Introduction

Model categories and their homotopy categories

A *model category* is Quillen's axiomatization of a place in which you can "do homotopy theory" [**52**]. Homotopy theory often involves treating homotopic maps as though they were the same map, but a homotopy relation on maps is not the starting point for abstract homotopy theory. Instead, homotopy theory comes from choosing a class of maps, called *weak equivalences*, and studying the passage to the *homotopy category*, which is the category obtained by localizing with respect to the weak equivalences, i.e., by making the weak equivalences into isomorphisms (see Definition 8.3.2). A model category is a category together with a class of maps called *weak equivalences* plus two other classes of maps (called *cofibrations* and *fibrations*) satisfying five axioms (see Definition 7.1.3). The cofibrations and fibrations of a model category allow for lifting and extending maps as needed to study the passage to the homotopy category.

The homotopy category of a model category. Homotopy theory originated in the category of topological spaces, which has unusually good technical properties. In this category, the homotopy relation on the set of maps between two objects is always an equivalence relation, and composition of homotopy classes is well defined. In the classical homotopy theory of topological spaces, the passage to the homotopy category was often described as "replacing maps with homotopy classes of maps". Most work was with CW-complexes, though, and whenever a construction led to a space that was not a CW-complex the space was replaced by a weakly equivalent one that was. Thus, weakly equivalent spaces were recognized as somehow "equivalent", even if that equivalence was never made explicit. If instead of starting with a homotopy relation we explicitly cause weak equivalences to become isomorphisms, then homotopic maps do become the same map (see Lemma 8.3.4) and in addition a cell complex weakly equivalent to a space becomes isomorphic to that space, which would not be true if we were simply replacing maps with homotopy classes of maps.

In most model categories, the homotopy relation does not have the good properties that it has in the category of topological spaces unless you restrict yourself to the subcategory of cofibrant-fibrant objects (see Definition 7.1.5). There are actually two different homotopy relations on the set of maps between two objects X and Y: *Left homotopy*, defined using cylinder objects for X, and *right homotopy*, defined using path objects for Y (see Definition 7.3.2). For arbitrary objects X and Y these are different relations, and neither of them is an equivalence relation. However, for cofibrant-fibrant objects, the two homotopy relations are the same, they are equivalence relations, and composition of homotopy classes is well defined (see Theorem 7.4.9 and Theorem 7.5.5). Every object of a model category is weakly equivalent to a cofibrant-fibrant object, and we could thus define a "homotopy category of cofibrant-fibrant objects" by taking the cofibrant-fibrant objects of the model category as our objects and homotopy classes of maps as our morphisms. Since a map between cofibrant-fibrant objects is a weak equivalence if and only if it is a homotopy equivalence (see Theorem 7.5.10 and Theorem 7.8.5), this would send weak equivalences to isomorphisms, and we define the *classical homotopy category* of a model category in exactly this way (see Definition 7.5.8).

The classical homotopy category is inadequate, though, because most work in homotopy theory requires constructions that create objects that may not be cofibrant-fibrant, even if we start out with only cofibrant-fibrant objects. Thus, we need a "homotopy category" containing *all* of the objects of the model category. We define the *Quillen homotopy category* of a model category to be the localization of the category with respect to the class of weak equivalences (see Definition 8.3.2). For the class of weak equivalences of a model category, this always exists (see Remark 8.3.3 and Theorem 8.3.5). Thus, the Quillen homotopy category of a model category contains all of the objects of the model category. The classical homotopy category is a subcategory of the Quillen homotopy category, and the inclusion of the classical homotopy category in the Quillen homotopy category is an equivalence of categories (see Theorem 8.3.9). We refer to the Quillen homotopy category as simply the *homotopy category*.

Homotopy function complexes. Homotopy theory involves the construction of more than just a homotopy category. Dwyer and Kan [**31, 32, 33**] construct the *simplicial localization* of a category with respect to a class of weak equivalences as the derived functor of the functor that constructs the homotopy category. This is a *simplicial category*, i.e., a category enriched over simplicial sets, and so for each pair of objects there is a simplicial set that is the "function complex" of maps between the objects. These function complexes capture the "higher order structure" of the homotopy theory, and taking the set of components of the function complex of maps between two objects yields the set of maps in the homotopy category between those objects.

Dwyer and Kan show that if you start with a *model category*, then simplicial sets weakly equivalent to those function complexes can be constructed using cosimplicial or simplicial resolutions (see Definition 16.1.2) in the model category. We present a self-contained development of these *homotopy function complexes* (see Chapter 17). Constructing homotopy function complexes requires making an arbitrary choice of resolutions, but we show that the category of possible choices has a contractible classifying space (see Theorem 17.5.28), and so there is a distinguished homotopy class of homotopy equivalences between the homotopy function complexes resulting from different choices (see Theorem 17.5.29 and Theorem 17.5.30).

