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Arithmeticity in the Theory of Automorphic Forms

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PREFACE

A preliminary idea of writing the present book was formed when I gave the Frank J. Hahn lectures at Yale University in March, 1992. The title of the lectures was "Differential operators, nearly holomorphic functions, and arithmetic." By "arithmetic" I meant the arithmeticity of the critical values of certain zeta functions, and I talked on the results I had on GL_2 and $GL_2 \times GL_2$. At that time the American Mathematical Society wrote me that they were interested in publishing my lectures in book form, but I thought that it would be desirable to discuss similar problems for symplectic groups of higher degree. Though I had satisfactory theories of differential operators and nearly holomorphic functions applicable to higher-dimensional cases, our knowledge of zeta functions on such groups was fragmentary and, at any rate, was not sufficient for discussing their critical values. Therefore I spent the next few years developing a reasonably complete theory, or rather, a theory adequate enough for stating general results of arithmeticity that cover the cases of all congruence subgroups of a symplectic group over an arbitrary totally real number field, including the case of half-integral weight.

On the other hand, I had been interested in arithmeticity problems on unitary groups for many years, and in fact had investigated some Eisenstein series on them. Therefore I thought that a book including the unitary case would be more appealing, and I took up that case as a principal topic of my NSF-CBMS lectures at the Texas Christian University in May, 1996. The expanded version of the lectures was eventually published by the AMS as "Euler products and Eisenstein series."

After this work, I felt that the time was ripe for bringing the original idea to fruition, which I am now attempting to do in this volume. To a large extent the present book may be viewed as a companion to the previous one just mentioned, and our arithmeticity concerns that of the Euler products and Eisenstein series discussed in it; I did not include the cases of GL_2 and $GL_2 \times GL_2$. Those cases are relatively well understood, and it is my wish to present something new. Though the arithmeticity in that sense is the main new feature, as will be explained in detail in the Introduction, I have also included some basic material concerning arithmeticity of modular forms in general, and also a treatment of analytic properties of zeta functions and Eisenstein series on symplectic groups which were not discussed in the previous book.

It is a pleasure for me to express my thanks to Haruzo Hida, who read the manuscript and contributed many useful suggestions.

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NOTATION AND TERMINOLOGY

We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ the ring of rational integers, the fields of rational numbers, real numbers, and complex numbers, respectively. We put

$$\mathbf{T} = \{\, z \in \mathbf{C} \mid |z| = 1 \,\}.$$

We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and for an algebraic number field K we denote by K_{ab} the maximum abelian extension of K. If p is a rational prime, $\mathbf{Z}_p$ and $\mathbf{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively.

For an associative ring R with identity element and an R-module M we denote by $R^\times$ the group of all its invertible elements and by M_n^m the R-module of all $m \times n$-matrices with entries in M; we put $M^m = M_1^m$ for simplicity. Sometimes an object with a superscript such as G^n in Section 23 is used with a different meaning, but the distinction will be clear from the context. For $x \in R_n^m$ and an ideal $\mathfrak{a}$ of R we write $x \prec \mathfrak{a}$ if all the entries of x belong to $\mathfrak{a}$. (There is a variation of this; see §1.8.)

The transpose, determinant, and trace of a matrix x are denoted by ${}^t x$, $\det(x)$, and $\mathrm{tr}(x)$. The zero element of R_n^m is denoted by 0_n^m or simply by 0, and the identity element of R_n^n by 1_n or simply by 1. The size of a zero matrix block written simply 0 should be determined by the size of adjacent nonzero matrix blocks. We put $GL_n(R) = (R_n^n)^\times$, and

$$SL_n(R) = \{\, \alpha \in GL_n(R) \mid \det(\alpha) = 1 \,\}$$

if R is commutative. If $x_1, \ldots, x_r$ are square matrices, $\mathrm{diag}[x_1, \ldots, x_r]$ denotes the matrix with $x_1, \ldots, x_r$ in the diagonal blocks and 0 in all other blocks. We shall be considering matrices x with entries in a ring with an anti-automorphism ρ (complex conjugation, for example), including the identity map. We then put $x^* = {}^t x^\rho$, and $\widehat{x} = (x^*)^{-1}$ if x is square and invertible.

