Contents

Preface

The field of mirror symmetry has exploded onto the mathematical scene in recent years. This is a part of an increasing connection between quantum field theory and many branches of mathematics.

It has sometimes been said that quantum field theory combines 20th century physics with 21st century mathematics. Physicists have gained much experience with mathematical manipulations in situations which have not yet been mathematically justified. They are able to do this in part because experiment can help them differentiate between which manipulations are feasible, and which are clearly wrong. Those manipulations that survive all known tests are presumed to be valid until evidence emerges to the contrary.

Based on this evidence, physicists are confident about the validity of mirror symmetry. One of the tools they use with great virtuosity is the Feynman path integral, which performs integration with complex measures over infinite dimensional spaces, such as the space of C^∞ maps from a Riemann surface to a Calabi-Yau threefold. This is not rigorous mathematics, yet these methods led to the 1991 paper of Candelas, de la Ossa, Green and Parkes [**CdGP**] containing some astonishing predictions about rational curves on the quintic threefold. These predictions went far beyond anything algebraic geometry could prove at the time.

The challenge for mathematicians was to understand what was going on and, more importantly, to prove some of the predictions made by the physicists. In this book, we will see that algebraic geometers have made substantial progress, though there is still a long way to go. The process of creating a mathematical foundation for aspects of mirror symmetry has given impetus to new fields of algebraic geometry. Examples include quantum cohomology, Kontsevich's definition of a stable map, the complexified Kähler moduli space of a Calabi-Yau threefold, Batyrev's duality between certain toric varieties, and Givental's notion of quantum differential equations. Mirror symmetry has also led to advances in deformation theory leading to the theory of the virtual fundamental class, as well as a previously unknown connection between algebraic and symplectic deformation theory. Even though we still don't know what mirror symmetry really "is", the predictions that mirror symmetry makes about Gromov-Witten invariants can now be proved mathematically in many cases.

Goal of the Book

Perhaps the greatest obstacle facing a mathematician who wants to learn about mirror symmetry is knowing where to start. Currently, many references are scattered throughout journals, and many mathematical ideas exist solely in the physics literature, which is difficult for mathematicians to read. Our primary goal is to give an introduction to the algebro-geometric aspects of mirror symmetry. We include sufficient detail so that the reader will have the major ideas and definitions spelled out, and explicit references to the literature when space constraints prohibit more detail. We explain both the rigorous mathematics as well as the intuitions borrowed from physics which are not yet theorems. We do this because we have two primary target audiences in mind: mathematicians wanting to learn about mirror symmetry, and physicists who know about mirror symmetry wanting to learn about the mathematical aspects of the subject.

Mirror symmetry is connected to several branches of mathematics (and there are even broader connections between physical theories in various dimensions and many areas of mathematics). We focus on the connection between algebraic geometry and mirror symmetry, although we discuss closely related areas such as symplectic geometry. By restricting our focus in this way, we hope to give a reasonably self-contained introduction to the subject.

The book begins with a general introduction to the ideas of mirror symmetry in Chapter 1. Then Chapter 2 discusses the quintic threefold and explains how mirror symmetry leads to the enumerative predictions of [**CdGP**]. Chapter 3 reviews toric geometry, and Chapter 4 describes mirror constructions due to Batyrev, Batyrev-Borisov, and Voisin-Borcea. The next four chapters (Chapters 5, 6, 7 and 8) flesh out the mathematics needed to formulate a precise version of mirror symmetry. These chapters cover maximally unipotent monodromy, Yukawa couplings, complex and Kähler moduli, the mirror map, Gromov-Witten invariants, and quantum cohomology. This will enable us to state some Mirror Conjectures at the end of Chapter 8.

The next three chapters (Chapters 9, 10 and 11) are dedicated to proving some instances of mirror symmetry. Equivariant cohomology and localization play a crucial role in the proofs, so that these are reviewed in Chapter 9. These methods also give powerful tools for computing Gromov-Witten invariants. In order to explain Givental's approach to the Mirror Theorem, we need the gravitational correlators and quantum differential equations discussed in Chapter 10. Finally, Chapter 11 describes the work of Lian, Liu and Yau [**LLY**] and Givental [**Givental2, Givental4**] on the Mirror Theorem.

The mathematics discussed in Chapters 1–11 is wonderful but highly nontrivial. Later in the preface we will give some guidance for how to read these chapters.

The book concludes with Chapter 12, which brings together all of the open problems mentioned in earlier chapters and discusses some of the many aspects of mirror symmetry not covered in the text. Finally, there are appendices on singular varieties and physical theories.

We tried to make the bibliography fairly complete, but it has been difficult to keep up with the amazing number of high-quality papers being written on mirror symmetry and related subjects. We apologize to our colleagues for the many recent papers not listed in the bibliography.

Relation to Physics

For mathematicians, one frustration of mirror symmetry is the difficulty of getting insight into the physicist's intuition. There is no question of the power of this intuition, for it is what led to the discovery of the mirror phenomenon. But getting access to it requires a substantial study of quantum field theory. A glance at Appendix B, which discusses some of the physical theories involved, will indicate the magnitude of this task. Understanding the physics literature on mirror symmetry requires an extensive background, more than provided in this book. Appendix B has the more modest goal of introducing the reader to some of the topics in the physics literature which are relevant to mirror symmetry.

While this book was written to address the mathematics of mirror symmetry, we also hope to show how the mathematics reflects the spirit of the physics. With this thought in mind, we begin Chapter 1 with a discussion of the physics which led to mirror symmetry. We use terminology from physics freely, though we don't assume that the reader knows any quantum field theory. The idea is to convey the sense that mirror symmetry is completely natural from the point of view of certain conformal field theories. This is the most "physical" chapter of the book. Subsequent chapters will concentrate on the mathematics, though we will pause occasionally to comment on the relationship between the mathematics and the physics.

An important aspect of the role of physics is that mathematically sophisticated physicists helped discover the mathematical foundation for mirror symmetry. Algebraic geometers can take pride in the wonderful theories they created to explain parts of mirror symmetry, but at the same time we should also recognize that physicists provided more than just predictions—they often suggested the appropriate objects to study, accompanied in some cases by mathematically rigorous descriptions. This will become clear by checking the references given in the text—a surprising number, even in the purely mathematical parts of the book, refer to physics papers. There is no question of the debt we owe to our colleagues in physics.

How to Read the Book

Mirror symmetry is a wonderful story, but its telling requires lots of details in many different areas of algebraic geometry. It is easy to get lost, especially if you try to read the book cover-to-cover. Fortunately, this isn't the only way to read the book.

Our basic suggestion is that you should begin with Chapters 1 and 2. As already mentioned, Chapter 1 explains some of the physics, and it also introduces two key ideas, the A-model of a Calabi-Yau manifold V, which encodes the enumerative information we want, and the B-model of the mirror V°, which we can compute using Hodge theory. Then Chapter 2 shows what this looks like in the case of the quintic threefold $V \subset \mathbb{P}^4$ and in the process derives the enumerative predictions made in [**CdGP**]. This chapter ends with a preview of the proofs of mirror symmetry from Chapter 11.

