

Preface

The task of the educator is to make the child's spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.

—Henri Poincaré

This course of analysis is radical; it returns to the roots of the subject. It is not a history of analysis. It is rather an attempt to follow the injunction of Henri Poincaré to let history inform pedagogy. It is designed to be a first encounter with real analysis, laying out its context and motivation in terms of the transition from power series to those that are less predictable, especially Fourier series, and marking some of the traps into which even great mathematicians have fallen.

This is also an abrupt departure from the standard format and syllabus of analysis. The traditional course begins with a discussion of properties of the real numbers, moves on to continuity, then differentiability, integrability, sequences, and finally infinite series, culminating in a rigorous proof of the properties of Taylor series and perhaps even Fourier series. This is the right way to view analysis, but it is not the right way to teach it. It supplies little motivation for the early definitions and theorems. Careful definitions mean nothing until the drawbacks of the geometric and intuitive understandings of continuity, limits, and series are fully exposed. For this reason, the first part of this book follows the historical progression and moves backwards. It starts with infinite series, illustrating the great successes that led the early pioneers onward, as well as the obstacles that stymied even such luminaries as Euler and Lagrange.

There is an intentional emphasis on the mistakes that have been made. These highlight difficult conceptual points. That Cauchy had so much trouble proving the mean value theorem or coming to terms with the notion of uniform convergence should alert us to the fact that these ideas are not easily assimilated. The student needs time with them. The highly refined proofs that we know today leave the mistaken impression that the road of discovery in mathematics is straight and sure. It is not. Experimentation and misunderstanding have been essential components in the growth of mathematics.

Exploration is an essential component of this course. To facilitate graphical and numerical investigations, *Mathematica* and *Maple* commands and programs as well as investigative projects are available on a dedicated website at www.macalester.edu/aratra.

The topics considered in this book revolve around the questions raised by Fourier's trigonometric series and the restructuring of calculus that occurred in the process of answering them. Chapter 1 is an introduction to Fourier series: why they are important and why they met with so much resistance. This chapter presupposes familiarity with partial differential equations, but it is purely motivational and can be given as much or as little emphasis as one wishes. Chapter 2 looks at the background to the crisis of 1807. We investigate the difficulties and dangers of working with infinite summations, but also the insights and advances that they make possible. More of these insights and advances are given in Appendix A. Calculus would not have revolutionized mathematics as it did if it had not been coupled with infinite series. Beginning with Newton's *Principia*, the physical applications of calculus rely heavily on infinite sums. The chapter concludes with a closer look at the understandings of late eighteenth century mathematicians: how they saw what they were doing and how they justified it. Many of these understandings stood directly in the way of the acceptance of trigonometric series.

In Chapter 3, we begin to find answers to the questions raised by Fourier's series. We follow the efforts of Augustin Louis Cauchy in the 1820s to create a new foundation to the calculus. A careful definition of differentiability comes first, but its application to many of the important questions of the time requires the mean value theorem. Cauchy struggled—unsuccessfully—to prove this theorem. Out of his struggle, an appreciation for the nature of continuity emerges.

We return in Chapter 4 to infinite series and investigate the question of convergence. Carl Friedrich Gauss plays an important role through his complete characterization of convergence for the most important class of power series: the hypergeometric series. This chapter concludes with a verification that the Fourier cosine series studied in the first chapter does, in fact, converge at every value of x .

The strange behavior of infinite sums of functions is finally tackled in Chapter 5. We look at Dirichlet's insights into the problems associated with grouping and rearranging infinite series. We watch Cauchy as he wrestles with the problem of the discontinuity of an infinite sum of continuous functions, and we discover the key that he was missing. We begin to answer the question of when it is legitimate to differentiate or integrate an infinite series by differentiating or integrating each summand.

Our story culminates in Chapter 6 where we present Dirichlet's proof of the validity of Fourier series representations for all "well behaved" functions. Here for the first time we encounter serious questions about the nature and meaning of the integral. A gap remains in Dirichlet's proof which can only be bridged after we have taken a closer look at integration, first using Cauchy's definition, and then arriving at Riemann's definition. We conclude with Weierstrass's observation that Fourier series are indeed strange creatures. The function represented by the series

$$\cos(\pi x) + \frac{1}{2} \cos(13\pi x) + \frac{1}{4} \cos(169\pi x) + \frac{1}{8} \cos(2197\pi x) + \dots$$

converges and is continuous at every value of x , but it is never differentiable.