For a complex number or more generally for a complex matrix α we denote by $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, and $\overline{\alpha}$ the real part, the imaginary part, and the complex conjugate of α. For complex hermitian matrices x and y we write $x > y$ and $y < x$ if $x - y$ is positive definite, and $x \geq y$ and $y \leq x$ if $x - y$ is nonnegative. For $r \in \mathbf{R}$ we denote by $[r]$ the largest integer $\leq r$.

Given a set A, the identity map of A onto itself is denoted by id_A or 1_A. To indicate that a union $X = \bigcup_{i \in I} Y_i$ is disjoint, we write $X = \bigsqcup_{i \in I} Y_i$. We understand that $\prod_{i=\alpha}^{\beta} = 1$ and $\sum_{i=\alpha}^{\beta} = 0$ if $\alpha > \beta$. For a finite set X we denote by $\#X$ or $\#(X)$ the number of elements in X. If H is a subgroup of a group G, we put $[G : H] = \#(G/H)$. However we use also the symbol $[K : F]$ for the degree of an algebraic extension K of a field F. The distinction will be clear from the context. By a *Hecke character* χ of a number field K we mean a continuous $\mathbf{T}$-valued character of the idele group of K trivial on $K^\times$, and denote by χ^* the ideal character associated with χ. By a *CM-field* we mean a totally imaginary quadratic extension of a totally real algebraic number field.

INTRODUCTION

Our ultimate aim is to prove several theorems of arithmeticity on the values of an Euler product $\mathcal{Z}(s)$ and an Eisenstein series $E(z, s)$ at certain critical points s. We take these $\mathcal{Z}$ and E to be those of the types we treated in our previous book "Euler Products and Eisenstein Series," referred to as [S97] here. They are defined with respect to an algebraic group G, which is either symplectic or unitary. To illustrate the nature of our problems, let us take a CM-field K and put

$$G(\varphi) = G^{\varphi} = \{\, \alpha \in GL_n(K) \mid \alpha\varphi \cdot {}^t\alpha^{\rho} = \varphi \,\}, \tag{0.1}$$

where ρ denotes complex conjugation and φ is an element of $GL_n(K)$ such that ${}^t\varphi^{\rho} = \varphi$. This group acts on a hermitian symmetric space which we write $\mathfrak{Z}^{\varphi}$. We shall often be interested in the special case where φ takes the form

$$\eta = \eta_q = \begin{bmatrix} 0 & 1_q \\ 1_q & 0 \end{bmatrix}. \tag{0.2}$$

In this case we write $\mathcal{H}_q$, or simply $\mathcal{H}$, instead of $\mathfrak{Z}^{\varphi}$ for the symmetric space.

Given a congruence subgroup Γ of G, a Hecke eigenform $\mathbf{f}$ of holomorphic type on $\mathfrak{Z}^{\varphi}$ with respect to Γ, and a Hecke character χ of K of algebraic type, but not necessarily of finite order, we can construct a "twisted Euler product" $\mathcal{Z}(s, \mathbf{f}, \chi)$, whose generic Euler p-factor for each rational prime p has degree $n[K : \mathbf{Q}]$. Then we shall eventually prove that

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) \in \pi^{\varepsilon}\mathfrak{q}\,\langle\, \mathbf{f}, \mathbf{f}\,\rangle\,\overline{\mathbf{Q}} \tag{0.3}$$

for σ_0 in a certain finite subset of $2^{-1}\mathbf{Z}$ and $\overline{\mathbf{Q}}$-rational $\mathbf{f}$. Here $\langle\, \mathbf{f}, \mathbf{f}\,\rangle$ is the inner product defined in a canonical way; ε is an integer determined by σ_0, the signature of φ, the weight of $\mathbf{f}$, and the archimedean factor of χ; $\mathfrak{q}$ is a certain "period symbol" determined by χ and φ. This is true for both isotropic and anisotropic φ, and even for a totally definite φ. In the simplest case in which $G = G^{\eta}$, we have $\mathfrak{q} = 1$.