After reading the first two chapters, there are various ways you can proceed, depending on your mathematical interests and expertise. To help you choose, here is a description of some of the highlights of the remaining chapters:

Chapter 3. Readers familiar with toric geometry can skip most of this chapter. Section 3.5 introduces reflexive polytopes, which are used in the Batyrev mirror construction.

Chapter 4. Section 4.1 describes the Batyrev mirror construction and gives some evidence for the mirror relation. Section 4.2 explains how this applies to the quintic threefold. K3 surfaces are used in Section 4.4 to construct some interesting mirror pairs of Calabi-Yau threefolds.

Chapter 5. Maximally unipotent monodromy is an important part of mirror symmetry and is defined in Section 5.2. Readers interested in computational techniques for projective hypersurfaces, toric hypersurfaces and hypergeometric equations should look at Sections 5.3, 5.4 and 5.5, while those interested in the Hodge theory of Calabi-Yau threefolds should read Section 5.6 very carefully.

Chapter 6. We consider complex moduli in Section 6.1 and Kähler moduli in Section 6.2. The two discussions are interwoven because of the relation between the two predicted by mirror symmetry. The main example we work out concerns toric hypersurfaces, so that the reader will need the Batyrev mirror construction from Chapter 4. Readers interested in moduli of Calabi-Yau manifolds, Kähler cones, and the global aspects of mirror symmetry will want to read these sections carefully. Section 6.3 discusses the mirror map and has more on hypergeometric equations, which are used to construct the mirror map in the toric case.

Chapter 7. With the exception of some examples, Chapter 7 is independent of the earlier chapters. The main objects of study are Gromov-Witten invariants. Sections 7.1.1, 7.1.2 and 7.3.1 are essential reading. Otherwise:

- The discussion of the virtual fundamental class in Sections 7.1.3–7.1.6 is more technical and can be skipped at the first reading. The one exception is Example 7.1.6.1, which gives an important formula for some Gromov-Witten invariants of the quintic threefold. The virtual fundamental class is used in various places in Chapters 9, 10 and 11.
- Readers interested in symplectic geometry should read Sections 7.2 and 7.4.4 carefully.
- Readers interested in enumerative geometry will want to look at Section 7.4. Some of the examples given here will be revisited in Chapter 8.

One surprise in Section 7.4.4 is the subtle relation between the instanton number n_{10} and the number of degree 10 rational curves on the quintic threefold.

Chapter 8. This chapter uses the Gromov-Witten invariants of Chapter 7 to define the two flavors of quantum cohomology, small and big. Everyone should read Section 8.1.1 for the small quantum product and Section 8.1.2 for some examples. Also, some knowledge of the Gromov-Witten potential is also useful. This is covered in Sections 8.2.2, 8.3.1 and 8.3.3. Then:

- Readers interested in enumerative geometry should read Sections 8.1, 8.2 and 8.3 carefully.
- Readers interested in Hodge theory will want to look at Section 8.5, which uses quantum cohomology to construct the A-variation of Hodge structure on the cohomology of a Calabi-Yau manifold.

A highlight of the chapter is Section 8.6, which formulates various Hodge-theoretic versions of the mirror conjecture.

Chapter 9. Sections 9.1 and 9.2.1 are required reading for anyone wanting to understand the proofs of mirror symmetry given in Chapter 11. This especially includes Example 9.2.1.3, which computes the Gromov-Witten invariant $\langle I_{0,0,d}\rangle$ using an equivariant version of the formula given in Example 7.1.6.1. Sections 9.2.2 and 9.2.3 prove some of the assertions about Gromov-Witten invariants of Calabi-Yau threefolds made in Section 7.4.4 and require a detailed knowledge of the virtual fundamental class.

Chapter 10. Readers only interested in the [**LLY**] approach to the Mirror Theorem can skip this chapter. For Givental's approach, however, the reader will need to read about gravitational correlators (Section 10.1.1–10.1.3), flat sections of the Givental connection (Section 10.2.1), and the J-function and quantum differential equations (Section 10.3). For readers with an interest in Hodge theory, the A-model variation of Hodge structure is discussed in Sections 10.2.2 and 10.3.2. This leads to a nice connection between Picard-Fuchs operators and relations in quantum cohomology.

Chapter 11. Here we discuss the recent proofs of the Mirror Theorem. There are two approaches to consider:

- The [**LLY**] approach to the Mirror Theorem for the quintic threefold is covered in Section 11.1. This requires knowledge of the essential sections of Chapters 7, 8 and 9 mentioned earlier.
- For Givental's version of the Mirror Theorem, one needs in addition the sections of Chapter 10 indicated above. In Sections 11.2.1, 11.2.3 and 11.2.4, we discuss the Mirror Theorem for nef complete intersections in $\mathbb{P}^n$, and then in Section 11.2.5 we consider what happens when the ambient space is a smooth toric variety.

In particular, we explain how both of these approaches prove all of the predictions for the quintic threefold made in Chapter 2. Another very interesting case is presented in Example 11.2.5.1, which concerns Calabi-Yau threefolds which are minimal desingularizations of degree 8 hypersurfaces in $\mathbb{P}(1,1,2,2,2)$. This example of a toric hypersurface makes numerous appearances in Chapters 3, 5, 6 and 8, so that the reader will need to look back at these earlier examples in order to fully appreciate what we do in Example 11.2.5.1.

We should also mention that in many cases, our proofs are not complete, for we often refer to the literature for certain details of the argument. The same applies to background material. For some topics (such as equivariant cohomology in Chapter 9) we review the basic facts, while for others (such as algebraic stacks in Chapter 7) we give references to the literature. We hope that this unevenness in the level is not too unsettling to the reader.

Acknowledgements

We are happy to thank numerous people who have assisted us in various ways. The authors would in particular like to express their appreciation to P. Aspinwall, P. Berglund, C. Borcea, P. Candelas, R. Dijkgraaf, B. Greene, T. Hübsch, A. Klemm, P. Mayr, G. Moore, D. R. Morrison, M. R. Plesser, D. van Straten, and C. Vafa, for numerous conversations over the past few years which helped us develop our views of mirror symmetry and understand the relevant mathematics and physics.

Thanks also to the Institut Mittag-Leffler for its hospitality during the spring of 1997. A seminar on mirror symmetry held that spring discussed portions of this book and helped us improve its exposition immensely. It is a pleasure to thank all of the participants. At the risk of inadvertently omitting someone, we in particular want to acknowledge P. Aluffi, V. Batyrev, K. Behrend, P. Belorousski, I. Ciocan-Fontanine, A. Elezi, G. Ellingsrud, C. Faber, B. Fantechi, W. Fulton, E. Getzler, L. Göttsche, T. Graber, B. Kim, A. Kresch, D. Laksov, J. Li, A. Libgober,

R. Pandharipande, K. Ranestad, E. Rødland, D. E. Sommervoll, S. A. Strømme, M. Thaddeus, and E. Tjøtta for their help.

During the process of writing the book, we sent draft chapters to various colleagues. In addition to some of the above people who helped with this, we also would like to thank E. Cattani, E. Ionel, B. Lian, K. Liu, A. Mavlyutov, T. Parker, G. Pearlstein, Y. Ruan, C. Taubes, G. Tian, P. Vermeire and S.-T. Yau for their helpful comments. We would also like to thank B. Dwork and J. Kollár for useful conversations. In addition, we would like to thank the reviewers of an earlier draft of the book for their feedback. We are also very grateful to A. Greenspoon for his careful reading of the final manuscript. Any errors which remain are, of course, our responsibility.

