CHAPTER

1 Points and Vectors

One of the real breakthroughs in mathematics happened when people realized that algebra could be joined with geometry. By setting up a coordinate system and assigning coordinates to points, mathematicians were able to describe geometric phenomena with algebraic equations and formulas.

This process allowed mathematicians and physicists to develop, over long periods of time, intuitions about geometric objects in dimensions greater than three. Through what this book refers to as *the extension program*, geometric ideas that are tangible in two (and three) dimensions can be extended to higher dimensions via algebra. Doing so will help you develop geometric intuitions for higher dimensions you cannot physically experience.

By the end of this chapter, you will be able to answer questions like these:

1. How can you describe adding and scaling vectors in geometric terms?

2. How can you use vectors to describe lines in space?

3. Let $A = (3, 2)$ and $B = (-1, 4)$.
   - **a.** How do you calculate and graph the following: $A + B$, $2A$, $-3B$, $2A - 3B$?
   - **b.** What is the value of $\|2A - 3B\|$?

You will build good habits and skills for ways to

- generalize from numerical examples
- use algebra to extend geometric ideas
- connect the rules of arithmetic to an algebra of points
- use different forms for different purposes

Vocabulary and Notation

- coordinates
- direction
- equivalent vectors
- extension program
- initial point (tail)
- length $\|X\|$
- linear combination
- magnitude
- n-dimensional Euclidean space
- opposite direction
- ordered n-tuple
- point
- same direction
- scalar multiple
- spanned
- terminal point (head)
- unit vector
- vector
- vector equation
- zero vector

1.1 Getting Started

In this chapter, you'll develop an "algebra of points." Before things get formal, here's a preview of coming attractions.

- To *add* two points in the coordinate plane, add the corresponding coordinates: $(3, 2) + (5, 7) = (8, 9)$ and, more generally, $(x, y) + (z, w) = (x + z, y + w)$.
- To *scale* a point in the coordinate plane by a number, multiply both coordinates of that point by that number: $5(3, 2) = (15, 10)$ and, more generally, $c(x, y) = (cx, cy)$.

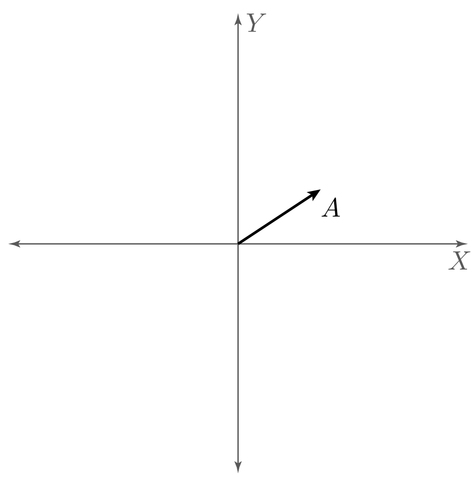
Exercises

1. Suppose $A = (1, 2)$. On one set of axes, plot these points:

a. $2A$ **b.** $3A$ **c.** $5A$
d. $(-1)A$ **e.** $(-3)A$ **f.** $(-6.5)A$

2. Here's a picture of a point A, with an arrow drawn from the origin to A.

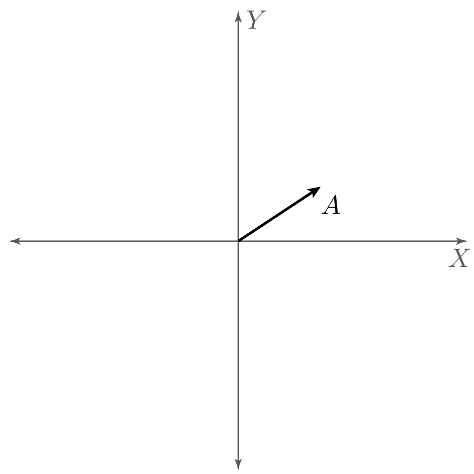
⟵ The arrow in the figure is called a *vector*.