Clearly such a result requires the definition of $\overline{\mathbf{Q}}$-rationality of automorphic forms. If G is of type G^{η}, then we can define the $\overline{\mathbf{Q}}$-rationality by the $\overline{\mathbf{Q}}$-rationality of the Fourier coefficients of a given automorphic form. If $[K : \mathbf{Q}] = 2$, for example, then $\mathcal{H}$ is a tube domain of the form $\mathcal{H} = \{\, z \in \mathbf{C}_q^q \mid i(z^* - z) > 0 \,\}$, and a holomorphic automorphic form f has an expansion

$$f(z) = \textstyle\sum_h c(h) \exp\big(2\pi i \cdot \mathrm{tr}(hz)\big) \qquad (z \in \mathcal{H}) \tag{0.4}$$

with $c(h) \in \mathbf{C}$, where h runs over all nonnegative hermitian matrices belonging to a $\mathbf{Z}$-lattice in K_q^q. Then for a subfield M of $\mathbf{C}$ we say that f is *M-rational* if $c(h) \in M$ for every h. This definition may look simplistic, but actually it is intrinsically the right definition. To explain about this point, we first note that $\Gamma\backslash\mathfrak{Z}^{\varphi}$ has a structure of algebraic variety that has a natural model W defined over $\overline{\mathbf{Q}}$. We call then a Γ-automorphic function (that is, Γ-invariant meromorphic function on $\mathfrak{Z}^{\varphi}$ satisfying the cusp condition) $\overline{\mathbf{Q}}$-*rational* (or *arithmetic*) if it corresponds to a $\overline{\mathbf{Q}}$-rational function on W in the sense of algebraic geometry. Now there are two basic facts:

(0.5) *The value of a* $\overline{\mathbf{Q}}$-*rational automorphic function at any CM-point of* $\mathfrak{Z}^{\varphi}$, *if finite, is algebraic.*

(0.6) *If* f *and* g *are* $\overline{\mathbf{Q}}$-*rational automorphic forms of the same weight, then* f/g *is a* $\overline{\mathbf{Q}}$-*rational automorphic function.*

Here a CM-point on $\mathfrak{Z}^{\varphi}$ is defined to be the fixed point of a certain type of torus contained in G. If $G = G^{\eta}$ and $q = 1$, then $\mathcal{H}$ is the standard upper half plane, and any point of $\mathcal{H}$ belonging to an imaginary quadratic field is a CM-point and vice versa. In such a special case, (0.5) and (0.6) follow from the classical theory of complex multiplication of elliptic modular functions. In more general cases, (0.5) was established by the author in the framework of canonical models. As for (0.6), it makes sense if $G = G^{\eta}$, and we can indeed give a proof, if nontrivial, of (0.6) in such a case. For φ of a more general type, however, (0.6) is a meaningful statement only when we have defined the $\overline{\mathbf{Q}}$-rationality of automorphic forms. Thus it is one of our main tasks to define the notion so that (0.6) holds.

