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PREFACE

The goal of this book is to provide background for applications of Teichmüller theory to dynamical systems and in particular to iteration of rational maps and conformal dynamics, to Kleinian groups and three-dimensional manifolds, to Fuchsian groups and Riemann surfaces, and to one-dimensional dynamics. Although Teichmüller theory is a theory of two-dimensional objects, it naturally impinges on three-dimensional topology through its relationship to Kleinian groups and on one-dimensional dynamics through the quasisymmetric boundary action of a quasiconformal self-map of a disc.

Teichmüller space is a universal classification space for complex structures on a surface of given quasiconformal type. It turns out that the space itself has a natural complex structure, and in applications operators on Teichmüller space are constructed that turn out to be holomorphic and contracting. There are, for example, Thurston's skinning map for the construction of hyperbolic structures on three-manifolds, Thurston's orbit finder for the construction of a rational map in the class of a critically finite map, and various proofs of rigidity for renormalization. None of these topics is dealt with in this book. Rather we focus on new developments in the theory in both the finite and infinite dimensional cases. Our program is to give an exposition of the main known theorems regardless of dimensionality, emphasizing techniques that apply generally, and hoping to provide background for more applications.

In order to give an overview, we need to give several definitions, and we begin with the definition of a Riemann surface. A Riemann surface is an oriented topological surface together with a system of local parameters or charts whose domains of definition cover the surface and that map these domains homeomorphically onto open sets in the complex plane. The charts have the property that for any two with overlapping domains of definition, ϕ_1 and ϕ_2, the transition map, $w = \phi_2 \circ \phi_1^{-1}(z)$, mapping the plane to the plane, is holomorphic.

Holomorphic homeomorphisms of plane domains are precisely those that are differentiable in the complex sense at every point. Alternatively, they are orientation-preserving and conformal. A map f is conformal if, on an infinitesimal level, it takes any family of ellipses with given inclination and eccentricity centered at the origin in the tangent space at z to another family with the same eccentricity and possibly different inclination centered at the origin in the tangent space at $f(z)$. This infinitesimal property has implications for the local behavior of the map; shapes but not necessarily sizes of tiny objects are nearly preserved by f, and in the limit as the objects become more and more tiny, shapes are exactly preserved.

A quasiconformal map allows these shapes to be distorted, but the distortion measured as a ratio of shape measurements is uniformly bounded. Although quasiconformal maps need not be everywhere differentiable, the quasiconformality condition is easiest to describe when they are. Any C^1 diffeomorphism between plane domains on an infinitesimal level maps a small circle to a small ellipse. The local dilatation K_z at a point z is the ratio of the length of the major axis of this ellipse to the length of its minor axis. The global dilatation or simply the dilatation of the map is the supremum K of the values K_z for all z in the domain. An orientation-preserving map for which K is finite is by definition quasiconformal. In order to make this definition apply to maps that are not of class C^1, one needs a considerable amount of analysis and the theory of derivatives in the sense of distributions. A summary of this analysis is presented in Chapter 1. Here we mention three essential facts. First of all, a quasiconformal map f is differentiable almost everywhere, and if $K(f) = 1$, f is conformal. Secondly, if f is K-quasiconformal, then it is Hölder continuous with Hölder exponent $1/K$, that is to say, $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^{1/K}$, where the constant C depends on a normalization. Although no assumption is made in the definition about the distortion of size, this Hölder exponent yields some control. Thirdly, any quasiconformal map f has a Beltrami coefficient $\mu(z) = f_{\overline{z}}(z)/f_z(z)$, and up to postcomposition by a conformal map f is uniquely determined by μ. The absolute value of $\mu(z)$ determines $K_z(f)$ by the formula $K_z(f) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$. Moreover, μ can be an arbitrary complex-valued L_∞-function with $||\mu||_\infty < 1$.