Draw these vectors, all on the same axes:

a. $2A$ **b.** $3A$ **c.** $5A$
d. $(-1)A$ **e.** $(-3)A$ **f.** $(-6.5)A$

3. Here's a picture of a point A, with an arrow drawn from the origin to A.



a. Describe in words the set of all multiples tA, where t ranges over all real numbers.

b. If $A = (r, s)$, find a coordinate equation for the set of multiples tA, where t ranges over all real numbers.

⟵
A **coordinate equation** in $\mathbb{R}^2$ is an equation of the form $ax + by = c$.

4. Suppose $O = (0, 0)$, $A = (5, 3)$, and $B = (3, -1)$. Show that O, A, B, and $A + B$ lie on the vertices of a parallelogram. It may be helpful to draw a picture.

5. Suppose $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Show that O, A, B, and $A+B$ lie on the vertices of a parallelogram. Again, it may be helpful to draw a picture.

⟵
$O = (0, 0)$,

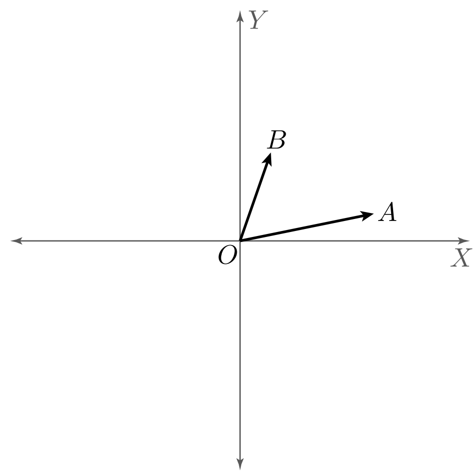
6. Suppose $A = (5, 3)$ and $B = (3, -1)$. Find and plot these points, all on the same axes:

a. $A + B$ **b.** $A + 3B$ **c.** $A + 5B$
d. $A + (-1B)$ **e.** $A + (-3B)$ **f.** $A + (-6.5B)$

Habits of Mind
Draw a picture!!

7. Suppose $A = (5, 3)$ and $B = (3, -1)$. Find a coordinate equation for the set of points X that is generated by $A + tB$, where t ranges over all real numbers.

8. Here's a picture of two points, A and B, with arrows drawn to each from the origin.

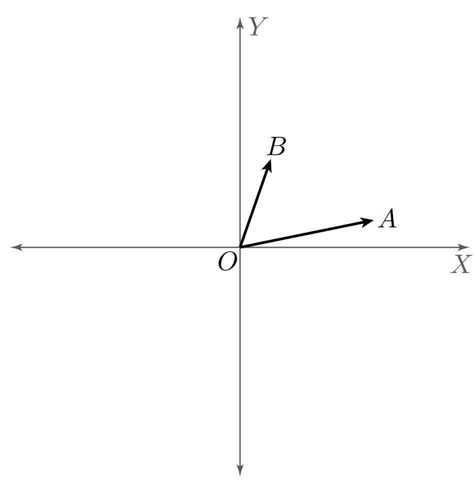


Draw these vectors, all on the same axes:

a. $A + B$ **b.** $A + 3B$ **c.** $A + 5B$

d. $A + (-1B)$ **e.** $A + (-3B)$ **f.** $A + (-6.5B)$

9. Here's a picture of two points, A and B, with arrows drawn to each from the origin.



Draw a picture of the set of all points X that is generated by $A+tB$, where t ranges over all real numbers.

⟵ You can think of the equation $X = A + tB$ as a *point-generator*: different numbers t generate different points X.

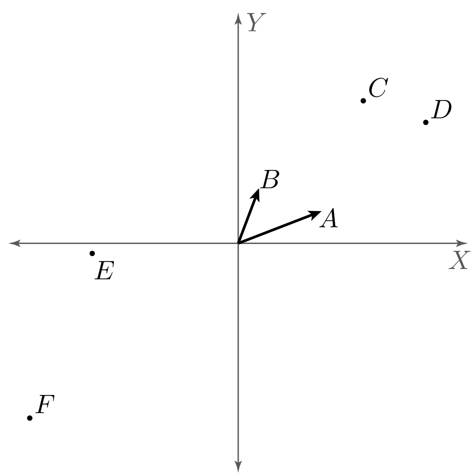
Habits of Mind

Draw a picture!!!