Turning our eyes to Eisenstein series, easily posable questions are as follows:

(i) *Assuming that* $E(z, \sigma_0)$ *is finite, is* $E(z, \sigma_0)$ *as a function of* z *holomorphic?*
(ii) *If that is so, is it* $\overline{\mathbf{Q}}$-*rational up to a well-defined constant?*

Here we take meromorphic continuation of $E(z, s)$ to the whole s-plane, as we proved in [S97], into account. Every researcher of automorphic forms should be able to accept such questions naturally, since the answers to them for $G = SL_2(\mathbf{Q})$ are well-known and fundamental. There is a marked difference between the $\overline{\mathbf{Q}}$-rationality here and the arithmeticity of $\mathcal{Z}(\sigma_0)$, since the latter concerns σ_0 in an interval which can be large, while $E(z, \sigma_0)$ can be holomorphic in z only at a single point σ_0. Now the interval, or rather the set of critical points belonging to the interval, is suggested by the functional equation for $\mathcal{Z}$, and we can find such a set even for $E(z, s)$ by means of its analytic properties. We cannot expect $E(z, \sigma_0)$ to be holomorphic in z for *every* critical point σ_0 in the set. We should also note a classical example in the elliptic modular case:

(0.7)
$$(-4\pi^2)^{-1} \lim_{s\to+0} \sum_{0\neq(c,d)\in\mathbf{Z}^2} (cz+d)^{-2}|cz+d|^{-s} = (4\pi y)^{-1} - 12^{-1} + 2\sum_{n=1}^{\infty}\Bigl(\sum_{a|n} a\Bigr)e^{2\pi i n z}.$$

This is a nonholomorphic modular form of weight 2, and there are similar nonholomorphic forms of weight $(n+3)/2$ with respect to a congruence subgroup of $Sp(n, \mathbf{Z})$. Therefore our next questions are:

(iii) *What is the analytic nature of these* $E(z, \sigma_0)$*?*
(iv) *Can we still speak of the* $\overline{\mathbf{Q}}$-*rationality of such* $E(z, \sigma_0)$*?*

One of the main purposes of this book is to answer these questions, which are not only meaningful by themselves, but also closely connected with the arithmeticity of

$\mathcal{Z}(\sigma_0)$. In fact, the answers to (iii) and (iv) are indispensable for the proof of (0.3) as we shall explain later, but first let us describe our answers.

We first define the notion of *nearly holomorphic function* on any complex manifold with a fixed Kähler structure. Without going into details in the general case, let us just say that a function on such a manifold $\mathfrak{Z}$ is called nearly holomorphic if it is a polynomial of some functions $r_1, \ldots, r_m$ on $\mathfrak{Z}$, determined by the Kähler structure, over the ring of all holomorphic functions on $\mathfrak{Z}$. If $\mathfrak{Z}$ is the above $\mathcal{H}$ of tube type with a G-invariant Kähler structure, then the r_i are the entries of $(z^* - z)^{-1}$, where z is a variable matrix on $\mathcal{H}$. If $\mathfrak{Z}$ is a hermitian symmetric space, there is also a characterization of such functions in terms of the Lie algebra of the transformation group on $\mathfrak{Z}$.

Now we can naturally define nearly holomorphic automorphic forms by replacing holomorphy by near holomorphy in the definition of automorphic forms. If $G = G^\eta$, then such a form f on $\mathcal{H}$ has an expansion

$$f(z) = \textstyle\sum_h p_h\big([\pi i(z^* - z)]^{-1}\big) \exp\big(2\pi i \cdot \mathrm{tr}(hz)\big) \qquad (z \in \mathcal{H}), \tag{0.8}$$

where $\sum_h$ is the same as in (0.4) and $p_h(Y)$ is a polynomial function in the entries of Y whose degree is less than a constant depending on f. We say that f is *M-rational* if p_h has all its coefficients in a field M for every h. For example, the function of (0.7) is a $\mathbf{Q}$-rational nearly holomorphic modular form. We shall show that $E(z, \sigma_0)$ is indeed nearly holomorphic and $\overline{\mathbf{Q}}$-rational in this sense, up to a constant, which is a power of π if $G = G^\eta$. Moreover, here is a noteworthy consequence of our definition:

(0.9) *If f and g are $\overline{\mathbf{Q}}$-rational nearly holomorphic automorphic forms of the same weight, then the value of f/g at any CM-point of $\mathcal{H}$, if finite, is algebraic.*