We also thank the seminar members at Oklahoma State University and the Valley Geometry Seminar for listening to many talks on mirror symmetry. Their feedback and support are greatly appreciated.

During the period when this book was being written, D.A.C. was partially supported by the NSF, and S.K. was supported by the NSF and NSA. We are grateful to these organizations for their support.

Finally, we would like to thank Sergei Gelfand and Sarah Donnelly of the AMS for their editorial guidance and Barbara Beeton of the AMS technical support staff for her help with AMS-LaTeX.

Our Hope

Mirror symmetry is an active area of research in algebraic geometry, with plenty to keep mathematicians busy for many years. This book should be regarded as a preliminary report on the current state of the subject—the definitive text on mirror symmetry has yet to be written. Nevertheless, we hope that the reader will find the connections explored here to be an exciting and continuing story.

November, 1998

David A. Cox
Sheldon Katz

CHAPTER 1

Introduction

Mirror symmetry has made some surprising predictions in algebraic geometry, ranging from the number of rational curves on a quintic threefold to the structure of certain moduli spaces. These are wonderful problems to work on and, as indicated in the preface, have led to some very interesting mathematics. Yet to understand where these predictions come from, the algebraic geometer must plunge into the language of physics, which is unfamiliar and sometimes frustratingly nonrigorous. Hence, to begin our survey of the algebraic geometry of mirror symmetry, we will start with the motivations for mirror symmetry and a discussion of what mirror symmetry means in physics. Our treatment will be somewhat incomplete, since it will involve many terms from physics which may be new to the reader. Nevertheless, we hope to convey some of the intuition behind this remarkable phenomenon.

We will then discuss three-point functions (which are crucial to the enumerative predictions of mirror symmetry) and the physical reasons why Calabi-Yau manifolds appear in the theory. Finally, at the end of the chapter, we will return to the more familiar world of mathematics and give the reader a preview of the algebraic geometry to be explored in the remaining chapters of the book.

1.1. The Physics of Mirror Symmetry

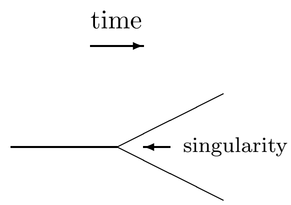
The goal of this section is to give the reader a feeling for why mirror symmetry should occur and what it should imply. From the point of view of physics, mirror symmetry arises naturally from standard constructions in supersymmetric string theory, and our discussion will begin with some elementary remarks about strings and supersymmetry. The reader should be assured that no previous knowledge of physics is assumed! Our aim is to convey the flavor of these physical theories and in the process enhance the reader's intuition for the resulting mathematics. A detailed understanding of the physics is not necessary, though later chapters will refer to the physics presented in this chapter. As general references for string theory, the reader can consult [**GSW, Polchinski2**].

In string theory, physical processes are described by the propagation of a string in spacetime. A propagating string traces out a surface, called the *world sheet* Σ of the string. Classical fields can be described as functions, sections of bundles, etc. on the world sheet, and quantizing leads to a two-dimensional quantum field theory. This theory has a generalized Hilbert space of states, together with a collection of observables, which become self-adjoint operators on the space of states. The other key ingredients of the theory are the *action* S obtained by integrating a Lagrangian over the world sheet Σ, and the *correlation functions*

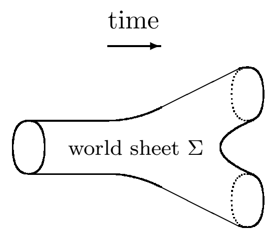
$$\langle \phi(x_1), \ldots, \phi(x_n) \rangle = \int [D\phi]\phi(x_1)\ldots\phi(x_n)e^{iS(\phi)}, \tag{1.1}$$

where ϕ is an observable. This Feynman integral is over all possible world sheets and is mathematically undefined at present. We discuss such theories in more detail in Appendix B. For now, let's concentrate on their general features.

String theories are nice because they eliminate some of the problems which occur when a particle splits into two particles. As the following picture shows, representing the particle by a point leads to a singularity, while the string representation is a smooth 2-manifold with boundary:



particle splitting,
represented as a point



particle splitting,
represented as a string

However, string theories still have some undesirable features, including many infinities which require renormalization. A remarkable discovery in recent times is that *supersymmetry* can eliminate many of these difficulties. Supersymmetry transforms bosons (particles with integer spins and symmetric wavefunctions) into fermions (particles with half-integer spins and antisymmetric wavefunctions) and vice-versa. Although supersymmetry has not been experimentally verified to date, supersymmetric theories have become very important in theoretical physics because of their nice behavior.

Another ingredient we need is that the world sheet Σ has a conformal structure, and our supersymmetric string theory needs to be equivalent under conformal equivalence. Hence this theory is a *superconformal field theory* (SCFT for short). The Lie algebra of the symmetry group of such a theory is a superconformal algebra. This algebra contains the conformal algebra (the Lie algebra of the group of conformal transformations of the world sheet) as a subalgebra, and it also contains the supersymmetry transformations.

The superstring theories come in four basic types: type I, type IIA, type IIB, and, of greatest interest to us, *heterotic*. Heterotic string theory is an $N = 2$ SCFT because there are two supersymmetries. In such a theory, the equations of motion for the fermions decouple into left- and right-moving solutions, which means that there are actually four supersymmetries, two left-moving and two right-moving. For this reason, heterotic string theories are more properly called $(2, 2)$ theories, as there are two independent supersymmetries in each of the left- and right-moving sectors of the theory.

The $N = 2$ superconformal algebra contains two copies of the usual superconformal algebra, and hence has two $u(1)$ subalgebras, one in the right-moving sector of the theory which infinitesimally rotates the two supersymmetries, and the other in the left-moving sector which acts similarly. A noncanonical choice of generator for each $u(1)$ can be made by ordering the two supersymmetry transformations. If the order of the supersymmetries is reversed, the result is to change the sign of the generator of the $u(1)$. The respective generators of these subalgebras are denoted by $(Q, \bar{Q})$ and are only well-defined up to sign. We regard $(Q, \bar{Q})$ as operators on the Hilbert space of states, which decomposes into eigenspaces under the action of $u(1) \times u(1)$. As we will see, these eigenspaces can be very interesting.

So far, we've discussed heterotic string theories in the abstract. The next step is to actually construct a such a theory, which is where algebraic geometry enters the picture. There are several ways this can be done, but for us, the most important is the *nonlinear sigma model* (sigma model for short) determined by a *Calabi-Yau threefold*[1] V and a *complexified Kähler class* $\omega = B + iJ$ on V. Here, B and J are elements of $H^2(V, \mathbb{R})$, with J a Kähler class.