Homotopy theory in model categories. Part 2 of this book studies model categories and techniques of homotopy theory in model categories. Part 2 is intended as a reference, and it logically precedes Part 1. We cover quite a bit of ground, but the topics discussed in Part 2 are only those that are needed for the discussion of localization in Part 1, fleshed out to give a reasonably complete development. We begin Part 2 with the definition of a model category and with the basic results that are by now standard (see, e.g., [**52, 54, 14, 35**]), but we give complete arguments in an attempt to make this accessible to the novice. For a more complete description of the contents of Part 2, see the summary on page 103 and the introductions to the individual chapters. For a description of Part 1, which discusses localizing model category structures, see below, as well as the summary on page 3.

Prerequisites. The category of simplicial sets plays a central role in the homotopy theory of a model category, even for model categories unrelated to simplicial sets. This is because a homotopy function complex between objects in a model category is a simplicial set (see Chapter 17). Thus, we assume that the reader has some familiarity with the homotopy theory of simplicial sets. For readers without the necessary background, we recommend the works by Curtis [**18**], Goerss and Jardine [**39**], and May [**49**].

Localizing model category structures

Localizing a model category with respect to a class of maps does not mean making the maps into isomorphisms; instead, it means making the images of those maps in the homotopy category into isomorphisms (see Definition 3.1.1). Since the image of a map in the homotopy category is an isomorphism if and only if the map is a weak equivalence (see Theorem 8.3.10), localizing a model category with respect to a class of maps means making maps into weak equivalences.

Localized model category structures originated in Bousfield's work on localization with respect to homology ([**8**]). Given a homology theory h_*, Bousfield established a model category structure on the category of simplicial sets in which the weak equivalences were the maps that induced isomorphisms of all homology groups. A space (i.e., a simplicial set) W was defined to be *local* with respect to h_* if it was a Kan complex such that every map $f\colon X \to Y$ that induced isomorphisms $f_*\colon h_*X \approx h_*Y$ of homology groups also induced an isomorphism $f^*\colon \pi(Y,W) \approx \pi(X,W)$ of the sets of homotopy classes of maps to W. In Bousfield's model category structure, a space was fibrant if and only if it was local with respect to h_*.

The problem that led to Bousfield's model category structure was that of constructing a *localization functor* for a homology theory. That is, given a homology theory h_*, the problem was to define for each space X a local space $\mathrm{L}_{h_*}X$ and a natural homology equivalence $X \to \mathrm{L}_{h_*}X$. There had been a number of partial solutions to this problem (perhaps the most complete being that of Bousfield and Kan [**14**]), but each of these was valid only for some special class of spaces, and only for certain homology theories. In [**8**], Bousfield constructed a functorial h_*-localization for an arbitrary homology theory h_* and for every simplicial set. In Bousfield's model category structure, a fibrant approximation to a space (i.e., a weak equivalence from a space to a fibrant space) was exactly a localization of that space with respect to h_*.

Some years later, Bousfield [**9, 10, 11, 12**] and Dror Farjoun [**20, 22, 24**] independently considered the notion of localizing spaces with respect to an arbitrary map, with a definition of "local" slightly different from that used in [**8**]: Given a map of spaces $f\colon A \to B$, a space W was defined to be f*-local* if f induced a weak equivalence of mapping spaces $f^*\colon \mathrm{Map}(B,W) \cong \mathrm{Map}(A,W)$ (rather than just a bijection on components, i.e., an isomorphism of the sets of homotopy classes of maps), and a map $g\colon X \to Y$ was defined to be an f*-local equivalence* if for every f-local space W the induced map of mapping spaces $g^*\colon \mathrm{Map}(Y,W) \to \mathrm{Map}(X,W)$

was a weak equivalence. An *f-localization* of a space X was then an f-local space $\mathrm{L}_f X$ together with an f-local equivalence $X \to \mathrm{L}_f X$. Bousfield and Dror Farjoun constructed f-localization functors for an arbitrary map f of spaces.

Given a map $f\colon A \to B$ of spaces, we construct in Chapters 1 and 2 an f-local model category structure on the category of spaces. That is, we construct a model category structure on the category of spaces in which the weak equivalences are the f-local equivalences, and in which an f-localization functor is a fibrant approximation functor for the f-local model category. In Chapter 4 we extend this to establish S-local model category structures for an arbitrary set S of maps in a left proper (see Definition 13.1.1) *cellular model category* (see page xiii and Chapter 12).

Constructing the localized model category structure. Once we've established the localized model category structure, a localization of an object in the category will be exactly a fibrant approximation to that object in the localized model category, but it turns out that we must first define a natural localization of every object in order to establish the localized model category structure. The reason for this is that we use the localization functor to identify the local equivalences: A map is a local equivalence if and only if its localization is a weak equivalence (see Theorem 3.2.18).

