Contents

Preface

It was at a C.I.M.E. conference at the Palazzo in Cortona during the summer of 1978 that my eyes were opened to Malliavin's multi-tiered mansion in which Brownian motion on a Riemannian manifold resides. There, in the Palazzo's beautiful ballroom with its tiny blackboard presided over by Cleopatra and her adder, Malliavin held his audience in thrall with tales whose comprehension demanded simultaneous appreciation of the "upstairs story," the "downstairs story," and the profound influence that events on either exercise on the other. I have to admit that I could not have said with certainty on exactly which "level" a given event was transpiring. Indeed, at first I thought that there were only two "levels:" the upper one where Wiener measure lives and the lower one which is the manifold where the Brownian motion is taking place.

My confusion about this critical point was a direct consequence of my nearly perfect ignorance of differential geometry. In particular, because I had no idea what it was, Malliavin's frequent references to an intermediate level called the "bundle of orthonormal frames" were lost on me. Such matters are not broached in the first ten pages of even the most ambitious introductory texts about Riemannian geometry, and the first ten pages is as far as I had ever penetrated into the many differential geometry books which I had failed to read. Nor were Malliavin's intriguing lectures sufficient to persuade me to mend my ways immediately. Indeed, another fifteen years passed before my joint work with first Shigeo Kusuoka and then Ognian Enchev finally convinced me that the pain resulting from not learning more differential geometry would inevitably exceed the pain of mastering more than the first ten pages of at least one differential geometry text. Thus, about five years ago I forced myself to come to terms with Bishop and Crittenden's remarkably concise text [2]. My choice was dictated by two considerations: first, my collaborator Enchev had already assimilated the material in this book and I did not want to fall too far behind; secondly, Bishop and Crittenden emphasize the role of the bundle of orthonormal frames, and Malliavin had already alerted me to the advantages of this perspective. Of course, once I had taken the plunge, I delved into several other sources. In fact, the citations in this text give a reasonably accurate map of where I learned what.

Having benefitted from the efforts of differential geometers to explain their subject to me through their writings, I decided to reciprocate by writing this book, which is my attempt to explain my subject to them. With this in mind, I have tried to minimize the weight of "probabilistic" baggage which my readers must bring to a reading of this book. Further, wherever the option was available, I have chosen to emphasize the geometric over the stochastic aspect of the topic at hand. In particular, I never have made explicit use here of Itô's stochastic calculus. In spite of the grand and beautiful edifice erected by L. Schwartz, R. Darling, P.A. Meyer, and others (cf. [14] for an excellent explanation of their ideas or Ikeda and Watanabe's famous [22] for a more standard treatment) to convince me and the world otherwise, I remain firmly convinced that Stratonovich calculus is the calculus of choice if one wants to maximize ones geometric insight into stochastic analysis on differentiable manifolds. Thus, I have, from the outset, solved all "my stochastic integral equations" (the quotation marks are here because this is the last time that the term "stochastic integral equation" makes an appearance in this book) by passing to limits after mollification. My hope was that this procedure will make the book more accessible to readers who have not been reared in the probabilistic tradition. My fear is that I may very well have produced a book which is incomprehensible equally to the probabilistic and differential geometric communities. Be that as it may, here is a summary of the material which I have tried to convey.

Because I did not want to assume that my reader is acquainted with Wiener measure, I have devoted Chapter 1 to the construction of Wiener measure and a brief resume of some of its properties. There are, by now, a myriad construction methods. The one which I have chosen is basically the one given by P. Lévy. Not only is Lévy's construction stunningly beautiful, it has the advantage that, in some sense, it sets the pattern for all the other constructions which follow.

Following Itô's ideas, but not his procedure, I use the techniques, orignally explained in [39], to show in Chapters 2 and 3 how one can massage Wiener paths into the paths of more general diffusions on $\mathbb{R}^N$. Chapter 2 covers the basic case, the one in which everything is sufficiently bounded that no problems about possible explosion ever arise. In Chapter 3, it is shown that much of what is done in Chapter 2 continues to hold even after the boundedness assumptions are removed. In addition, Chapter 3 addresses several other topics of importance, chief of which are subordination and invariant measures.