From the input data (V, ω), there is a geometric construction of an $N = 2$ SCFT which explicitly gives distinct roles for the various supersymmetries; hence in this context there is a canonical choice for $(Q, \bar{Q})$. This gives an explicit choice of the $u(1) \times u(1)$ representation on our Hilbert space, and one can compute that for $p, q \geq 0$, $(Q, \bar{Q})$ has eigenspaces:

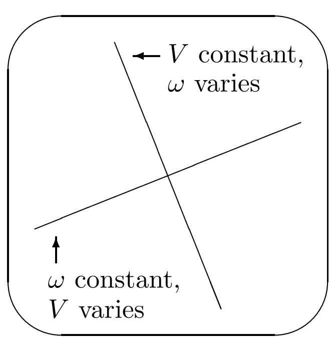
$$\begin{aligned} (p,q) \text{ eigenspace} &\simeq H^q(V, \wedge^p T_V) \\ (-p,q) \text{ eigenspace} &\simeq H^q(V, \Omega^p_V). \end{aligned} \tag{1.2}$$

Appendix B.2 describes more fully what it means to be a nonlinear sigma model and Section 1.3 explains how the Calabi-Yau condition arises from the physics.

The most important fields in a heterotic string theory are associated to elements of $H^1(V, T_V)$ and $H^1(V, \Omega^1_V)$, corresponding to eigenvalues $(1, 1)$ and $(-1, 1)$ respectively. The operators corresponding to elements in these spaces are called *marginal operators*. These are important partly because they are closely related to the moduli of sigma models coming from (V, ω). Intuitively, SCFT moduli are obtained by simultaneously varying the complex structure on V and the complexified Kähler class $\omega = B + iJ$, although there are extra discrete SCFT identifications on the moduli spaces which we will ignore for the moment. While readers should be familiar with the complex moduli, the idea of "complexified Kähler moduli" may be new. We will describe this in more detail in Section 1.4.

It follows from this description that the SCFT moduli space has two foliations, one of whose leaves can be described as "V constant", while the leaves of the other are "ω constant." This leads to the following picture:

(1.3)



SCFT moduli near (V, ω)

[1] For now, a Calabi-Yau threefold is a smooth compact connected threefold with vanishing first Betti number and trivial canonical class. Later, we will allow certain singularities.

In spite of this picture, we should emphasize that the SCFT moduli space is *not* a product of the complex structure and Kähler structure moduli spaces, not even locally. In fact, the Kähler moduli space of ω can depend on the complex structure of V [**Wilson2**]. In general, this is only well-defined if we have in mind a fixed complex structure on V. On the other hand, it follows from [**Wilson2**] that for a sufficiently generic Calabi-Yau threefold V, the Kähler moduli of ω is independent of the complex structure of V. These issues will be discussed in more detail in Chapter 6. Hence, although the above picture is useful at a conceptual level, it does not reflect the subtleties of the SCFT moduli space.

Now that we have a better idea of how Calabi-Yau threefolds and complexified Kähler classes gives interesting physical theories, it is time to explain where mirror symmetry comes from. The basic starting point lies in the sign indeterminacy of $(Q, \bar{Q})$. We mentioned above that $(Q, \bar{Q})$ are only well-defined up to sign, yet the sigma model coming from (V, ω) makes a very specific choice. If we changed Q to $-Q$ and left $\bar{Q}$ as is, we would interchange the (p, q) and $(-p, q)$ eigenspaces, which by (1.2) would interchange $H^q(V, \wedge^p T_V)$ and $H^q(V, \Omega_V^p)$. This is not possible since these are vector spaces of different dimensions in general. Yet from the physical point of view, such a sign change is reasonable. This asymmetry suggests that maybe the sign change corresponds to the sigma model arising from a *different* pair (V°, ω°). If such a pair (V°, ω°) exists, we say that (V, ω) and (V°, ω°) are a *mirror pair*. More formally, we have the following definition from physics.

Physics Definition 1.1.1. *(V, ω) and (V°, ω°) form a* mirror pair *if their sigma models induce isomorphic superconformal field theories whose $N = 2$ superconformal representations are the same up to the above sign change.*

Note that this is not a mathematical definition since the SCFT associated to (V, ω) is not rigorously defined. However, in Chapter 8, we will give a careful definition of a *mathematical mirror pair*. This definition will incorporate many of the properties predicted by mirror symmetry.

It is these properties to which we now turn our attention. If (V, ω) and (V°, ω°) are a mirror pair, then we get isomorphic SCFT's. But what does this mean about the mathematics? One of the major goals of this book is to understand the mathematical consequences of mirror symmetry.

To see what mirror symmetry tells us about V and V°, first note that if we combine (1.2) with the eigenvalue change $(p, q) \leftrightarrow (-p, q)$, we get isomorphisms

$$\begin{aligned} H^q(V, \wedge^p T_V) &\simeq H^q(V^\circ, \Omega_{V^\circ}^p) \\ H^q(V, \Omega_V^p) &\simeq H^q(V^\circ, \wedge^p T_{V^\circ}). \end{aligned} \tag{1.4}$$

Since V is Calabi-Yau, it has a nonvanishing holomorphic 3-form Ω, and cup product with Ω gives a (noncanonical) isomorphism $H^q(V, \wedge^p T_V) \simeq H^q(V, \Omega_V^{3-p})$. The same is true for V°, so that (1.4) can be written as

$$\begin{aligned} H^q(V, \Omega_V^{3-p}) &\simeq H^q(V^\circ, \Omega_{V^\circ}^p) \\ H^q(V, \Omega_V^p) &\simeq H^q(V^\circ, \Omega_{V^\circ}^{3-p}). \end{aligned} \tag{1.5}$$

These isomorphisms have a nice interpretation in terms of the Hodge diamond. Since V is a smooth threefold, its Hodge numbers $h^{p,q}(V) = \dim H^q(V, \Omega_V^p)$ have the symmetries $h^{p,q}(V) = h^{q,p}(V) = h^{3-p,3-q}(V) = h^{3-q,3-p}(V)$, and since V is Calabi-Yau, we also have $b_1(V) = 0$ and $\Omega_V^3 \simeq \mathcal{O}_V$. This implies that $h^{1,0}(V) = 0$ and $h^{3,0}(V) = 1$. Furthermore, $h^{0,2}(V) = \dim H^2(V, \mathcal{O}_V) = \dim H^1(V, \mathcal{O}_V) =$

$h^{0,1}(V) = 0$, where the second equality follows from Serre duality and $\Omega_V^3 \simeq \mathcal{O}_V$. Thus the Hodge diamond of V is as follows:

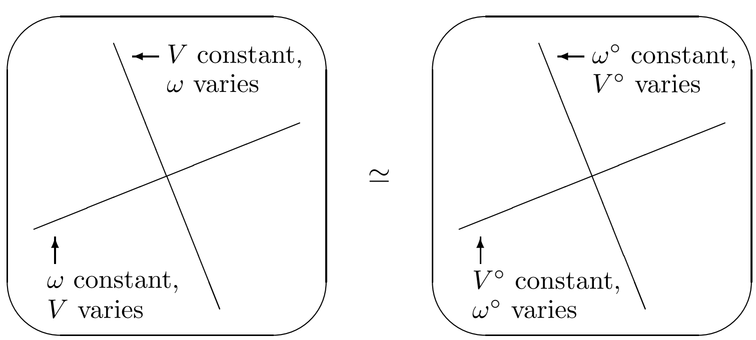
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1}(V) & & 0 & \\ 1 & & h^{2,1}(V) & & h^{2,1}(V) & & 1 \\ & 0 & & h^{1,1}(V) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

If we now compare the Hodge diamonds of a mirror pair V and V°, (1.5) implies that $h^{p,q}(V) = h^{3-p,q}(V^\circ)$, which shows that the Hodge diamond of V° is the reflection (or mirror image) of the Hodge diamond of V about a 45° line. This is where the name "mirror symmetry" comes from.