The model categories with which we work are all *cofibrantly generated* model categories (see Definition 11.1.2). That is, there is a set I of cofibrations and a set J of trivial cofibrations such that

- a map is a trivial fibration if and only if it has the right lifting property with respect to every element of I,
- a map is a fibration if and only if it has the right lifting property with respect to every element of J, and
- both of the sets I and J *permit the small object argument* (see Definition 10.5.15).

For example, in the category Top of (unpointed) topological spaces (see Notation 1.1.4), we can take for I the set of inclusions $\mathrm{S}^{n-1} \to \mathrm{D}^n$ for $n \geq 0$ and for J the set of inclusions $\bigl|\Lambda[n,k]\bigr| \to \bigl|\Delta[n]\bigr|$ for $n > 0$ and $0 \leq k \leq n$. The left Bousfield localization $\mathrm{L}_f\mathrm{Top}$ of Top with respect to a map f in Top (see Definition 3.3.1) will have the same class of cofibrations as the standard model category structure on Top, and so the set I of generating cofibrations for Top can serve as a set of generating cofibrations for $\mathrm{L}_f\mathrm{Top}$. The difficulty lies in finding a set J_f of generating trivial cofibrations for $\mathrm{L}_f\mathrm{Top}$.

A first thought might be to let J_f be the collection of all cofibrations that are f-local equivalences, since the fibrations of $\mathrm{L}_f\mathrm{Top}$ are defined to be the maps with the right lifting property with respect to all such maps, but then J_f would not be a set. The problem is to find a subcollection J_f of the class of *all* cofibrations that are f-local equivalences such that

- a map has the right lifting property with respect to every element of J_f if and only if it has the right lifting property with respect to every cofibration that is an f-local equivalence, and
- the collection J_f forms a set.

That is the problem that is solved by the Bousfield-Smith cardinality argument.

The Bousfield-Smith cardinality argument. Every map in Top has a cofibrant approximation (see Definition 8.1.22) that is moreover an inclusion of cell complexes (see Definition 10.7.1 and Proposition 11.2.8). Since Top is left proper (see Definition 13.1.1), this implies that for a map to have the right lifting property with respect to all cofibrations that are f-local equivalences, it is sufficient that it have the right lifting property with respect to all inclusions of cell complexes that are f-local equivalences (see Proposition 13.2.1).

If we choose a fixed cardinal γ, then the collection of homeomorphism classes of cell complexes of size no larger than γ forms a set. The cardinality argument shows that there exists a cardinal γ such that a map has the right lifting property with respect to all inclusions of cell complexes that are f-local equivalences if and only if it has the right lifting property with respect to all such inclusions between cell complexes of size no larger than γ. Thus, we can take as our set J_f a set of representatives of the isomorphism classes of such "small enough" inclusions of cell complexes.

Our localization functor L_f is defined by choosing a set of inclusions of cell complexes $\overline{\Lambda\{f\}}$ and then attaching the codomains of the elements of $\overline{\Lambda\{f\}}$ to a space by all possible maps from the domains of the elements of $\overline{\Lambda\{f\}}$, and then repeating this an infinite number of times (see Section 1.3). In order to make the cardinality argument, we need to find a cardinal γ such that

(1) if X is a cell complex, then every subcomplex of its localization $\mathrm{L}_f X$ of size at most γ is contained in the localization of a subcomplex of X of size at most γ, and
(2) if X is a cell complex of size at most γ, then $\mathrm{L}_f X$ is also of size at most γ.

We are able to do this because

(1) every map from a closed cell to a cell complex factors through a finite subcomplex of the cell complex (see Corollary 10.7.7), and
(2) given two cell complexes, there is an upper bound on the cardinal of the set of continuous maps between them, and this upper bound depends only on the size of the cell complexes

(see Section 2.3).

Cellular model categories. Suppose now that $\mathcal{M}$ is a cofibrantly generated model category and that we wish to localize $\mathcal{M}$ with respect to a set S of maps in $\mathcal{M}$ (see Definition 3.3.1). If I is a set of generating cofibrations for $\mathcal{M}$, then

- we define a *relative cell complex* to be a map built by repeatedly attaching codomains of elements of I along maps of their domains (see Definition 10.5.8),
- we define a *cell complex* to be the codomain of a relative cell complex whose domain is the initial object of $\mathcal{M}$, and
- we define an *inclusion of cell complexes* to be a relative cell complex whose domain is a cell complex.