It should be noted that this is anything but a direct consequence of (0.6). Also, for a general type of φ we cannot use (0.8). However, once we have the $\overline{\mathbf{Q}}$-rationality of holomorphic automorphic forms, we can at least define the $\overline{\mathbf{Q}}$-rationality of nearly holomorphic automorphic forms by property (0.9), though it is of course nontrivial to show that such a definition is indeed meaningful. So far we have taken G to be unitary, but the symplectic case can be handled too; in fact it is similar to and easier than G^η, though the case of half-integral weight requires special consideration.

Having thus presented our problems in rough forms, we can now set our program as follows:

(1) We first define the $\overline{\mathbf{Q}}$-rationality of automorphic forms so that (0.6) holds.

(2) We define nearly holomorhic automorphic forms and their $\overline{\mathbf{Q}}$-rationality so that (0.9) holds.

(3) We prove the near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z, \sigma_0)$ up to a power of π in the easiest cases, namely, when G is symplectic or of type G^η, and E is defined with respect to a parabolic subgroup whose unipotent radical is a commutative group of translations on $\mathcal{H}$. Let us call such an E a series of split type.

(4) We prove (0.3) by using the result of (3).

(5) Finally we prove the near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z, \sigma_0)$ up to a well-defined constant in the most general case.

Let us now briefly describe the technical aspect of how these can be achieved. One important point is that certain differential operators on $\mathcal{H}$ are essential to (2) and (3). In the above we tacitly assumed that our automorphic forms are scalar-valued, but in order to use differential operators effectively, it is necessary to consider vector-valued forms. If $[K : \mathbf{Q}] = 2$ and $G = G(\eta_q)$, such a form is defined relative to a representation $\{\rho, X\}$ of a group

$$\mathfrak{K} = \{ (a, b) \in GL_q(\mathbf{C}) \times GL_q(\mathbf{C}) \mid \det(a) = \det(b) \},$$

where X is a finite-dimensional complex vector space and ρ is a rational representation of $\mathfrak{K}$ into $GL(X)$. Put $T = \mathbf{C}_q^q$ and view it as a global holomorphic tangent space of $\mathcal{H}_q$; define a representation $\{\rho \otimes \tau, \mathrm{Hom}(T, X)\}$ of $\mathfrak{K}$ by $[(\rho \otimes \tau)(a, b)h](u) = \rho(a, b)h({}^t aub)$ for $(a, b) \in \mathfrak{K}$, $h \in \mathrm{Hom}(T, X)$, and $u \in T$. For a function $g : \mathcal{H} \to X$ we define $\mathrm{Hom}(T, X)$-valued function Dg and $D_\rho g$ on $\mathcal{H}$ by

$$(Dg)(u) = \textstyle\sum_{i,j=1}^q u_{ij}\partial g/\partial z_{ij} \qquad (u \in T),$$
$$(D_\rho g)(z) = \rho\big(\Xi(z)\big)^{-1} D\big[\rho\big(\Xi(z)\big)g(z)\big],$$

where $z = (z_{ij})_{i,j=1}^q \in \mathcal{H}$ and $\Xi : \mathcal{H} \to \mathfrak{K}$ is defined by $\Xi(z) = \big(i(\overline{z} - {}^t z), i(z^* - z)\big)$. These can also be defined on $\mathfrak{Z}^\varphi$ for φ of a general type. Then we can show that if g is an automorphic form of weight ρ, then $D_\rho g$ is a form of weight $\rho \otimes \tau$. If $q = 1$, then $\mathcal{H}$ is the standard upper half plane, $G^\eta \cap SL_2(K) = SL_2(\mathbf{Q})$, $\mathfrak{K} = \mathbf{C}^\times$, $\rho(a) = a^k$ for $a \in \mathbf{C}^\times$ with $k \in \mathbf{Z}$, and $\Xi(z) = (2y, 2y)$ where $y = \mathrm{Im}(z)$; we can easily identify Dg with $\partial g/\partial z$, so that $D_\rho g = y^{-k}(\partial/\partial z)(y^k g)$, and $(\rho \otimes \tau)(a) = a^{k+2}$. Thus D_ρ is the well-known operator that sends a form of weight k to a form of weight $k + 2$.