The isomorphisms (1.4) and (1.5) are actually the first of a series of increasingly impressive consequences of mirror symmetry. The next interesting implication of being a mirror pair concerns moduli spaces. To see where moduli enter the picture, note that (1.4) gives isomorphisms $H^1(V, T_V) \simeq H^1(V^\circ, \Omega^1_{V^\circ})$ and $H^1(V, \Omega^1_V) \simeq H^1(V^\circ, T_{V^\circ})$. This naturally identifies the tangent space to the complex moduli of V with the tangent space to the Kähler moduli of ω° (see Section 1.4 for a definition), and similarly identifies the tangent space to the Kähler moduli of ω with the tangent space to the complex moduli of V°. Thus the complex moduli space of V is locally isomorphic to the Kähler moduli space of ω°, and similarly the complex moduli space of V° is locally isomorphic to the Kähler moduli space of ω. These local isomorphisms are collectively called *the mirror map*. In Chapter 6, we will study complex and Kähler moduli in more detail and give a careful definition of the mirror map.

Recall from our (slightly inaccurate) picture (1.3) that the SCFT moduli space of (V, ω) has two foliations, one where the leaves are "V constant" and the other with leaves "ω constant." This means that if (V, ω) and (V°, ω°) are a mirror pair, then we get the following local isomorphism of SCFT moduli spaces:

(1.6)



SCFT moduli near (V, ω) SCFT moduli near (V°, ω°)

This picture also clarifies why it makes no sense to speak of "the" mirror manifold of a Calabi-Yau manifold V: given V, we can vary ω freely in one leaf of the foliation of the SCFT moduli space, which on the mirror side would cause ω° to be fixed and V° to vary freely. So if anything, "the mirror of V" should be the class ω°, together with the moduli space of those deformations of V° on which ω° makes sense as a complexified Kähler class.

In addition to what we've discussed so far, the existence of a mirror pair has further consequences, not all of which are understood yet. The basic idea is that *any* quantity that can be defined in terms of the SCFT can in principle be computed using two different constructions for the SCFT. In the best cases, these quantities can be computed in terms of the geometry of V and V°. Note that due to the sign change in Q and the fact that $H^q(V, \Omega_V^p)$ corresponds to $H^q(V^\circ, \wedge^p T_{V^\circ})$, it is expected that the geometric calculations will be different for the different models. The best example is given by the correlation functions of the SCFT, already mentioned in (1.1). These will be discussed in more detail in Section 1.2 and are the key to the enumerative predictions made by mirror symmetry. We will see that a computation on the mirror family can yield amazing results about the original Calabi-Yau manifold.

In the physics literature, mirror symmetry is a rich phenomenon. In addition to the mirror symmetry for nonlinear sigma models discussed so far, mirror symmetry has also been observed for some non-geometric types of SCFT's, including *minimal models* and *Landau-Ginzburg orbifolds*. To explain how mirror symmetry works in these cases, one needs to take the "orbifold" of a SCFT by a finite group. This begins with the subtheory consisting of invariant fields, but since the resulting subtheory is not stable under the flow of time (i.e., is not unitary), extra fields are added to get a unitary theory which is again a SCFT (actually, the physics is a bit more subtle—see Appendix B.4 for the details). If we quotient out by a carefully chosen group action, we get the same physical theory we started with, but with a change in the sign of the Q eigenvalues. This version of "mirror symmetry" predates the discovery of mirror symmetry for nonlinear sigma models.

Early evidence for mirror symmetry of Calabi-Yau threefolds was given by lists of Calabi-Yau hypersurfaces in weighted projective spaces (or their quotients by finite groups). The Hodge numbers of these hypersurfaces exhibited a striking (but far from perfect) symmetry. For some of these hypersurfaces, mirror symmetry was demonstrated in [**GPl**] by first showing mirror symmetry for certain Landau-Ginzburg theories (as mentioned above) and then relating these theories to the sigma models of the hypersurfaces. As we will see in Chapter 4, all of these weighted projective hypersurfaces are a subclass of those that arise from Batyrev's reflexive polytope construction [**Batyrev4**], as observed in [**CdK**]. It is conjectured (and widely believed) that Calabi-Yau threefolds coming from reflexive polytopes are mirror symmetric, and more generally, that the larger class of toric complete intersections [**Borisov1**] is mirror symmetric. Much evidence has been given in the last few years [**CdGP, Font, Morrison1, CdFKM, HKTY1, CFKM, HKTY2, BK1, AGM1, ES1, BKK, Kontsevich2, MP1, Givental2, Givental4, LLY**]. In Chapter 11, we will outline two related approaches to the *Mirror Theorem*, which establishes the equality of certain correlation functions of Calabi-Yau toric complete intersections and their conjectured mirrors.

1.2. Three-Point Functions

The correlation functions defined in (1.1) are objects of intrinsic interest in a SCFT. In physics, they arise naturally in the study of successive generations of particles. The most common correlation function is the *three-point function*, which describes interactions between particles from three generations, not necessarily distinct. In the Standard Model of elementary particle physics, a *generation* of particles is a collection of particles with particular types of interactions under the electric, weak nuclear, and strong nuclear forces. Experiments indicate that there are 3 generations of particles. One of these generations includes the most familiar particles, namely the electron and its accompanying neutrino, and the up and down quarks (which are the constituents of the proton and the neutron). The other known generations contain the more exotic quarks and leptons.

We will consider three-point functions for the nonlinear sigma model coming from a Calabi-Yau threefold V and a complexified Kähler class ω. The most interesting three-point functions are the *Yukawa couplings*, which come from the marginal operators discussed in the previous section. These correspond to $H^1(V,\Omega^1_V)$ and $H^1(V,T_V)$, which gives two types of Yukawa coupling to consider.

We begin with the Yukawa coupling coming from $H^1(V,\Omega^1_V)$. To each element of $H^1(V,\Omega^1_V)$, the sigma model associates a 27-dimensional vector space of fields, which form an irreducible representation of E_6. In order to connect string theory to the physical world, this vector space is presumed to contain a generation of elementary particles. The Yukawa coupling between three generations corresponding to elements ω_i of $H^1(V,\Omega^1_V)$, $i=1,2,3$, is a physically important coupling between three particles, one from each of the respective generations of particles.