(If $\mathcal{M} = \mathrm{Top}$, the category of topological spaces, our set I of generating cofibrations is the set of inclusions $\mathrm{S}^{n-1} \to \mathrm{D}^n$ for $n \geq 0$, and so a cell complex is a space built by repeatedly attaching disks along maps of their boundary spheres.) In such a model category, every map has a cofibrant approximation (see Definition 8.1.22) that is an inclusion of cell complexes (see Proposition 11.2.8). Thus, if we assume that $\mathcal{M}$ is left proper (see Definition 13.1.1), then for a map to have the right lifting property with respect to all cofibrations that are S-local equivalences, it is sufficient that it have the right lifting property with respect to all inclusions of cell complexes that are S-local equivalences (see Proposition 13.2.1). In order to make the cardinality argument, though, we need to assume that maps between cell complexes in $\mathcal{M}$ are sufficiently well behaved; this leads us to the definition of a *cellular model category* (see Definition 12.1.1).

A cellular model category is a cofibrantly generated model category with additional properties that ensure that

- the intersection of a pair of subcomplexes (see Definition 12.2.5) of a cell complex exists (see Theorem 12.2.6),
- there is a cardinal σ (called the *size of the cells* of $\mathcal{M}$; see Definition 12.3.3) such that if X is a cell complex of size τ, then any map from X to a cell complex factors through a subcomplex of size at most $\sigma\tau$ (see Theorem 12.3.1), and
- if X is a cell complex, then there is a cardinal η such that if Y is a cell complex of size ν ($\nu \geq 2$), then the set $\mathcal{M}(X, Y)$ has cardinal at most ν^η (see Proposition 12.5.1).

Fortunately, these properties follow from a rather minimal set of conditions on the model category $\mathcal{M}$ (see Definition 12.1.1), satisfied by almost all model categories that come up in practice.

Left localization and right localization. There are two types of morphisms of model categories: *Left Quillen functors* and *right Quillen functors* (see Definition 8.5.2). The localizations that we have been discussing are all *left localizations*, because the functor from the original model category to the localized model category is a left Quillen functor that is initial among left Quillen functors whose total left derived functor takes the images of the designated maps into isomorphisms in the homotopy category (see Definition 3.1.1). There is an analogous notion of *right localization.*

Given a CW-complex A, Dror Farjoun [**20, 21, 23, 24**] defines a map of topological spaces $f\colon X \to Y$ to be an *A-cellular equivalence* if the induced map of function spaces $f_*\colon \mathrm{Map}(A, X) \to \mathrm{Map}(A, Y)$ is a weak equivalence. He also defines the class of *A-cellular spaces* to be the smallest class of cofibrant spaces that contains A and is closed under weak equivalences and homotopy colimits. We show in Theorem 5.1.1 and Theorem 5.1.6 that this is an example of a *right localization*, i.e., that there is a model category structure in which the weak equivalences are the A-cellular equivalences and in which the cofibrant objects are the A-cellular spaces. In fact, we do this for an arbitrary right proper cellular model category (see Theorem 5.1.1 and Theorem 5.1.6).

The situation here is not as satisfying as it is for left localizations, though. The left localizations that we construct for left proper cellular model categories are again left proper cellular model categories (see Theorem 4.1.1), but the right localizations that we construct for right proper cellular model categories need not even be cofibrantly generated if not every object of the model category is fibrant. However, if every object is fibrant, then a right localization will again be right proper cellular with every object fibrant; see Theorem 5.1.1.

Acknowledgments

I am grateful to the M.I.T. Mathematics Department for putting up with me over all these years, and making me always feel welcome. Although there has been a gradual change in the group of topologists there since I was a graduate student, what has been constant is the warm, supportive attitude that has allowed me always to think of it as home.

This project grew out of my reading an early draft of Emmanuel Dror Farjoun's book [**24**]; I am grateful to him for his preprints and helpful communications.

Drafts of this book been have been circulating for a number of years, originally under the title *Localization, Cellularization, and Homotopy Colimits* and more recently as *Localization of Model Categories*, and a number of people have provided helpful comments. Paul Goerss helped correct a problem with the proof of the existence of the localization model category structure for cellular model categories. Garth Warner gave several early drafts an amazingly careful review; he found many obscurities and outright errors, and provided a level of analysis that usually takes several years and a group of reviewers. Daniel Biss, Dan Christensen, Dan Dugger, Bill Dwyer, Lars Hesselholt, Mike Hopkins, Mark Hovey, Dan Isaksen, Haynes Miller, George Peschke, Charles Rezk, and Jeff Smith offered corrections and helpful suggestions. Of course, any mistakes or omissions that remain are entirely my responsibility.

Above all, though, I owe a great deal of thanks to Dan Kan. He is responsible for my first becoming interested in this subject, and I am indebted to him for countless helpful conversations and suggestions. His view of mathematics has been a lasting influence upon me, and it is a pleasure to thank him for his support and encouragement over the years.