Now iteration of operators of this type, such as $D_{\rho\otimes\tau} D_\rho$, produces an automorphic form with values in a representation space of $\mathfrak{K}$ of a large dimension if $q > 1$, even if we start with $X = \mathbf{C}$. Decomposing the space into irreducible subspaces and looking particularly at the irreducible subspaces of dimension one, we can define a natural differential operator Δ that sends scalar-valued automorphic forms to scalar-valued forms of increased weight. The significance of these iterated operators and Δ are explained by the following fact, which is formulated only for Δ for simplicity:

(0.10) *If Δ is of total degree p in terms of $\partial/\partial z_{ij}$, then $\pi^{-p}\Delta$ preserves near holomorphy and $\overline{\mathbf{Q}}$-rationality.*

If $G = G^\eta$, this can be derived from our definition in terms of expression (0.8). Now property (0.9), if true, would imply that for a $\overline{\mathbf{Q}}$-rational holomorphic automorphic forms f and g such that Δf and g have the same weight, the value of $(\pi^{-p}\Delta f)/g$ at any CM-point, if finite, is algebraic. This is highly nontrivial, and in fact we first prove this special case of (0.9), and derive the general case from that result.

As for problem (3), we first investigate the Fourier expansion of $E(z, s)$ of split type. In fact, this was done in [S97], but here we examine the behavior of the Fourier coefficients at a critical value of s. Employing their explicit forms, we find that $E(z, \sigma)$ is holomorphic in z and $\overline{\mathbf{Q}}$-rational, or is of the type (0.7), if the weight of E and σ belong to certain special types. For a more general weight and a general σ_0, we prove that $cE(z, \sigma_0) = \Delta E'(z, \sigma)$ with a suitable Δ, a nonzero constant c, and a suitable E' belonging to those special types. Then (0.10) settles problem (3) for $E(z, \sigma_0)$.

To treat problems (4) and (5), let us now go back to the Euler product $\mathcal{Z}(s, \mathbf{f}, \chi)$ of (0.3) on G^φ; we refer the reader to [S97] for its precise definition. We consider G^ψ with $\psi = \mathrm{diag}[\varphi, \eta]$, where η is as in (0.2). Then $G^\varphi \times G^\eta$ can be embedded in G^{ψ}, and G^{ψ} has a parabolic subgroup whose reductive factor is $G^{\varphi} \times GL_q(K)$. Given a suitable congruence subgroup Γ' of G^{ψ}, we can define an Eisenstein series $E(z,\, s;\, \mathbf{f},\, \chi)$ for $(z,\, s) \in \mathfrak{Z}^{\psi} \times \mathbf{C}$ with respect to that parabolic subgroup and the set of data $(\mathbf{f},\, \chi,\, \Gamma')$. Now we easily see that $\mathrm{diag}[\psi,\, -\varphi]$ is equivalent to η_{n+q}, so that $G^{\psi} \times G^{\varphi}$ can be embedded into $G(\eta_{n+q})$, and $\mathfrak{Z}^{\psi} \times \mathfrak{Z}^{\varphi}$ can be embedded into $\mathcal{H}_{n+q}$. Pulling back an Eisenstein series on $\mathcal{H}_{n+q}$ of split type to $\mathfrak{Z}^{\psi} \times \mathfrak{Z}^{\varphi}$, we obtain a function $H(z,\, w;\, s)$ of $(z,\, w;\, s) \in \mathfrak{Z}^{\psi} \times \mathfrak{Z}^{\varphi} \times \mathbf{C}$, with which we proved in [S97] an equality that takes the form

