PREFACE

This volume is intended as an introduction to group-theoretic methods in analysis on spaces that possess certain amounts of mobility and symmetry.

The role of group theory in elementary classical analysis is a rather subdued one; the motion group of R^3 enters rather implicitly in standard vector analysis, and the conformal groups of the sphere and of the unit disk become important tools primarily after the Riemann mapping theorem has been established.

In contrast, our point of view here is to place a natural transformation group of a given space in the foreground. We use this group as a guide for the principal concepts (like that of an invariant differential operator) and as motivation for the leading problems in analysis on the space. For examples of problems that arise naturally in such a framework, we call attention here to the Eigenfunction Problems (A, B, and C in Introduction §1, No. 1, p. 2), the Radon-inversion and Orbital Integral Problems (A–D in Chapter I, §3, p. 147), and the Invariant Differential Operator Problems (A–E in Chapter II, §4, p. 277).

The present volume is intended as a textbook and a reference work on the three topics in the subtitle. Its length is caused by a rather leisurely style with many applications and digressions, sometimes in the form of exercises (with solutions).

The introductory chapter deals with the two-dimensional case, requiring only elementary methods and no Lie group theory. While this example has considerable interest in itself, it serves here to connect our main topics to classical analysis. Its primary purpose, however, is to introduce techniques and theories that generalize to semisimple Lie groups and symmetric spaces, including Harish-Chandra's c-functions (treated in generality in Chapter IV), the author's work on harmonic analysis on symmetric spaces (to be treated in generality in another volume), and group-theoretic analysis of the eigenfunctions of the Laplacian, including classical boundary value properties of harmonic functions.

In Chapter I we give an exposition of invariant integration on homogeneous spaces and relate this to the structure theory of semisimple Lie groups. We formulate general analytic integral–geometric problems (Radon transforms and orbital integrals) for a double fibration of a homogeneous space and develop their solutions in an elementary fashion for some simple classes of homogeneous spaces.

In Chapter II we discuss the effect on differential operators of a group action on a manifold. This gives rise to a separation of variables for differential operators, projections, transversal parts, orbital parts, and radial parts of differential operators; all these are useful for problems with built-in invariance conditions.

Chapter III deals with linear group action on a vector space and with the corresponding invariant polynomials and harmonic polynomials. The results, together with a description of the orbits, find analytic applications in the case of the linear isotropy representation of a symmetric space.

In Chapter IV we study the (zonal) spherical functions, that is, the K-invariant eigenfunctions of the G-invariant differential operators on the Riemannian homogeneous space G/K. Their theory, as well as that of the corresponding spherical transform, is worked out in considerable detail for the case of symmetric space (and its tangent space). Here the connection with representation theory is particularly simple and important.

In the investigation of the spherical function's behavior at ∞, Harish-Chandra's beautiful $\mathbf{c}$-function emerges. Eventually this function is expressed by the formula

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\langle i\lambda, \alpha_0 \rangle}\Gamma(\langle i\lambda, \alpha_0 \rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))},$$

the notation being explained at the end of §6. As indicated there, each detail in this formula has its own significance. In particular, the location of the singularities of $\mathbf{c}(\lambda)$ is crucial for the proof of the Paley–Wiener-type theorem for the spherical transform, which in turn enters into the proof of the corresponding inversion and Plancherel formula.

Chapter V deals with harmonic analysis on compact homogeneous spaces U/K. Because of the intimate connection with finite-dimensional representation theory for U, the chapter begins with a detailed exposition of the weight theory and the character theory for U.

Since this book is in part intended as a textbook, we now give some description of its level. The first third of the book is introductory and has on occasion been used as a textbook for first-year graduate students without background in Lie group theory. The remainder of the book requires some standard functional analysis results. The Lie-theoretic tools needed can for example be found in my book "Differential Geometry, Lie Groups, and Symmetric Spaces" (abbreviated [DS]), of which the present book can be considered an "analytic continuation." Thus, our aim has been to provide complete proofs of all the results in the book. Although this process of unification and consolidation has at times led to some simplifications of proofs, we have in the exposition been more concerned with clarity than brevity.

Each chapter begins with a short summary and ends with historical notes giving references to source material: an effort has been made to give appropriate credit to authors of individual results. At the same time we have tried to make these notes reflect the fact that the logical order of the exposition often differs drastically from the order of the historical development.

Much of the material in this book has been the subject of lectures at the Massachusetts Institute of Technology in recent years; some of the content of my lecture notes [1980c, 1981] has been incorporated in the Introduction and in Chapter I. Parts of the manuscript have been read and commented on by M. Baum, M. Cowling, M. Flensted-Jensen, F. Gonzales, A. G. Helmink, G. F. Helmink, B. Hoogenboom, A. Korányi, M. Mazzarello, J. Orloff, F. Richter, H. Schlichtkrull, and G. Travaglini. I am particularly indebted to T. Koornwinder for his numerous useful suggestions.