenodes. This is the subject of Chapter 4.

§4. Homoclinic trajectories of nonhyperbolic cycles

This section is parallel to the previous one but contains more pictures. A homoclinic trajectory of a cycle is the trajectory tending to the cycle both in direct and in reverse time. As usual, we operate with the Poincaré map P of a cycle and its iterate rather than with trajectories of the vector field itself. For example, a homoclinic trajectory intersects the global transversal section (if such a section exists) in an orbit $\{x_t\}|_{t=\dots,-1,0,1,\dots}$ of the Poincaré map P, $x_k = P^k(x_0)$, $k \in \mathbb{Z}$, such that

$$\lim_{k\to\pm\infty} x_k = 0.$$

In this section we also formulate the Birkhoff–Smale theorem, which gives the key argument for recognizing systems that do not belong to the MS-boundary. Several new modifications of this theorem are presented.

4.1. Types of nonhyperbolic fixed points generic in codimension 1. The three types of degeneracies possibly occurring in generic one-parameter families of diffeomorphisms are the following:

1. Saddlenode (one multiplier equal to $+1$).
2. Flip (one multiplier equal to -1).
3. Andronov–Hopf point: a pair of nonreal multipliers on the unit circle, $\exp(\pm i\phi)$, $\phi \in \mathbb{R}$, $\phi \notin \pi \cdot \mathbb{Z}$.

We call the corresponding cycle saddlenode, flip, or Andronov–Hopf cycle, respectively. The stable and unstable sets for fixed points of maps or for limit cycles of vector fields are defined as in §3 for singular points of vector fields.

4.2. Saddlenode fixed points and cycles. The Poincaré map for a nonhyperbolic saddlenode is topologically equivalent to the map

$$\begin{gathered}(x, y, z) \mapsto (x', y', z'),\\ x' = x + x^2, \qquad x \in (\mathbb{R}^1, 0),\\ y' = Ay, \quad z' = Bz, \qquad y \in \mathbb{R}^s,\ z \in \mathbb{R}^u,\\ \|A\| < 1,\ \|B^{-1}\| < 1.\end{gathered} \tag{4.1}$$

Hence, the "hyperbolic part" (y', z') in (4.1) has one of the four normal forms (2.9), where A is a linear contraction, and B a linear dilatation. In fact, there are only two topological normal forms for each operator A or B, depending on the preservation or reversal of orientation.

Thus the stable and unstable sets of the fixed point O for the map (4.1) are homeomorphic to the closed half-spaces of dimensions $s+1$ and $u+1$ respectively. The intersection of their interiors is generically a one-dimension manifold having up to an infinite number of connected components, which are no longer phase curves of a flow, but rather invariant curves of the diffeomorphism. The corresponding vector field has invariant surfaces. Some possibilities are shown in Figure 1.15.

Now consider a vector field with a saddlenode cycle. The invariant curves of the Poincaré map correspond to invariant surfaces of the vector field. These surfaces are filled with trajectories homoclinic to the same cycle. These surfaces may be homeomorphic to the 2-torus and the Klein bottle (as in case (a), Figure 1.15). Any number of surfaces may occur simultaneously in case (b). The union of all homoclinic trajectories and cycles may be nonarcwise connected in case (c). It may have a complicated topology, case (d). The homoclinic surface may be not smooth,

