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Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials

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Preface

Since the early part of the twentieth century, with the work of Fredholm, Hilbert, Riesz, et al., the use of singular integral operators has developed into a range of tools for the study of partial differential equations. This includes the use of single and double layer potentials on planar curves to treat classical boundary problems for the Laplace operator on a planar region and higher-dimensional extensions. It also includes the construction of parametrices for elliptic PDE with variable coefficients. Fourier integral representations of these operators have provided many useful insights, though this method has not entirely supplanted the singular integral representation. When the use of the Fourier integral representation is emphasized, the operators are often referred to as pseudodifferential operators. Paradifferential operators form a singular class of pseudodifferential operators, particularly suited for applications to nonlinear PDE.

Treatments of pseudodifferential operators most frequently concentrate on operators with smooth coefficients, but there has been a good bit of work on operators with symbols of minimal smoothness, with applications to diverse problems in PDE, from nonlinear problems to problems in nonsmooth domains. In this monograph we discuss a number of facets of the operator calculi that have arisen from the study of pseudodifferential operators, paradifferential operators, and layer potentials, with particular attention to the study of nonsmooth structures.

In Chapter I we study pseudodifferential operators whose symbols have a limited degree of regularity. We consider various cases, including measures of regularity just barely better (or just barely worse) than merely continuous, measures either a little better or a little worse than Lipschitz, and others. Function spaces used to describe the degree of regularity of symbols include

$$C^\omega, \quad C^{(\lambda)}, \quad L^\infty \cap \mathrm{vmo}, \quad B^s_{p,1}.$$

Here C^ω consists of functions with modulus of continuity ω. The space $C^{(\lambda)}$, with $\lambda(j) = \omega(2^{-j})$, is defined in terms of estimates on a Littlewood-Paley decomposition of a function. These spaces coincide for Hölder-Zygmund classes of functions, but they diverge in other cases. The space vmo is the space of functions of vanishing mean oscillation, and $B^s_{p,1}$ are certain Besov spaces. The interplay between some of these function spaces is itself a significant object of study in this chapter.

The class of paradifferential operators, introduced in [**Bon**], has had a substantial impact on nonlinear analysis. In Chapter II we make use of paradifferential operator calculus to establish various nonlinear estimates, some of which have previously been established from other points of view. My interest in organizing some of this material, particularly in §§1–5, was stimulated by correspondence with T. Kato.

Other material in Chapter II includes investigations of paradifferential operators on the new function spaces $C^{(\lambda)}$.

Chapter III gives a sample of applications of some of the results of Chapters I–II to topics in PDE. We treat some linear PDE with rough coefficients, including some natural differential operators arising on Riemannian manifolds with nonsmooth metric tensors. We consider the method of layer potentials on domains that are not smooth (though not so rough as those considered in Chapter IV). We also treat a couple of topics in nonlinear PDE, including inviscid, incompressible fluid flow on rough planar domains and wave equations with quadratic nonlinearities. We also discuss various div-curl estimates, including a number of estimates of [**CLMS**]. Some of the work in this section, especially variable-coefficient results, grew out of correspondence with P. Auscher, following up on our work in [**AT**]. Other topics studied in Chapter III include the construction of harmonic coordinates on Riemannian manifolds with limited smoothness, regularity results for the metric tensor of a Riemannian manifold when one has estimates on the Ricci tensor, and propagation of singularities for PDE whose coefficients are more singular than $C^{1,1}$, but which still have well defined null bicharacteristics by virtue of Osgood's theorem.

Chapter IV deals with the method of layer potentials on Lipschitz domains. We establish the fundamental estimates of Cauchy integrals on Lipschitz curves of [**Ca2**] and [**CMM**] (via a method of [**CJS**]) and extensions to higher dimensions from [**CDM**]. We then discuss the Dirichlet problem for Laplace equations and variants on Lipschitz domains. We consider operators with variable coefficients, hence Lipschitz domains in Riemannian manifolds. Our treatment of this follows [**MT**], though here we restrict attention to the simpler case of smooth coefficients, whereas [**MT**] treats cases arising from C^1 metric tensors. This extends earlier work of [**Ve**] and others on the flat Laplacian on Lipschitz domains in Euclidean space.

Prerequisites for this work include an acquaintance with basic results on pseudodifferential operators and some methods from harmonic analysis, including the Littlewood-Paley theory. Sufficient material on these prerequisites could be obtained from either [**T2**] or Chapters 7 and 13 of [**T5**]. Indeed, this present work can be viewed as a companion to [**T2**].

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