Differentiable manifolds make their initial appearance in Chapter 4, where they appear as an embedded submanifold M of $\mathbb{R}^N$. First it is shown that quite general diffusions on M can be viewed as special cases of the diffusions constructed in Chapters 2 and 3. Second, when M is given the Riemannian structure which it inherits from $\mathbb{R}^N$, it is shown that the Brownian motion on M can be realized by "projecting" Wiener paths from the ambient $\mathbb{R}^N$ onto M.[1] The unabashedly extrinsic ideas initiated in Chapter 4 are developed further in Chapters 5 and 6. Specifically, curvature considerations are introduced in Chapter 5, where, in connection with Yau's non-explosion criterion,

[1] So far as I know, the first time that such a construction of Brownian motion appears is when, as Itô pointed out, I had stumbled upon it in [38] for the 2-sphere in $\mathbb{R}^3$. Subsequently, John Lewis [26] realized that the same construction works in general, although he lost the interpretation in terms of a projection. Nonetheless, the projection reappeared in the treatment given by Chris Rogers and David Williams [33].

I present the first evidence that Ricci curvature has a lot to say about the behavior of Brownian paths. Further evidence of the same fact is provided in Chapter 6, where I prove Bochner's identity in an integrated form which leads to a beautiful interpretation given by J.-M. Bismut in [3].

The rest of the book takes an intrinsic point of view. In Chapter 7, it is explained how the material in Chapters 2, 3, and 4 transfers, without difficulty, to the setting of an abstract differentiable manifold M. In particular, Chapter 7 ends with a "dirty," hands-on construction of Brownian motion via localization. In order to prepare the way for the intrinsic construction of Brownian motion due to Eells, Elworthy, and Malliavin (cf. [11] and [29]), Chapter 8 starts with a quick summary of the basic facts about the bundle $\mathcal{O}(M)$ of orthonormal frames, gives the E-E-M construction of Brownian motion as the projection from $\mathcal{O}(M)$ to M of the diffusion on $\mathcal{O}(M)$ associated with Bochner's Laplacian, and ends with a demonstration that all the essentially intrinsic results proved earlier about Brownian motion on a submanifold are, if anything, easier to understand in this abstract setting.

Chapter 9 is something of a digression. The idea is to expose how systematic use of normal coordinates enters into the study of Brownian motion on a manifold. Not surprisingly, the applications are strictly local. For example, it is shown how familiar expansions of the metric in normal coordinates are manifested in the computation of the exit time and exit place of Brownian motion from very small balls.

Finally, in the concluding chapter I take up the topic which originally stimulated my own interest in Brownian paths on Riemannian manifolds. Namely, for many years I worked on a set of ideas which I dubbed the *Malliavin calculus.* The essential, unifying theme of these ideas is that useful analytic information can be obtained from doing differential calculus in pathspace. More precisely, by perturbing paths and examining the infinitesimal response of their distribution to the perturbation, one can gain insight into various analytic quantities which are representable in terms of distribution of those paths. I had (most successfully with Shigeo Kusuoka) practiced this art in the Euclidean context. Around the same time, Jean–Michel Bismut (cf. [3]) was taking the initial steps which are necessary if one wants to do the same thing in a differential geometric setting. Somewhat later, Bismut's program was given an enormous boost by Bruce Driver's key article [9]. Motivated, at least in part, by the desire not to read all 104 pages of Driver's paper, Ognian Enchev and I embarked on a program to obtain Driver's conclusions on our own, and Chapter 10 is derived from the paper [15] which grew out of our efforts.

Finally, I have to recognize the critical role that my friend S.–T. Yau has played in all this. In particular, Yau consistently challenged me to come up with something that probability theory could do that Yau himself could not. Of course, I knew all along that such an example does not exist, but I was damned if I would tell Yau. Now I have.

Daniel W. Stroock, June 1999