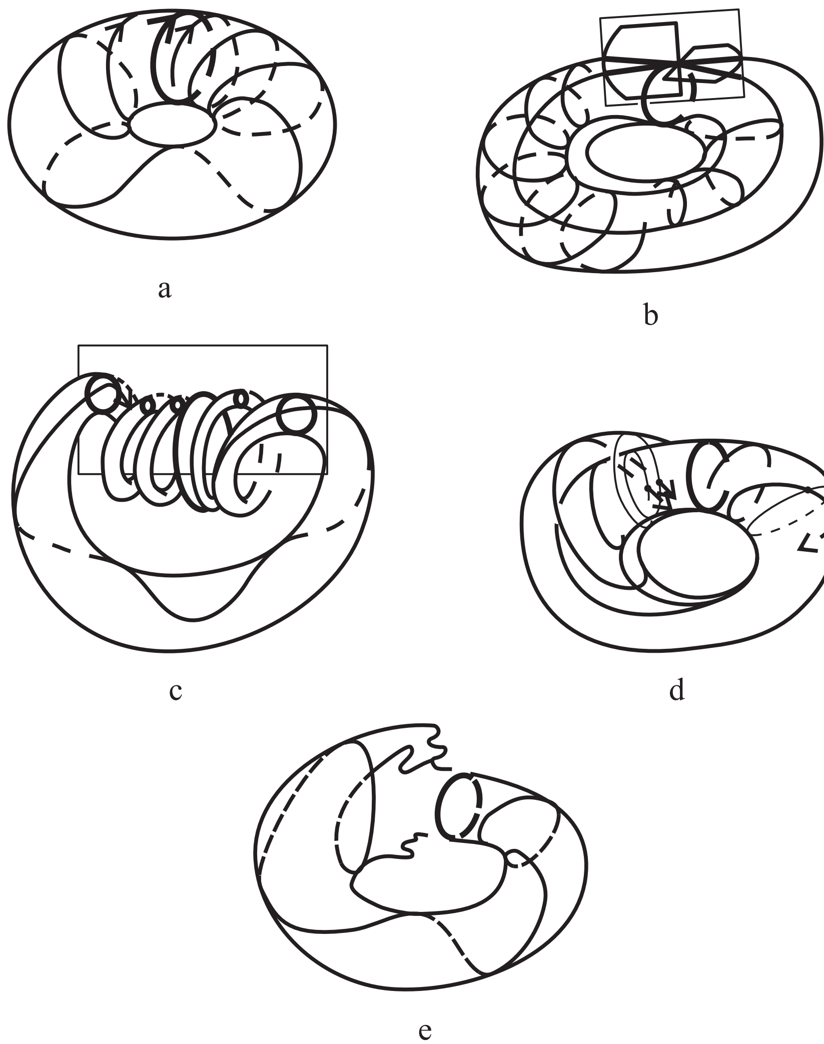


FIGURE 1.15. Homoclinic surfaces of the saddlenode cycle. (a) A homoclinic torus. (b) Two homoclinic tori. (c) A snake. (d) A twisted homoclinic torus. (e) Nonsmooth homoclinic torus.

case (e). The bifurcations in cases (a), (b), (d) are presented in Chapter 5. Case (e) is studied in Chapter 6. In case (c), a blue sky catastrophe may occur; see [**ST**].

4.3. Homoclinic intersections and the Birkhoff–Smale theorem. In this subsection we formulate and comment on the famous theorem due to Birkhoff and Smale, which shows that the set of Morse–Smale systems is not dense in the space of all vector fields on manifolds of dimension greater than 2.

THEOREM 4.1 (Birkhoff–Smale theorem). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 2$, be a diffeomorphism such that the origin is a hyperbolic fixed point and there exists a point*

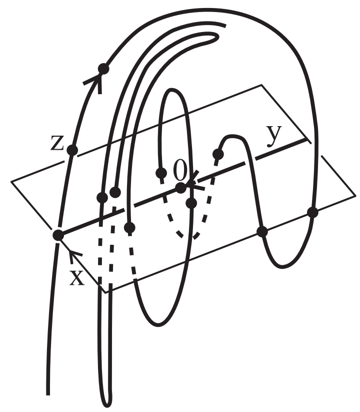


FIGURE 1.16. Intersection of the stable and unstable sets of a fixed point in the flip case.

$p \neq 0$ of transversal intersection of the stable and unstable manifolds of the origin. Then f has an infinite number of hyperbolic periodic orbits.

We derive this statement from a more general theorem of hyperbolic theory in Chapter 8 under slightly different assumption that the map f near the origin is C^1-equivalent to its linear part.