Since Riemann surfaces have conformal structure, it makes sense to speak of the conformality and quasiconformality of maps between Riemann surfaces. In particular, if $w = f(z)$, the constant $K_z(f)$ is defined independently of the selected charts ϕ_1 defined in a neighborhood of z in R_1 and ϕ_2 defined in a neighborhood of w in R_2. Therefore, we can define $K(f)$ to be the essential supremum over all z in R_1 of the quantity $K_z(f)$.

Riemann surfaces R_1 and R_2 are called conformal if there is a conformal homeomorphism from R_1 onto R_2. In general, Riemann surfaces can be quasiconformal without being conformal, and for a given Riemann surface, its quasiconformal deformation theory is the study of the conformal equivalence classes of Riemann surfaces in the same quasiconformal class. The space of such conformally distinct surfaces in the quasiconformal class of R is called moduli space $\mathcal{M}(R)$. If R is compact, $\mathcal{M}(R)$ is a finite dimensional complex variety, but not a manifold.

The study of moduli is simplified by introducing an equivalence relation on a larger set of objects. The larger set of objects is the set of orientation-preserving quasiconformal maps from a fixed base Riemann surface onto a variable Riemann surface. The equivalence relation is easiest to describe if we assume the fixed surface is compact and without boundary. Two maps f_0 and f_1 from R to R_0 and to R_1 are equivalent if there is a conformal map c from R_0 onto R_1 such that $c \circ f_0$ is homotopic to f_1 by a homotopy g_t consisting of quasiconformal maps. Since a quasiconformal map f is, up to postcomposition by a conformal map, uniquely determined by its Beltrami coefficient $\mu = f_{\overline{z}}/f_z$, this equivalence relation also induces an equivalence relation on the space of complex-valued L_∞-Beltrami differentials μ with $||\mu||_\infty < 1$.

The set $[f]$ of quasiconformal maps equivalent to a given map f is called a Teichmüller equivalence class, and the space of all equivalence classes is the quasiconformal Teichmüller space $T(R)$. Because of basic properties of quasiconformal maps, within any equivalence class $[f]$ there is always a representative f_0

such that $K_0 = K(f_0)$ is minimal. If two equivalence classes $[f]$ and $[g]$ are given, the distance in Teichmüller's metric between these two classes is $\frac{1}{2}\log K_0$, where K_0 is the minimal dilatation of a map in the class of $f \circ g^{-1}$. In the compact case it is superfluous to modify the term *Teichmüller space* with the word *quasiconformal* because any two homeomorphic compact Riemann surfaces of the same genus are automatically quasiconformal (even diffeomorphic). Teichmüller showed that when R is compact $T(R)$ is a complete metric space homeomorphic to a cell of real dimension $6g - 6$ if the genus g of R is more than 1. When the genus is 1, it has real dimension 2.

The group of homotopy classes of quasiconformal self-maps of the base surface R is called the mapping class group, $MCG(R)$. It acts naturally as a group of isometries on $T(R)$ and identifies points that correspond to conformally equivalent Riemann surfaces. Thus, factoring Teichmüller space by the mapping class group yields moduli space; $\mathcal{M}(R)$ is equal to $T(R)$ factored by $MCG(R)$.

A Riemann surface is of finite analytic type if it can be obtained by removing a finite number of points from a compact Riemann surface. We note that this property is quasiconformally invariant. That is, if f is a quasiconformal map from such a Riemann surface, R, to another, $f(R)$, then necessarily $f(R)$ is also of finite analytic type. A surface of infinite analytic type can be obtained by taking a compact surface and deleting any infinite closed set. For example, an interesting infinite-type Riemann surface is the Riemann sphere minus the standard middle-thirds Cantor set in the unit interval. Any surface of infinite genus has infinite analytic type. It turns out that $T(R)$ is infinite dimensional if and only if R is of infinite analytic type.