The material presented within this book is not of uniform difficulty. There are computational inquiries that should engage all students and refined arguments that will challenge the best. My intention is that every student in the classroom and each individual reader striking out alone should be able to read through this book and come away with an understanding of analysis. At the same time, they should be able to return to explore certain topics in greater depth.

Historical Observations

In the course of writing this book, unexpected images have emerged. I was surprised to see Peter Gustav Lejeune Dirichlet and Niels Henrik Abel reveal themselves as the central figures of the transformation of analysis that fits into the years from 1807 through 1872. While Cauchy is associated with the great theorems and ideas that launched this transformation, one cannot read his work without agreeing with Abel's judgement that "what he is doing is excellent, but very confusing." Cauchy's seminal ideas required two and a half decades of gestation before anyone could begin to see what was truly important and why it was important, where Cauchy was right, and where he had fallen short of achieving his goals.

That gestation began in the fall of 1826 when two young men in their early 20s, Gustav Dirichlet and Niels Henrik Abel, met to discuss and work out the implications of what they had heard and read from Cauchy himself. Dirichlet and Abel were not alone in this undertaking, but they were of the right age to latch onto it. It would become a recurring theme throughout their careers. By the 1850s, the stage was set for a new generation of bright young mathematicians to sort out the confusion and solidify this new vision for mathematics. Riemann and Weierstrass were to lead this generation. Dirichlet joined Gauss as teacher and mentor to Riemann. Abel died young, but his writings became Weierstrass's inspiration.

It was another twenty years before the vision that Riemann and Weierstrass had grasped became the currency of mathematics. In the early 1870s, the general mathematical community finally understood and accepted this new analysis. A revolution had taken place. It was not an overthrow of the old mathematics. No mathematical truths were discredited. But the questions that mathematicians would ask and the answers they would accept had changed in a fundamental way. An era of unprecedented power and possibility had opened.


Changes to the Second Edition

This second edition incorporates many changes, all with the aim of aiding students who are learning real analysis. The greatest conceptual change is in Chapter 2 where I clarify that the Archimedean understanding of infinite series *is* the approach that Cauchy and the mathematical community has adopted. While this chapter still has a free-wheeling style in its use of infinite series—the intent being to convey the power and importance of infinite series—it also begins to introduce rigorous justification of convergence. A new section devoted entirely to geometric series has been added. Chapter 4, which introduces tests of convergence, has been reorganized.

I have also trimmed some of the digressions that I found led students to lose sight of my intent. In particular, the section on the Newton–Raphson method and the proof of Gauss’s test for convergence of hypergeometric series have been taken out of the text. Because I feel that this material is still important, though not central, these sections and much more are available on the web site dedicated to this book.

Web Resource: When you see this box with the designation “Web Resource”, more information is available in a pdf file, *Mathematica* notebook, or *Maple* worksheet that can be downloaded at www.macalester.edu/aratra. The box is also used to point to additional information available in **Appendix A**.

I have added many new exercises, including many taken from *Problems in Mathematical Analysis* by Kaczor and Nowak. Problems taken from this book are identified in Appendix C. I wish to acknowledge my debt to Kaczor and Nowak for pulling together a beautiful collection of challenging problems in analysis. Neither they nor I claim that they are the original source for all of these problems.

All code for *Mathematica* and *Maple* has been removed from the text to the website. Exercises for which these codes are available are marked with the symbol . The appendix with selected solutions has been replaced by a more extensive appendix of hints.

I considered adding a new chapter on the structure of the real numbers. Ultimately, I decided against it. That part of the story properly belongs to the second half of the nineteenth century when the progress described in this book led to a thorough reappraisal of integration. To everyone’s surprise this was not possible without a full understanding of the real numbers which were to reveal themselves as far more complex than had been thought. That is an entirely other story that will be told in another book, *A Radical Approach to Lebesgue’s Theory of Integration*.

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