4.4. Flip maps and cycles. The topological normal form in the flip case is

$$\begin{gathered}(x, y, z) \mapsto (x', y', z'),\\ x' = -x \pm x^3, \quad x \in (\mathbb{R}^1, 0), \qquad y' = Ay, \quad z' = By,\end{gathered} \tag{4.2$^\pm$}$$

where y, z, A, and B have the same meaning as in (4.1). For the minus sign the germs of stable and unstable sets of the origin O are homeomorphic to s- and $(u+1)$-dimensional open balls, hence generically they intersect outside the origin in a set consisting of isolated points (see Figure 1.16). The case of the other sign exhibits a similar behavior, the center manifold now being part of the unstable rather than the stable set. Afraimovich (1985) conjectured that a vector field with a homogeneous orbir of a flip cycle cannot occur on the boundary of the MS-set. The main reason for that is in the similarity of this case with the case covered by the Birkhoff–Smale theorem. The only difference with the hyperbolic case is in the much slower rate of convergence of a point from the stable set to the fixed point. In the hyperbolic case, the Birkhoff–Smale theorem guarantees the existence of an infinite nonwandering set for the system under consideration and, at the same time, for all sufficiently close systems. In the flip case, similar arguments show that the same effect holds true for flip systems with homoclinic intersection of stable and unstable sets. This implies that such a system cannot occur on the MS-boundary.

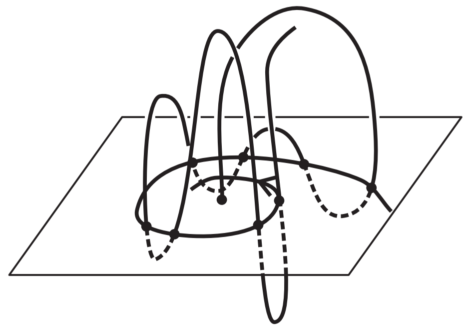


FIGURE 1.17. Intersection of the stable and unstable sets of a fixed point in the Andronov–Hopf case.

4.5. Andronov–Hopf points and cycles. The topological normal form in this case is

$$\begin{gathered}(z, y, p) \mapsto (z', y', p'),\\ z' = \exp(i\phi)z \pm z|z|^2, \qquad z \in (\mathbb{C}^1, 0),\\ y' = Ay, \quad p' = Bp, \qquad (y, p) \in \mathbb{R}^s \times \mathbb{R}^u,\\ \|A\| < 1, \quad \|B^{-1}\| < 1,\end{gathered} \tag{4.3$^\pm$}$$

with the same properties of the hyperbolic part. For the minus sign case, the germs of stable and unstable manifolds are open balls of dimensions $s + 2$ and u. The dimensions are complementary, hence in the generic case the intersection consists of isolated points (see Figure 1.17). We expect that the same arguments as in 4.4 will show that such type of behavior never occurs on the boundary of the set of Morse–Smale systems (Afraimovich, 1985). As far as we know, the detailed proof of this statement and the previous one has not been published yet. We have presented these conjectures here to explain why the homoclinic orbits of flip cycles and Andronov–Hopf cycles are beyond the scope of this book.

§5. Homoclinic loops of hyperbolic fixed points and other contours

The study of bifurcations of homoclinic loops of hyperbolic saddles in $\mathbb{R}^3$ signaled the starting point of multidimensional bifurcation theory.

For generic vector fields, the stable and unstable manifolds of a hyperbolic fixed point in $\mathbb{R}^3$ do not intersect. Indeed, they have supplementary dimensions s and u, $s + u = n$, and are foliated by the phase curves. If these manifolds intersect, then their projections along the phase curves onto a suitable cross-section intersect as well. These intersections have dimensions $s - 1$ and $u - 1$, $(s - 1) + (u - 1) = n - 2$, while the dimension of the cross-section is $n - 1$. Two transversal submanifolds of a domain in $\mathbb{R}^{n-1}$ of total dimension $n - 2$ do not intersect at all.

5.1. Two kinds of hyperbolic fixed points. There are two kinds of generic hyperbolic fixed points. Vector fields with loops homoclinic to fixed points of the first kind may be found on the boundary of the Morse–Smale set. Unfoldings of these loops generate a unique cycle.

Vector fields with homoclinic loops of points of the second kind never appear on the boundary of the Morse–Smale set. They have infinitely many periodic orbits, accumulating to the homoclinic loop. All the neighboring vector fields have an infinite number of periodic orbits as well and, generically, no homoclinic loop.