In this situation, the Yukawa coupling is calculated by Feynman path integral techniques to be

$$\begin{aligned}\langle\omega_1,\omega_2,\omega_3\rangle = &\int_V \omega_1\wedge\omega_2\wedge\omega_3 \;+ \\ &\sum_{\beta\neq 0} n_\beta \textstyle\int_\beta\omega_1\int_\beta\omega_2\int_\beta\omega_3\,\dfrac{e^{2\pi i\int_\beta\omega}}{1-e^{2\pi i\int_\beta\omega}},\end{aligned} \tag{1.7}$$

where the sum is over homology classes $\beta\in H_2(V,\mathbb{Z})$ and n_β is naively the number of rational curves in the homology class β. A careful definition of n_β requires an understanding of *Gromov-Witten invariants*, which will be discussed in detail in Chapter 7. The theory predicts that the n_β don't change if we deform V, so that this Yukawa coupling depends on ω but not on V. In Chapter 8, we will see that the coupling (1.7) is closely related to the quantum cohomology ring of V.

The Yukawa coupling just described is sometimes called the *A-model correlation function*. The latter terminology arises from the *A-model* described in Appendix B.2. This is a "twisted" version of the SCFT, which means that certain fields are locally the same as in the sigma model, but globally are sections of certain twists of the bundles that they were originally sections of. These couplings are identified with the corresponding three-point functions in the A-model.

Let us also say a few words about where (1.7) comes from. Notice that the first term of this formula is just cup product:

$$\int_V \omega_1\wedge\omega_2\wedge\omega_3. \tag{1.8}$$

From the physics point of view, two things happen in going from (1.8) to the A-model correlation function (1.7). The first is a *non-renormalization theorem*, which says that from a perturbative point of view, there are no quantum corrections needed. As mentioned in the previous section, this is one of the nice consequences of supersymmetry. However, there are also world sheet non-perturbative corrections to be considered, which in this case are the *holomorphic instantons*. These are nonconstant holomorphic maps $\Sigma \to V$, where Σ is a compact Riemann surface. When we treat the A-model correlation function more carefully in Chapters 7 and 8, we will see that Σ can have nodal singularities and more than one component. In the A-model correlation function, the only instantons needed are those where Σ has genus 0. Naively, these are what the n_β count in formula (1.7). In the terminology of Chapter 7, we call n_β an *instanton number*.

The second Yukawa coupling to consider comes from $H^1(V, T_V)$. Here, elements of $H^1(V, T_V)$ correspond to the conjugate 27-dimensional representation of E_6, and we get a Yukawa coupling associated to three generations corresponding to elements θ_i of $H^1(V, T_V)$, $i = 1, 2, 3$. The Yukawa coupling in this case is given by

$$\langle \theta_1, \theta_2, \theta_3 \rangle = \int_V \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega), \tag{1.9}$$

where Ω is a holomorphic 3-form on V. The expression $\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega$ is defined by the composition

$$S^3 H^1(V, T_V) \otimes H^0(V, \Omega^3_V) \mapsto H^3(V, \wedge^3 T_V \otimes \Omega^3_V) \simeq H^3(V, \mathcal{O}_V) \simeq H^{0,3}(V).$$

Alternatively, one can think of this as

$$\int_V \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega),$$

where ∇ is the Gauss-Manin connection.[2] Note that $\langle \theta_1, \theta_2, \theta_3 \rangle$ is not yet well-defined, since Ω can be multiplied by any constant. There is however a natural normalization which we will describe later. This Yukawa coupling is clearly independent of the complexified Kähler class ω but depends on the complex structure of V (since Ω is a holomorphic 3-form).

The Yukawa coupling (1.9) is sometimes called the *B-model correlation function*, since it is identical with the corresponding three-point function in a different twisted theory, the *B-model* described in Appendix B.2.

To explain where the formula (1.9) comes from, we proceed as in the A-model case. Beginning with the "cup product"

$$\int_V \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega), \tag{1.10}$$

the same non-renormalization theorem applies, and we have the same holomorphic instantons as before. The crucial observation is that these instantons enter via cup product. In the A-model case, ω_i appears in (1.7) via $\int_\beta \omega_i$, which can be thought of as the cup product $g_\beta \cup \omega_i \in H^6(V, \mathbb{C}) \simeq \mathbb{C}$, where $g_\beta \in H^4(V, \mathbb{C}) = H^{2,2}(V)$ is the Poincaré dual of β. But in the B-model case, we have $\theta_i \in H^1(V, T_V) \simeq H^1(V, \Omega^2_V) = H^{2,1}(V)$, and since $g_\beta \in H^{2,2}(V)$, their cup product $g_\beta \cup \theta_i$ lies in $H^{4,3}(V) = 0$. Hence the instantons don't interact with (1.10), which is the crude reason why (1.10) equals the B-model correlation function (1.9).

[2] This is how the Yukawa coupling appears in [**CdGP**]. As we will see in Proposition 5.6.1, Hodge theory leads us to introduce an additional minus sign into the definition of the Yukawa coupling. There are also physical reasons for adding a minus sign, as discussed in [**Cd**]. Since Ω can be multiplied by an arbitrary constant, the sign is of little concern for now. On the other hand, in Section 5.6.4, we will fix a choice of Ω so will have to be careful about the sign. Our definition of the *normalized Yukawa coupling* will include this sign.

The absence of instanton corrections in (1.9) is extremely important. It means that we can compute the B-model Yukawa coupling *exactly* using standard methods of algebraic geometry. This procedure will be explained in Chapter 5.

Suppose now that (V, ω) and (V°, ω°) are a mirror pair, i.e., the sigma model associated to (V, ω) gives the same SCFT as the sigma model associated to (V°, ω°) with the appropriate sign change. This gives a natural isomorphism $H^1(V, \Omega^1_V) \simeq H^1(V^\circ, T_{V^\circ})$, and since three-point functions are intrinsic to the SCFT, it should follow that the A-model correlation function arising from the sigma model on (V, ω) coincides with the (appropriately normalized) B-model correlation function arising from the sigma model on (V°, ω°). This is one of the major mathematical consequences of mirror symmetry. The actual details of this identification are a bit more complicated because of the role of the mirror map, but this equality of A-model and B-model correlation functions is certainly plausible.

Note that the properties of the correlation functions are consistent with mirror symmetry. We have already observed that the A-model correlation function associated to (V, ω) depends on ω but not on V. It follows from mirror symmetry and the local identification of moduli spaces discussed above that the B-model correlation function associated to (V°, ω°) should depend on V° but not on ω°. As noted above, the B-model does indeed have this property.

As we will see with the example of the quintic threefold, being able to identify the three-point function of the A-model with the three-point function of the B-model of its mirror was used in [**CdGP**] to make some remarkable predictions for numbers of rational curves on the quintic threefold. This example is discussed in detail in the next chapter. In general, some of the most important consequences of mirror symmetry arise from the combination of the two following facts:

- The equality of the A-model and B-model correlation functions.
- The ability to compute the B-model function exactly.