Defining the equivalence relation on quasiconformal maps f from a Riemann surface R of infinite analytic type to a variable Riemann surface $f(R)$ is a technical matter. The essential idea is that two maps f_0 and f_1 are equivalent if there is a conformal map $c : f_0(R) \to f_1(R)$ such that $c \circ f_0$ and f_1 are homotopic by a homotopy that pins down the boundary points of R. The difficulty is in how to define boundary points and how to determine whether a quasiconformal homotopy extends to those points. One way to handle the difficulty is to use hyperbolic geometry and the uniformization theorem. One obtains the so-called ideal boundary for any Riemann surface whose universal covering group is Fuchsian. The question of how to define boundary points natural for quasiconformal deformations is an interesting problem.

We also study a new type of Teichmüller space that concerns only infinite-type Riemann surfaces and that explicitly deals with the asymptotic geometrical behavior of the Riemann surface at the boundary. The definition of asymptotic Teichmüller space $AT(R)$ is the same as that of ordinary Teichmüller space except in the definition of equivalence classes the word *conformal* is replaced by *asymptotically conformal*. A class $[f]$ of maps in $T(R)$ is asymptotically conformal if one can make $K_z(f_0)$ arbitrarily close to 1 for z in $R \setminus C$ by choosing a suitable representative f_0 of $[f]$ and by choosing a sufficiently large compact subset C of R. On surfaces of finite analytic type all Teichmüller classes of quasiconformal maps are asymptotically conformal, and thus in this case $AT(R)$ consists of just one point. In all other cases $AT(R)$ is infinite dimensional. Many of the results concerning asymptotic Teichmüller space presented in the text represent our joint work with Cliff Earle.

Study of $AT(R)$ automatically leads to the notion of the boundary dilatation $H([f])$ of a class $[f]$ in $T(R)$. For any compact set $C \subset R$ one looks for a representative f_0 of the class $[f]$ such that the essential supremum of $K_z(f_0)$ for z in $R \setminus C$ is as small as possible. The boundary dilatation $H([f])$ of $[f]$ is the infimum of all of these numbers over all compact subsets C of R. It can happen that $H([f]) = K_0([f])$. The Teichmüller metric on $AT(R)$ is given by boundary dilatation; the distance between two classes in $AT(R)$ represented by maps f and g is $\frac{1}{2} \log H([f \circ g^{-1}])$.

$T(R)$ has a natural basepoint, which is the equivalence class of the identity map on R, and the tangent space to $T(R)$ at this basepoint is isomorphic to the Banach dual space of the complex Banach space $A(R)$, the space of integrable holomorphic quadratic differentials $\varphi(z)dz^2$ on R. The norm of φ in $A(R)$ is given by $||\varphi|| = \iint_R |\varphi(z)|dxdy$.

The pairing $(\mu, \varphi) = \iint_R \mu(z)\varphi(z)\ dxdy$ induces a second notion of equivalence of Beltrami coefficients. Two Beltrami coefficients μ and ν are called infinitesimally equivalent if $(\mu - \nu, \varphi) = 0$ for all φ in $A(R)$ and μ is infinitesimally trivial if μ is infinitesimally equivalent to 0. When in addition $||\mu||_\infty$ and $||\nu||_\infty$ are both less than 1, μ and ν are globally equivalent if the quasiconformal maps f^μ and f^ν with Beltrami coefficients μ and ν represent the same point in $T(R)$. We say μ is globally trivial (or just trivial) if f^μ is equivalent to the identity map. Finally, μ is called extremal in its global class if f^μ has minimal dilatation in its Teichmüller class, and similarly, μ is extremal in its infinitesimal class if $||\mu||_\infty$ takes the smallest possible value in its infinitesimal class. We will see that the interplay of the infinitesimal and global equivalence relations on Beltrami coefficients is an essential element of Teichmüller theory.

Some of the main results we prove are:

- For any Riemann surface R, $T(R)$ and $AT(R)$ are complex manifolds modeled on Banach spaces.