$$c(s)\mathcal{Z}(s,\, \mathbf{f},\, \chi)E(z,\, s;\, \mathbf{f},\, \chi) = \Lambda(s) \int_{\Gamma\backslash\mathfrak{Z}^{\varphi}} H(z,\, w;\, s)\mathbf{f}(w)\delta(w)^m \mathbf{d}w \tag{0.11}$$

in the simplest case, where Γ is a congruence subgroup of G^{φ}, c is an easy product of gamma functions, Λ is a product of some L-functions, $\mathbf{d}w$ is a G^{φ}-invariant measure on $\mathfrak{Z}^{\varphi}$, and $\delta(w)^m$ is a factor, similar to y^k in the one-dimensional case, that makes the integral meaningful. If $\psi = \varphi$, then (0.11) takes the form

$$c'(s)\mathcal{Z}(s,\, \mathbf{f},\, \chi)\mathbf{f}(z) = \Lambda'(s) \int_{\Gamma\backslash\mathfrak{Z}^{\varphi}} H'(z,\, w;\, s)\mathbf{f}(w)\delta(w)^m \mathbf{d}w. \tag{0.12}$$

We evaluate (0.11) and (0.12) at $s = \sigma_0$ for σ_0 belonging to a certain "critical set," and observe that $H(z,\, w;\, \sigma_0)$ is nearly holomorphic in $(z,\, w) \in \mathfrak{Z}^{\psi} \times \mathfrak{Z}^{\varphi}$, and even $\overline{\mathbf{Q}}$-rational up to a power of π and a factor $\mathfrak{q}$ as in (0.3). Then we can show that

$$\Lambda(\sigma_0)H(z,\, w;\, \sigma_0) = \pi^{\alpha}\mathfrak{q}\sum_i g_i(z)\overline{h_i(w)}$$

with some $\alpha \in \mathbf{Z}$, and functions g_i on $\mathfrak{Z}^{\psi}$ and h_i on $\mathfrak{Z}^{\varphi}$, which are nearly holomorphic and $\overline{\mathbf{Q}}$-rational. The same is true for $\Lambda'H'$; both g_i and h_i are defined on $\mathfrak{Z}^{\varphi}$ then. This fact applied to (0.12) produces a proportionality relation

$$\mathcal{Z}(\sigma_0,\, \mathbf{f},\, \chi) \in \pi^{\beta}\mathfrak{q}\,\langle\, \mathbf{p}',\, \mathbf{f}\,\rangle\,\overline{\mathbf{Q}}$$

with some $\beta \in \mathbf{Z}$ and a $\overline{\mathbf{Q}}$-rational nearly holomorhic $\mathbf{p}'$. Now we can show that $\mathcal{Z}(s,\, \mathbf{f},\, \chi) \neq 0$ for $\mathrm{Re}(s) > 3q/2$ if $G = G(\eta_q)$ and for $\mathrm{Re}(s) > n$ if $G = G^{\varphi}$ with φ of a general type. There is one more crucial technical fact that we can replace $\mathbf{p}'$ by a $\overline{\mathbf{Q}}$-rational holomorphic cusp form $\mathbf{p}$ that belongs to the same Hecke eigenvalues as $\mathbf{f}$. Choosing σ_0 so that $\mathcal{Z}(\sigma_0,\, \mathbf{f},\, \chi) \neq 0$, we can show that $\langle\, \mathbf{p},\, \mathbf{f}\,\rangle\,/\,\langle\, \mathbf{f},\, \mathbf{f}\,\rangle \in \overline{\mathbf{Q}}$, and eventually (0.3) for σ_0 belonging to an appropriate set. Strictly speaking, (0.12) is true only under a consistency condition on $(\mathbf{f},\, \chi)$, and the proof of (0.3) in the most general case is more complicated.