Let us now describe the two kinds of hyperbolic singular points. Consider the restriction of the linearization of a vector field at the hyperbolic singular points to its stable manifold. For a generic vector field, this restriction has exactly one maximum real eigenvalue λ or exactly two complex conjugate eigenvalues with maximum real part, which we also denote by λ. Obviously, $\lambda < 0$. The corresponding real line in the first case or real plane in the second are called *stable real* or *complex leading directions*. In the same way, the *unstable real* and *complex leading directions* are defined. The corresponding real eigenvalue or the real part of the complex eigenvalue is denoted by μ, where $\mu > 0$. Generically, $\lambda + \mu \neq 0$. We call the stable leading direction *subordinate* if $\lambda + \mu > 0$, and the unstable leading direction *subordinate* if $\lambda + \mu < 0$.

Vector fields of the first kind are those for which the subordinate leading direction is real. *Vector fields of the second kind* have subordinate complex leading direction.

We give a complete description of the bifurcation of the homoclinic loops for the singular points of the first kind. The 3-dimensional and n-dimensional cases are treated in §§7.1 and 7.3 respectively. For singular points of the second kind, we only study the 3-dimensional case, in §7.2. Formally, this study is beyond the scope of this book, because the vector fields under consideration do not belong to the boundary of the Morse–Smale set.

5.2. Definitions and examples of contours. A homoclinic trajectory of a singular point or a cycle is a particular case of so-called *contours*. In this section we give a general definition and describe contours that may occur on the MS-boundary.

Terminological remark. *Contours* are often called *cycles* in other sources. We reserve the term "cycle" for a periodic orbit of a vector field for the multidimensional case as well as for the theory of vector fields on the plane, where it is of common usage. Instead, we use a different term, "contour", for spatial analogs of planar separatrix polygons.

A separatrix polygon on the plane is the finite union of cyclically enumerated singular points and separatrices connecting them in the prescribed order (see Figure 1.18). A contour is a multidimensional analog of this concept with an additional flexibility: some singularities may be replaced by cycles.

Definition 5.1. A *contour* is a finite union of cyclically ordered singular points and cycles (together referred to as *elements*) such that there exist trajectories of the field connecting them in the prescribed order: for any two elements Q_k and Q_{k+1} there exists a trajectory tending to Q_k as $t \to -\infty$ and to Q_{k+1} as $t \to +\infty$. The total number of elements is called the (*combinatorial*) *length* of the contour.

The simplest example of a contour is the homoclinic orbit of a hyperbolic singular point, which will be studied in Chapters 3 and 7. Some different examples of contours of length 1 were presented in §§3 and 4 (in this case the trajectories are homoclinic orbits of a nonhyperbolic singular point or a cycle). Here we discuss contours of length $\geqslant 2$.

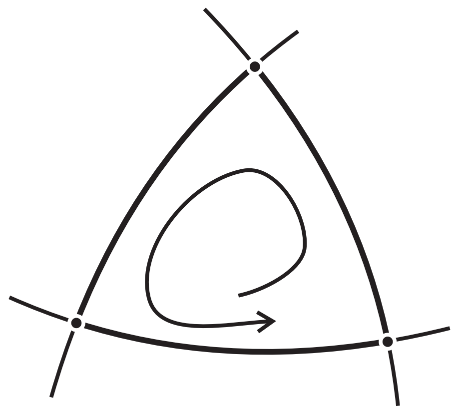


FIGURE 1.18. Separatrix polygon in the plane.

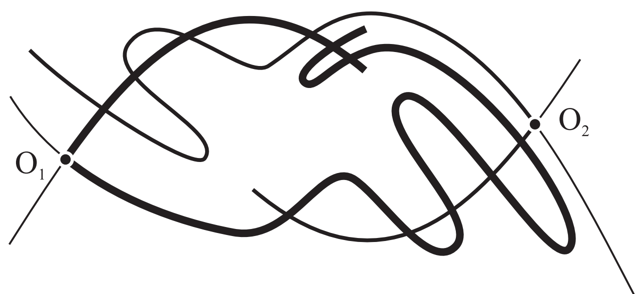


FIGURE 1.19. Contour of length one near a contour of greater length.

