Contents

To the Student xiii
To the Instructor xvii

0 Paradoxes? 1

1 Logical Foundations 5

2 Proof, and the Natural Numbers 17

3 The Integers, and the Ordered Field of Rational Numbers 27

4 Induction and Well-Ordering 37

5 Sets 45

6 Functions 57

7 Inverse Functions 71

8 Some Subsets of the Real Numbers 79

9 The Rational Numbers Are Denumerable 87

10 The Uncountability of the Real Numbers 93

11 The Infinite 97

12 The Complete, Ordered Field of Real Numbers 111

13 Further Properties of Real Numbers 121

14 Cluster Points and Related Concepts 125

15 The Triangle Inequality **131**

16 Infinite Sequences **135**

17 Limits of Sequences **141**

18 Divergence: The Non-Existence of a Limit **149**

19 Four Great Theorems in Real Analysis **157**

20 Limit Theorems for Sequences **167**

21 Cauchy Sequences and the Cauchy Convergence Criterion **175**

22 The Limit Superior and Limit Inferior of a Sequence **181**

23 Limits of Functions **187**

24 Continuity and Discontinuity **201**

25 The Sequential Criterion for Continuity **213**

26 Theorems About Continuous Functions **219**

27 Uniform Continuity **227**

28 Infinite Series of Constants **237**

29 Series with Positive Terms **251**

30 Further Tests for Series with Positive Terms **263**

31 Series with Negative Terms **273**

32 Rearrangements of Series **283**

33 Products of Series **291**

34 The Numbers e and γ **303**

35 The Functions $\exp x$ and $\ln x$ **313**

36 The Derivative **319**

37 Theorems for Derivatives **331**

38 Other Derivatives **341**

39 The Mean Value Theorem **351**

40 Taylor's Theorem **359**

41 Infinite Sequences of Functions **367**

42 Infinite Series of Functions **377**

43 Power Series **389**

44 Operations with Power Series **399**

45 Taylor Series **415**

46 Taylor Series, Part II **423**

47 The Riemann Integral **433**

48 The Riemann Integral, Part II **449**

49 The Fundamental Theorem of Integral Calculus **461**

50 Improper Integrals **475**

51 The Cauchy-Schwarz and Minkowski Inequalities **485**

52 Metric Spaces **489**

53 Functions and Limits in Metric Spaces **501**

54 Some Topology of the Real Number Line **509**

55 The Cantor Ternary Set **517**

Appendix A Farey Sequences **527**

Appendix B Proving that $\sum_{k=0}^{n} \frac{1}{k!} < (1 + \frac{1}{n})^{n+1}$ **531**

Appendix C The Ruler Function Is Riemann Integrable **535**

Appendix D Continued Fractions **539**

Appendix E L'Hospital's Rule **545**

Appendix F Symbols, and the Greek Alphabet **555**

Annotated Bibliography **557**
Solutions to Odd-Numbered Exercises **561**
Index **655**