Next, we evaluate (0.11) at a critical σ_0 in a similar way, to find that

$$\mathcal{Z}(s_0,\, \mathbf{f},\, \chi)E(z,\, \sigma_0;\, \mathbf{f},\, \chi) = \pi^{\gamma}\mathfrak{q}\,\langle\, \mathbf{r},\, \mathbf{f}\,\rangle\, g(z)$$

with some $\overline{\mathbf{Q}}$-rational nearly holomorphic function g on $\mathfrak{Z}^{\psi}$ and some $\mathbf{r}$ of the same type as the above $\mathbf{p}$. Dividing this equality by $\langle\, \mathbf{f},\, \mathbf{f}\,\rangle$ and employing (0.3), we obtain the desired near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z,\, \sigma_0;\, \mathbf{f},\, \chi)$, which is the final main result of this book.

Since the title of each section can give a rough idea of its contents, we shall not describe them here for every section. However, there are some points which are not discussed in the above, and on which special attention may be paid. Let us note here some of the noteworthy aspects.

(A) As to the arithmeticity of automorphic forms, we stated only (0.6) as a basic requirement. However, there are other natural questions about arithmeticity whose answers become necessary in various applications. Let us mention here only a few facts we shall prove in this connection: (i) all automorphic forms are spanned by the $\overline{\mathbf{Q}}$-rational forms; (ii) the group action (defined relative to a fixed weight) preserves $\overline{\mathbf{Q}}$-rationality; (iii) in these statements $\overline{\mathbf{Q}}$ can be replaced by a smaller field such as $\mathbf{Q}$ or $\mathbf{Q}_{\mathrm{ab}}$ if the group and the weight are of special types.

(B) In Sections 19 through 25 we give a detailed treatment of $\mathcal{Z}(s, \mathbf{f}, \chi)$ and $E(z, s; \mathbf{f}, \chi)$ in the symplectic case, as well as in the case $G = G^{\eta}$. These cases were mentioned but not discussed in detail in the previous book [S97]. Also, in the symplectic case we can define $\mathcal{Z}$ and E even with respect to a half-integral weight, and we believe that the subject acquires the status of a complete theory only when that case is included. Therefore in this book we treat both integral and half-integral weights, and present the main results for both, though at a few points the details of the proof for a half-integral weight are referred to some papers of the author.

(C) We have spoken of a CM-point, which is naturally related to an abelian variety with complex multiplication. Thus it is necessary to view $\Gamma\backslash\mathfrak{Z}^{\varphi}$ as a space parametrizing a family of abelian varieties. This will be discussed in Sections 4 and 6. The topic was treated in [S98], but we prove here something which was not fully explained in that book. Namely, in Section 9, we prove the reciprocity-law for the value of an automorphic function at a CM-point, when $\Gamma\backslash\mathfrak{Z}^{\varphi}$ is associated with a PEL-type.

(D) In the elliptic modular case it is well-known that the space of all holomorphic modular forms is the direct sum of the space of cusp forms and the space of Eisenstein series. In Section 27 we prove several results of the same nature for symplectic and unitary groups. For example, we show that the orthogonal complement of the space of cusp forms in the space of all holomorphic automorphic forms is spanned by certain Eisenstein series, and the direct sum decomposition can be done $\overline{\mathbf{Q}}$-rationally. This will be proven for the weights with which the series are defined beyond the line of convergence.

(E) Though we are mainly interested in the higher-dimensional cases, in Section 18 we give an elementary theory of Eisenstein series in the Hilbert modular case, which leads to arithmeticity results on the critical values of an L-function of a CM-field. Also, in the Appendix we include some material of expository nature such as theta functions of a quadratic form and the estimate of the Fourier coefficients of a modular form. Many of them are well-known when the group is $SL_2(\mathbf{Q})$ or even $Sp(n, \mathbf{Q})$ for some statements, but the researchers have often had difficulties in finding references for the results on a more advanced level. Therefore we have expended conscious efforts in treating such standard topics in a rather general setting.