Together, these allow one to compute Gromov-Witten invariants on Calabi-Yau threefolds, which in turn give a wealth of enumerative information. All of this, of course, depends on our ability to prove mathematically that these physical consequences are indeed correct.

What we have just described can be thought of as the "classical" approach to the consequences of mirror symmetry, where the A-model and B-model correlation functions are the primary objects of interest. However, in the years since the discovery of mirror symmetry, the focus has shifted a bit. In particular, the relation between the B-model correlation function and the Gauss-Manin connection is much better understood, which on the A-model side has led to the development of quantum cohomology and the A-variation of Hodge structure. Also, the work of Givental and of Lian, Liu, and Yau on the Mirror Theorem has introduced other new objects of interest—we will see that equivariant cohomology and localization play an important role in the proofs of the Mirror Theorem. We will address all of these ideas in subsequent chapters.

We close this section with a final observation about the formulas (1.7) and (1.9). They seem asymmetric since the first is much more complicated than the second. Fortunately, mirror symmetry easily accounts for this discrepancy. We noted above that the A-model correlation function should vary with complexified Kähler class ω, while the B-model function should vary with complex structure on V. Since the cup product in (1.8) is a purely topological invariant, it is clear that other terms are needed if we are to have a nontrivial dependence on ω. On the other hand, the cup product in (1.10) already depends on the complex structure, since Ω is a holomorphic 3-form on V. So it is conceivable that no further terms are needed.

1.3. Why Calabi-Yau Manifolds?

Our next task is to explain why mirror symmetry applies only to Calabi-Yau manifolds, especially Calabi-Yau threefolds. A full explanation of why nonlinear sigma models need Calabi-Yau manifolds would require a considerable detour into physics. We give a partial treatment of this topic in Appendices B.2 and B.3. For now, we will content ourselves with sketching some of the ideas. The starting point is the assumption that space-time should be a 10-dimensional manifold with a semi-Riemannian metric of signature $(9, 1)$. This manifold should locally be a product $M_{3,1} \times V$, where $M_{3,1}$ is the usual space-time of special relativity, and V is a compact 6-dimensional Riemannian manifold. The basic intuition is that V is too small to be seen at macroscopic scales but is essential for the quantum aspects of the theory. In any dimension other than 10, the conformal symmetry discussed earlier in the section does not survive the process of quantization.

As usual, the physics is nontrivial and involves some unfamiliar terms. The idea is to approach a nonlinear sigma model by first considering other theories which are more elementary from the physics point of view. In particular, one starts with an $N = 1$ supergravity theory in the low energy limit. This gets rid of the fermionic fields, and then supersymmetry and other consistency requirements impose strong conditions on the metric on Riemannian manifold V. In particular, one finds that the holonomy group of the metric equals $SU(3)$. This has some nice consequences:

- (differential geometry) The metric on V is Ricci flat, i.e., its Ricci curvature tensor vanishes identically. This implies $b_1(V) = 0$.
- (algebraic geometry) V has a complex structure such that $c_1(V) = 0$, and the metric is Kähler for this complex structure.

Hence we see that V is Calabi-Yau, as desired.

So far, we only have $N = 1$ supersymmetry in spacetime. After some further work, this theory can be reinterpreted as an $N = 2$ SCFT on the world sheet, although to preserve superconformal invariance, we need to deform the above Ricci flat metric. A precise description of this deformation is not known (this is still an open question in physics), but one can show that the new metric lies in the same Kähler class as the old one, so that we still have a Calabi-Yau threefold.

For a more complete description of how the Ricci curvature arises (from the physics point of view) and references to the literature, the reader should consult Chapter 0 of [**Hübsch**].

1.4. The Mathematics of Mirror Symmetry

From a mathematical point of view, the formulation of mirror symmetry given in Section 1.1 poses serious problems. For example, the definition of an $N = 2$ SCFT involves integrals over the space of *all* maps $\Sigma \to V$. Such integrals have yet to be defined rigorously. So a mathematical proof of mirror symmetry would involve an isomorphism between objects (the sigma models of (V, ω) and (V°, ω°)) which aren't well-defined mathematically. Even the $N = 2$ SCFT moduli spaces pictured in Section 1.1 are not well-defined!

What are mathematicians to do in this situation? One approach would be to avoid SCFT's altogether by concentrating on careful definitions of Kähler moduli spaces, Gromov-Witten invariants, etc. and then trying to prove that these objects behave as predicted by mirror symmetry. Another approach would be to embrace the physics and use its intuitions to see more deeply into the algebraic geometry, notably in the predictions mirror symmetry makes concerning the Gromov-Witten invariants of Calabi-Yau toric complete intersections. A third would be to formulate a purely mathematical version of mirror symmetry. For example, [**Kontsevich3**] proposes that the mirror of a complex manifold V is a certain symplectic manifold V° and that mirror symmetry might be formulated as an equivalence of derived categories relating coherent sheaves on V to a category built from Lagrangian submanifolds of V°. Another fascinating although still somewhat speculative approach is to attempt to geometrically *construct* the mirror manifold as a moduli space of special Lagrangian submanifolds [**SYZ**].[3] In practice, all of these approaches have been used, which is why mirror symmetry is such an exciting field.

In this book, we will follow mainly the first approach, with occasional comments on the physics of the situation. Thus, our goal is to discuss the algebraic geometry involved in understanding the mathematical aspects of Sections 1.1 and 1.2. One difference is that, unlike the physical theories, we will work with Calabi-Yau manifolds of arbitrary dimension, not just dimension three. However, when we try to formulate a mathematical definition of mirror pair in Chapter 8, we will be most successful in the case of Calabi-Yau threefolds.

We begin with a careful definition of Calabi-Yau. Since the mirror symmetry constructions to be studied in Chapter 4 sometimes produce singular varieties, we need a definition which allows certain types of singularities.

DEFINITION 1.4.1. *A* Calabi-Yau variety *is a d-dimensional normal compact variety V which satisfies the following conditions:*

(*i*) *V has at most Gorenstein canonical singularities.*
(*ii*) *The dualizing sheaf of V is trivial, i.e., $\widehat{\Omega}^d_V \simeq \mathcal{O}_V$.*
(*iii*) $H^1(V, \mathcal{O}_V) = \cdots = H^{d-1}(V, \mathcal{O}_V) = \{0\}$.

If in addition V has at most Gorenstein $\mathbb{Q}$-factorial terminal singularities, we say that V is a minimal Calabi-Yau variety.

In Appendix A, we review the dualizing sheaf $\widehat{\Omega}^d_V$ and the definitions of Gorenstein, canonical and terminal singularities. The Calabi-Yau threefolds considered in Section 1.1 certainly satisfy this definition.