5.3. Contours in generic systems. The simplest example of a contour is a homoclinic curve of a hyperbolic cycle. The Birkhoff–Smale theorem cited in 4.3 applies to the corresponding Poincaré map, so such contours cannot occur on the MS-boundary. The same is true for similar contours of any length.

A vector field is said to be of *Kupka–Smale type* if and only if it has hyperbolic singular points and periodic orbits only, and their stable and unstable manifolds intersect transversally.

THEOREM 5.1 (Birkhoff–Smale). *Suppose that a contour of length more than one with only hyperbolic cycles appears for a vector field of Kupka–Smale type. Then this vector field has a homoclinic orbit of a hyperbolic cycle.*

COROLLARY 5.1. *The vector field in Theorem* 5.1 *has an infinite number of periodic orbits, and the same is true for all* C^1*-close vector fields.*

SKETCH OF THE PROOF OF THEOREM 5.1. The existence of a contour of length one near a contour of length more than one is clarified by Figure 1.19 for the case of two hyperbolic cycles replaced by two hyperbolic fixed points of the map. Denote these points by O_1, O_2.

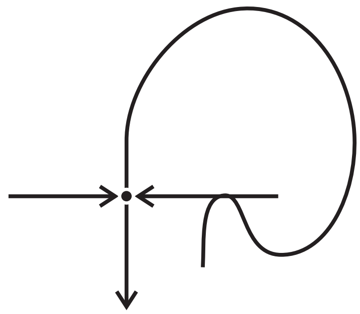


FIGURE 1.20. Homoclinic tangency of a saddle fixed point.

Take a small ball on the unstable manifold of O_1 containing the point of the heteroclinic orbit going from O_1 to O_2. The hyperbolicity of the point O_2 implies that the iterates of this ball accumulate to the unstable manifold of O_2. They stretch wider and wider along this manifold until they cross the stable manifold of O_1. The intersection point belongs to the required homoclinic orbit of O_1.

For contours of greater length the reasoning is the same.

REMARK. The same construction gives a countable number of intersections of $W^u_{O_1}$ and $W^s_{O_1}$. In general, they belong to different orbits of the map. Hence, the existence of a long contour formed by periodic orbits generically implies the existence of a countable set of homoclinic orbits for any of these cycles.

5.4. Contours with hyperbolic elements and nontransversal intersections of invariant manifolds. Contours of such type may appear in generic one-parameter families of vector fields only if:

1. All elements of the contour are hyperbolic cycles (not singular points).
2. There is only one hyperbolic singular point, and all the rest are hyperbolic cycles.

The first case has been the subject of intensive study in the last three decades. Recently the book by J. Palis and F. Takens [**PT**] devoted to this subject has appeared. Here we mention only several phenomena relevant to that case.

Ω-*explosion.* The term "explosion" means a sudden increase of the size of the nonwandering set triggered by a small perturbation of a system.

A contour exhibiting tangency between invariant manifolds in $\mathbb{R}^3$ may appear on the MS-boundary. A small perturbation produces a vector field with transversal intersection of the stable and unstable manifolds of the hyperbolic cycle. By the Birkhoff–Smale theorem, this field has an infinite nonwandering set.

Countable number of stable periodic orbits. Let f_ε be a generic one-parameter family of diffeomorphisms of $\mathbb{R}^2$ exhibiting a *homoclinic tangency* of a dissipative saddle fixed point for $\varepsilon = 0$: for $\varepsilon = 0$ one has a hyperbolic fixed point p with $|\det Df(p)| < 1$ and with tangent stable and unstable invariant curves as in Figure 1.20.

If we perturb a vector field having a Poincaré map of the above form, unfolding it in a generic one-parameter family $v_\varepsilon(\,\cdot\,)$, then a surprising phenomenon occurs. Arbitrarily close to the value $\varepsilon = 0$ on the parameter axis, there exist intervals carrying residual subsets (i.e., countable intersections of open dense sets) of parameters ε for which the field v_ε has an infinite number of periodic attracting orbits. This difficult result was established by S. Newhouse and is described in detail in the book [**PT**].