- $T(R)$ and $AT(R)$ are complete with respect to Teichmüller's metric; $AT(R)$ is a quotient space of $T(R)$ with quotient metric induced by the natural projection from $T(R)$ to $AT(R)$, and this metric is given by boundary dilatation.

- These metrics have infinitesimal forms and are equal to the integrals of their infinitesimal forms.

- Teichmüller's metric on $T(R)$ is equal to Kobayashi's metric, a metric defined purely in terms of the family of holomorphic functions from the unit disc into $T(R)$.

- When R has finite genus with either finite or infinite analytic type, every holomorphic automorphism of $T(R)$ is induced by a quasiconformal self-map of R. Moreover, except for a few low-dimensional Teichmüller spaces, all occurring in genus 2 and lower, $MCG(R)$ acts as the full automorphism group of $T(R)$. For compact surfaces this result was proved by Royden.

- A quasiconformal map f has minimal dilatation in its Teichmüller class if and only if its Beltrami coefficient μ is extremal in its infinitesimal class. This result is called the Hamilton-Krushkal, Reich-Strebel necessary-and-sufficient condition for extremality.

- A quasiconformal map f is nearly extremal in its Teichmüller class if and only if its Beltrami coefficient μ is nearly extremal in its infinitesimal class.

This result is called Teichmüller contraction. There is also a version of Teichmüller contraction for $AT(R)$.

- A quasiconformal map f is the uniquely extremal representative in its Teichmüller class if and only if its Beltrami coefficient μ is uniquely extremal in its infinitesimal class.
- When R is the Riemann sphere $\overline{\mathbb{C}}$ minus a closed set, there is an alternative notion of equivalence on Beltrami coefficients. The resulting Teichmüller space is a complex manifold universal for holomorphic motions of the closed set.
- When R is a plane domain, the boundary dilatation of any class $[f]$ in $T(R)$ is realized at some point of the set-theoretic boundary of R. When R is the unit disc, this result is called Fehlmann's theorem on the existence of substantial points.

The foundation for these results is the study of the space $A(R)$ of integrable holomorphic quadratic differentials. Both the space $A(R)$ and each one of its elements play a geometric rôle. The geometric rôle of the space enters through the Teichmüller infinitesimal Banach norm on $A(R)$ and the main variational lemma that relates infinitesimally trivial Beltrami coefficients to trivial ones. The lemma says μ is infinitesimally trivial if, and only if, there is a curve μ_t of trivial Beltrami coefficients with the property that $||\mu_t - t\mu||_\infty/t$ approaches zero as $t \to 0$.

The geometric rôle of each element of $A(R)$ enters through the length-area method, also called Grötzsch's argument. This method shows how any holomorphic quadratic differential defined on a subdomain of R can be used to measure shape distortion of a quasiconformal map. The argument applied to the horizontal trajectories of a globally defined integrable holomorphic quadratic differential yields the main inequality of Reich and Strebel. Eventually, one learns that elements of $A(R)$ yield well-defined functionals on $T(R)$ and measure shape distortion of a Teichmüller class.

In later chapters we explore a variety of topics including measured foliations, heights mappings, a generalization of classical slit mapping theorems, and a construction of earthquakes based in the idea of applying a limiting process to finite earthquakes.

Much of the material in Chapters 1 and 2 on quasiconformal mapping and Riemann surfaces is presented without proof. Where results in these chapters are not fully proved, it is hoped the reader will agree to work with the theorems as stated and proceed directly to the subject at hand.

At the end of each chapter we provide exercises and sometimes open problems. There is a long list of results known for finite dimensional Teichmüller spaces but not known for infinite dimensional cases, and another list known for infinite dimensional cases but not for asymptotic Teichmüller spaces. Many research problems are posed by this observation.

The purpose of the bibliography is to provide pointers to persons wishing to pursue related subjects further. Although quite long, it is by no means complete. It is meant only to steer persons to the large and expanding body of literature in the field.

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