Introduction

Welcome to *Linear Algebra and Geometry*. You probably have an idea about the "Geometry" in the title, but what about "Linear Algebra"?

It's not so easy to explain what linear algebra is about until you've done some of it. Here's an attempt:

You may have studied some of these topics in previous courses:

- Solving systems of two linear equations in two unknowns and systems of three linear equations in three unknowns
- Using matrices to solve systems of equations
- Using matrices and matrix algebra for other purposes
- Using coordinates or vectors to help with geometry
- Solving systems of equations with determinants
- Working with reflections, rotations, and translations

Linear algebra ties all these ideas together and makes connections among them.

And it does much more than this. Much of high school algebra is about developing tools for solving equations and analyzing functions. The equations and functions usually involve one, or maybe two, variables. Linear algebra develops tools for solving equations and analyzing functions that involve *many* variables, all at once. For example, you'll learn how to find the equation of a plane in space and how to get formulas for rotations about a point in space (these involve three variables). You'll learn how to analyze systems of linear equations in many variables and work with matrices of any size—many applications involve matrices with thousands or even millions of entries.

Here are some quotes from two people who use linear algebra in their professions:

> *Linear algebra is a powerful tool in finance. Innovations are often developed within the world's most sophisticated financial firms by those fluent in the language of vectors and matrices.*

> *Linear algebra is not only a valuable tool in its own right, but it also facilitates the applications of tools in multivariate calculus and multivariate statistics to many problems in finance that involve risks and decisions in multiple dimensions. Studying linear algebra first, before those subjects, puts one in the strongest position to grasp and exploit their insights.*
>
> — Robert Stambaugh,
> Professor of Finance
> Wharton School

> *Students who will take such a course have probably had the equivalent of two years of algebra and a year of geometry, at least if they come from a fairly standard program. They will have seen some analytic geometry, but not enough to give them much confidence in the relationship between algebraic and geometric thinking in the plane, and even less in three-space. Linear algebra can bring those subjects together in ways that reinforce both. That is a goal for all students, whether or not they have taken calculus, and it can form a viable alternative to calculus in high school. I would love to have students in a first-year course in calculus who already had thought deeply about the relationships between algebra and geometry.*
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> — Thomas Banchoff,
> Professor of Mathematics
> Brown University

⟵ Dr. Banchoff is one of the core consultants to this book.

When you finish the core program (Chapters 1–5), you'll have the language, the tools, and the habits of mind necessary to understand many questions in advanced mathematics and its applications to science, engineering, computer science, and finance.

It takes some time, effort, and practice to develop these skills.

- The **language** of linear algebra speaks about two kinds of mathematical objects—*vectors* and *matrices*—as well as special functions—*linear mappings*—defined on these objects. One of the core skills in the language of linear algebra is to learn how to use geometric and algebraic images interchangeably. For example, you'll refer to the set of solutions to the equation $x - 2y + 3x + w = 0$ as a "hyperplane in four dimensions."

⟵ If you don't know what a vector or matrix is, don't worry—you soon will.

- The **tools** of linear algebra involve developing a new kind of algebraic skill—you'll be calculating with vectors and matrices, solving equations, and learning about algorithms that carry out certain processes. You may have met matrix multiplication or matrix row reduction in other courses. These are examples of the kinds of tools you'll learn about in this course.

⟵ If you don't know about these operations, don't worry—you soon will.

- The **habits of mind** in linear algebra are the most important things for you to develop. These involve being able to imagine a calculation—with matrices, say—without having to carry it out, making use of a general kind of distributive law (that works with vectors and matrices), and extending an operation from a small set to a big set by preserving the rules for calculating. An example of the kind of mathematical thinking that's important in linear algebra is the ability to analyze the following question *without finding two points on the graph*:

> If (a, b) and (c, d) are points on the graph of $3x + 5y = 7$, is $(a + c, b + d)$ on the graph?

Mathematical habits are just that—habits. And they take time to develop. The best way to develop these habits is to work carefully through all the problems.

This book contains many problems, more than in most courses. That's for a reason. All the main results and methods in this book come from generalizing numerical examples. So, a problem set that looks like an extensive list of calculations is there because, if you carefully work through the calculations and ask yourself what's common among them, a general result (*and* its proof) will be sitting right in front of you.

The authors of this book took care never to include extraneous problems. Usually, the problems build on each other like the stories of a tower, so that you can climb to the top a little at a time and then see something of the whole landscape. Many of these problem sets have evolved over several decades of use in high school classrooms, gradually polished every year and influenced by input from a couple of generations of students.

This is all to say that linear algebra is an important, useful, and beautiful area of mathematics, and it's a subject at which you can become very good by working the problems—and analyzing your work—in the chapters ahead.

Before you start, the authors of this program have some advice for you:

$\longleftarrow$ The authors include teachers, mathematicians, education professionals, and students; most of them fit into more than one of these categories.

> *The best way to understand mathematics is to work really hard on the problems.*

If you work through these problems carefully, you'll never wonder why a new fact is true; you'll know because you discovered the fact for yourself. Theorems in linear algebra spring from calculations, and the problem sets ask you to do lots of calculations that highlight these theorems.

The sections themselves provide examples and ideas about the ways people think about the mathematics in the chapters. They are designed to give you a reference, but they probably won't be as complete as the classroom discussions you'll have with your classmates and your teacher. In other words, you still have to pay attention in class.

But you'll have to pay attention a lot less if you do these problems carefully. That's because many of the problems are previews of coming attractions, so doing them and looking for new ideas will mean fewer surprises when new ideas are presented in class.

This approach to learning has been evolving for more than 40 years—many students have learned, *really* learned, linear algebra by working through these problems. You are cordially invited to join them.