As far as we know, for the second type of contours only the three-dimensional case was studied in full detail [**BLMP**]. In this example, an Ω-explosion may also occur.

5.5. Nonhyperbolic contours. A contour occurring in a generic one-parameter family may have no more than one nonhyperbolic element, and when this happens, no other degeneracies are allowed. Therefore, the stable and unstable sets of all singular points and cycles intersect transversally. As was claimed in 5.2, the existence of a hyperbolic contour of combinatorial length $\geqslant 2$ implies the existence of a hyperbolic homoclinic loop (a contour of length 1). A similar effect can be observed in the nonhyperbolic case.

THEOREM 5.2. *Suppose that a nonhyperbolic contour of length $\geqslant 2$ occurs for a vector field belonging to the intersection of a generic one-parameter family with the MS-boundary. Then*:

(i) *If the nonhyperbolic element is a singular point, then it is a saddlenode, a saddle with respect to the hyperbolic variables, and it has a homoclinic curve.*
(ii) *If the nonhyperbolic element is a cycle, than it is a saddlenode cycle and has a homoclinic trajectory.*

SKETCH OF THE PROOF OF THEOREM 5.2. The existence of a homoclinic orbit for the nonhyperbolic element in both cases is proved in the same way as in the previous theorem. If this element is a nonhyperbolic singular point, and the contour corresponds to a generic point on the MS-boundary, then this singular point is a saddlenode; see §3. If this element is a nonhyperbolic cycle, then, under the same assumption, it is a saddlenode cycle; see 4.4, 4.5.

This result reduces the investigation of nonhyperbolic cycles to the case of loops (cycles of length 1) in the sense that any behavior occurring in bifurcations of the latter will be also present in the former general case. For example, the result from Chapter 4 implies that Ω-explosion will typically occur in unfoldings of type (i). However, this reduction is one-way: some special effects eventually may be observed for long contours only. However, this study is at its very beginning, so we will focus on the case of cycles of length 1, which were already described in §§3, 4, and 5.1 above.

§6. Summary of results

The central role in this book is played by the middle chapters. In this section we summarize the contents of these chapters and briefly describe the subject matter of the others.

6.1. Architecture of the book. Nonlocal spatial bifurcations are described in Chapters 4–7. The corresponding results are summarized in the table of 6.3. Chapter 3 deals with planar bifurcations. The technique of local normal forms turns these theorems into simple exercises. Chapters 8–10 form the foundation of our study. The theory of normal forms for local families is presented in Chapters 9 and 10. This theory allows one to replace the cumbersome asymptotic analysis

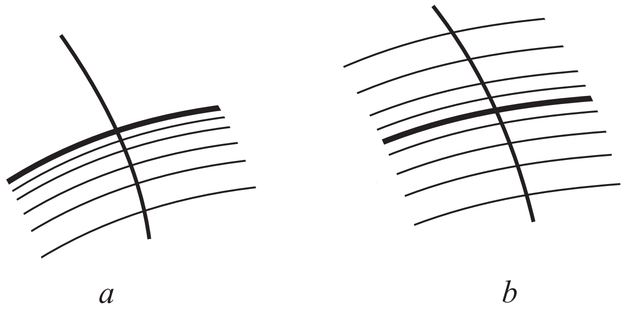


FIGURE 1.21. Accessible and nonaccessible boundary points of the MS-set.

used in earlier investigations by explicit formulas. These formulas describe the behavior of phase curves near singular points. The principal characteristic of this behavior is the so-called correspondence map, which may be explicitly calculated in the normalizing chart. The result allows for a heuristic description of bifurcations of the corresponding homoclinic curves.

The correspondence map has strong hyperbolic properties. Some basic theorems of hyperbolic theory allow us to use these properties to confirm the heuristic description of bifurcations. Chapter 8 contains the detailed proofs of these theorems.