Some other terms used in Section 1.1 deal with moduli of various sorts. The space of all complex structures on a given manifold V is a well known object in algebraic geometry, but the idea of *Kähler moduli* may be unfamiliar. Recall from Section 1.1 that we had a Calabi-Yau threefold V with a complexified Kähler class $\omega = B + iJ \in H^2(V, \mathbb{C})$ such that J was Kähler. However, if we change ω by an integral class, we don't change the physical theory, since the definition of nonlinear sigma model only uses $\exp(2\pi i \int_\Sigma \phi^*(\omega))$ for maps $\phi : \Sigma \to V$.[4] This quantity is unchanged if we change ω by an element of $H^2(V, \mathbb{Z})$. Thus, in defining Kähler moduli, we should mod out by the image of $H^2(V, \mathbb{Z})$. This leads to the following definition (which as before allows some singularities).

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[3] In general, symplectic geometry plays an important role in mirror symmetry, though we will concentrate more on the algebro-geometric aspects.

[4] You can see this in the A-model correlation function (1.7). The full details can be found in Appendix B.2.

Definition 1.4.2. *Let V be a projective orbifold with $h^{2,0}(V) = 0$. Then:*

(*i*) *The* Kähler cone *is the subset $K(V) \subset H^2(V,\mathbb{R}) = H^{1,1}(V,\mathbb{R})$ consisting of all Kähler classes.*

(*ii*) *The* complexified Kähler space *is the quotient*

$$K_{\mathbb{C}}(V) = \{\omega \in H^2(V,\mathbb{C}) : \mathrm{Im}(\omega) \in K(V)\}/\mathrm{im}\, H^2(V,\mathbb{Z}),$$

where $\mathrm{im}\, H^2(V,\mathbb{Z})$ *is the image of the natural map* $H^2(V,\mathbb{Z}) \to H^2(V,\mathbb{C})$.

(*iii*) *The* complexified Kähler moduli space *is the quotient* $K_{\mathbb{C}}(V)/\mathrm{Aut}(V)$.

Actually, the Kähler moduli space[5] as it arises in SCFT differs from this slightly. The Kähler moduli space as we have defined it receives *quantum corrections* which will modify its properties slightly. In particular, the theory may not converge for all ω, and so we will be forced to restrict our attention to complexified Kähler classes $\omega = B + iJ$ where J is sufficiently positive. Nevertheless, the larger space that we defined here is mathematically interesting and will be one our primary objects of study. We will consider some of the related subtleties in Chapter 6.

In Appendix A, we will review the definitions of orbifold and Kähler class on an orbifold. Since $h^{2,0} = 0$, the Kähler cone $K(V)$ is an open convex cone in $H^2(V,\mathbb{R})$. This tells us that the complexified Kähler space $K_{\mathbb{C}}(V)$ is a well-behaved object. However, in order to determine the structure of the Kähler moduli space $K_{\mathbb{C}}(V)/\mathrm{Aut(V)}$, we need to know how the automorphism group $\mathrm{Aut}(V)$ acts on the Kähler cone. We will return to this subject in Chapter 6.

Notice that Definitions 1.4.1 and 1.4.2 seem to involve slightly different types of singularities. Fortunately, by a result of [**Reid1**], a Gorenstein orbifold has canonical singularities. This means that any orbifold satisfying the second and third parts of Definition 1.4.1 is automatically Calabi-Yau. Hence, for the purposes of mirror symmetry, *Calabi-Yau orbifolds* are a natural class to work with.

1.5. What's Next?

We now describe how the next ten chapters will take us from here to a proof of the Mirror Theorem. We begin in Chapter 2 with a careful description of the mirror of a smooth quintic threefold $V \subset \mathbb{P}^4$. By carrying out the strategy outlined in Section 1.2, we will get some explicit predictions for the number of rational curves on V of given degree. We will revisit this example several times during subsequent chapters as we develop more mathematical background and eventually justify all of the computations which appear in Chapter 2.

In trying to generalize the example of the quintic, it was soon realized that toric geometry had an important role to play in mirror symmetry. Hence Chapter 3 will explore various ways of describing toric varieties, and then Chapter 4 will describe the known mirror symmetry constructions, many of which use toric geometry. This will give us a large supply of examples which should satisfy mirror symmetry and provide a good testing ground for the mathematics. We should also mention that there are even physical theories (the *gauged linear sigma models* of [**Witten5**] to be described in Appendix B.5) which explicitly use toric varieties. Support for mirror symmetry in the context of the gauged linear sigma model is given in [**MP1**].

Our next task is to describe and compute the B-model Yukawa coupling. This will be done in Chapter 5. It turns out that finding the correct coordinates for

[5] In discussing the Kähler moduli space, we frequently drop the adjective "complexified" when the meaning is clear from context.

calculating this Yukawa coupling requires a good understanding of the moduli space at certain boundary points. Hence, in Chapter 6 we will study the structure and compactifications of these moduli spaces. As mentioned above, we will also consider Kähler moduli. In fact, we will see that certain basic facts about Kähler moduli give insight into what the compactification of the usual moduli space should look like. In Chapter 7, we will discuss Gromov-Witten invariants. Definitions have been given in both the symplectic and algebraic categories, and either can be used to give a mathematical definition of the A-model Yukawa coupling.

By the end of Chapter 7, we will have everything needed to give a precise formulation of "classical" mirror symmetry, which asserts that certain correlation functions are compatible via the mirror map. But starting in Chapter 8, we will explore a deeper understanding of the subject. Two new ingredients are *quantum cohomology*, which can be thought of as working out the algebraic and enumerative implications of Gromov-Witten invariants, and the *A-model variation of Hodge structure*, which is a natural consequence of quantum cohomology. The basic idea is that mirror symmetry really involves an isomorphism, via the mirror map, of two variations of Hodge structure: one on the B-model (the usual VHS coming from complex moduli), and the other on the A-model (the A-variation of Hodge structure). This version of the Mirror Theorem will be formulated at the end of Chapter 8. It turns out that the desired equality of correlation functions follows immediately from this isomorphism of variations of Hodge structure.

The Mirror Theorem, when formulated using variations of Hodge structure, is still an open question, although recent work of Givental [**Givental2, Givental4**] and Lian, Liu and Yau [**LLY**] represents substantial progress toward proving this form of the theorem. We will discuss the ideas of these papers in Chapter 11. As we will see, this will require the introduction of new techniques and new objects of study. In particular, *equivariant cohomology* and *localization* will play an important role in the proof. In Chapter 9, we will review some of the basic definitions and theorems, and we will use these methods to prove some interesting results about Gromov-Witten invariants. Then Chapter 10 discusses an extension of Gromov-Witten invariants called *gravitational correlators*. These invariants will enable us to describe the flat sections of the A-model connection. We will also define the Givental J-function and explain its relation to *quantum differential equations*. Finally, in Chapter 11, we will discuss the Mirror Theorems stated in [**Givental2, Givental4**] and [**LLY**]. A brief preview of what is involved will be given at the end of Chapter 2, and then the full details for the quintic threefold will be presented in Chapter 11.

These chapters cover a lot of mathematics, and reading them straight through would be a somewhat daunting task. We suggest that the reader start with Chapter 2. From here, there are several ways to proceed, depending on the reader's interests. The preface offers some guidance for what to read, and a glance at the table of contents may also be useful. Our hope is that once you start reading, the intrinsic interest of the subject will draw you in. Mirror symmetry is a fascinating story, and it is fun to see how the various pieces fit together.