Chapter 2 contains the summary of results of the last three chapters together with some heuristic explanations. It makes it possible to use the principal tools of the last chapters before the corresponding results are rigorously proved. Moreover, Chapter 2 contains an elementary description of the Smale horseshoe, a study of the fractal dimension of attractors and the concept of prevalence, i.e., the notion of "almost everywhere in function spaces". The last two topics may be used for the future study of nonlocal bifurcations.

In order to describe the table below, we need several definitions.

6.2. Accessible boundary classes and one-parameter families leading out of the Morse–Smale set. Both properties named in the title characterize the structure of the set of vector fields of Morse–Smale type (Morse–Smale sets for brevity) near a generic boundary point of this set. The characteristics is given in terms of generic one-parameter families passing through the boundary point.

DEFINITION 6.1. A class of boundary points of a Morse–Smale set is called *accessible from one side* (*respectively, from two sides*) if for any generic one-parameter family crossing this class, there exists a parametrization such that the intersection point of the family with the class corresponds to the zero value of the parameter, and all sufficiently small negative (respectively, nonzero) values of the parameter correspond to Morse–Smale systems. In the opposite case, the class is called *nonaccessible* (see Figure 1.21).

DEFINITION 6.2. A one-parameter family is *not leading out of the Morse–Smale set* if there exist a parametrization such that the zero value of the parameter

The Main Table

Class	Subclass	Acc.	Out	Max. No.	Hyperbolic set	Str. attr.	Ref.
Nonhyperbolic singular point	One homoclinic orbit	++	+	1	−	−	4.1 4.3
	More than one homoclinic orbit	+−	−	∞	Ω	−	4.4
Nonhyperbolic cycle, compact noncritical case	One homoclinic surface of type T^2 or K^2	+−	+	∞	−	−	5.4 5.5
	More than one homoclinic surface of type T^2 or K^2	+−	−	∞	Partially hyperbolic	−	5.7
	Twisted homoclinic surface	+−	+	∞	Ω	+	5.8
Nonhyperbolic cycle, compact semicritical case	One homoclinic surface of type T^2	+−	−	∞	Ω	+	6.1
Hyperbolic singular point with a homoclinic loop	A loop in $\mathbb{R}^3$	++	+	1	−	−	7.1
	A loop in $\mathbb{R}^n$	++	+	1	−	−	7.3
	Subordinate complex leading direction in $\mathbb{R}^3$	*	*	∞	Ω	−	7.2

corresponds to a boundary point of the Morse–Smale set, and an open dense set in some neighborhood of zero on the parameter axes such that any point of this set corresponds to a Morse–Smale system. In the opposite case, the point is *leading out of the Morse–Smale set*.

6.3. Description of the table. The concepts of genericity and attractors used below may be specified in many different ways. This specification is discussed in §§1, 2 of Chapter 2. Hyperbolic sets are defined in §4 of Chapter 2. The heuristic meaning of genericity and attractors is clear anyway. The main example of a hyperbolic set is the nonwandering set of the Smale horseshoe map, the simplest example of which is given in §3 of Chapter 2.

The table consists of 8 columns. The first column, called *Class*, gives the name of the degeneration. The *Subclass* column distinguishes the subsets of the corresponding class with similar properties of unfoldings. In column 3, the sign ++ means that the subclass is accessible from two sides, and +− that it is accessible from one side in the sense of Definition 6.1. The sign + in column 4 means that the generic one-parameter family crossing the subclass is not leading out of the Morse–Smale set, and the sign − means that it is. The column *Maximum number of cycles* contains information on the periodic orbits generated by bifurcations of the generic equations from the corresponding subclass. The integer in the column gives the explicit number of cycles generated in typical one-parameter families crossing the subclass. The symbol ∞ means either that an infinite number of cycles may be generated in time or that an arbitrary large, though finite, number may be generated. The symbol Ω in column 6 means that a nontrivial hyperbolic set is generated, and the sign $-$ means that it is not. The signs $+$ and $-$ in column 7 mean the same for the strange attractor. *Strange attractor* here means an attractor which is not a finite union of smooth manifolds; see Chapter 2 for details. In the column *References*, theorems formalizing the brief description summarized in the preceding part of the same line are indicated. The first digit in the reference means the Chapter, and the second the section.