10. Suppose $A = (5,4)$ and $B = (-1,3)$. Find numbers c_1 and c_2 if

a. $c_1A + c_2B = (4,7)$ **b.** $c_1A + c_2B = (9,11)$
c. $c_1A + c_2B = (11,5)$ **d.** $c_1A + c_2B = (-13,1)$
e. $c_1A + c_2B = (1, \frac{-5}{2})$ **f.** $c_1A + c_2B = (-\frac{11}{5}, -1)$

11. Here's a picture of two points, A and B, with arrows drawn to each from the origin, as well as some other points.



Estimate the values for c_1 and c_2 if

a. $c_1A + c_2B = C$ **b.** $c_1A + c_2B = D$
c. $c_1A + c_2B = E$ **d.** $c_1A + c_2B = F$

12. Find the length of each vector.

⟵ Draw a picture.

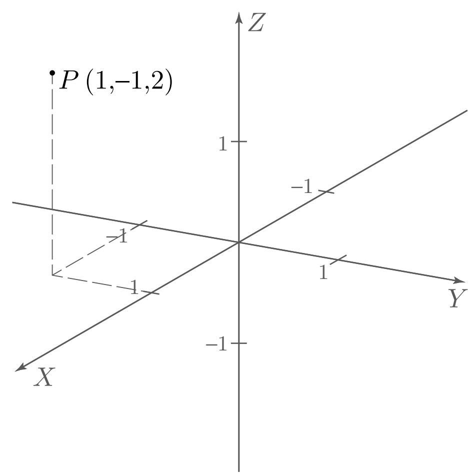
a. $A = (5,12)$ **b.** $B = (3,4)$ **c.** $C = (-2,10)$
d. $P = (4,1,8)$ **e.** $P = (4,1,9)$ **f.** $D = (a,b)$
g. $Q = (a,b,c)$

⟵ In each part the vector goes from the origin to the labeled point.

13. Find the lengths of the sides of $\triangle AOB$ if

a. $A = (5,12), B = (-27,36)$
b. $A = (21,20), B = (21,220)$
c. $A = (48,64), B = (15,8)$
d. $A = (4,4), B = (4,-4)$
e. $A = (4,0), B = (2, 2\sqrt{3})$
f. $A = (3\sqrt{2}, 3\sqrt{2}), B = (-4\sqrt{2}, 4\sqrt{2})$
g. $A = (-14,29,22), B = (-126,45,-18)$

14. Here's a picture of a *three*-dimensional coordinate system.



Find the equation of

a. the x–y plane
b. the x–z plane
c. the y–z plane
d. the plane parallel to the x–y plane that contains the point $(1, -1, 2)$

Remember

Equations are point-testers: a point is on the graph of your equation if and only if the coordinates of the point make the equation true.

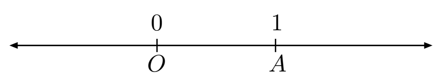
1.2 Points

In most of your high school work so far, the equations and formulas have expressed facts about the *coordinates* of points—the variables have been placeholders for numbers. In this lesson, you will begin to develop an *algebra of points*, in which you can write equations and formulas whose variables are points in two and three dimensions.

In this lesson, you will learn how to

- locate points in space and describe objects with equations
- use the algebra of points to calculate, solve equations, and transform expressions, all in $\mathbb{R}^n$
- understand the geometric interpretations of adding and scaling

You probably studied the method for building number lines (or "coordinatized lines") in previous courses. Given a line, you can pick two points O and A and assign the number 0 to O and 1 to A.

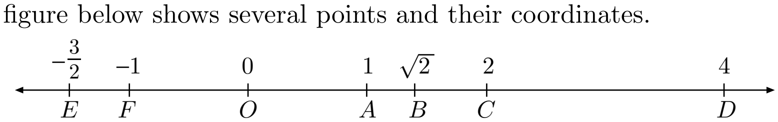


This sets the "unit" of the number line, and you can now set up a one-to-one correspondence between the set of real numbers, denoted by $\mathbb{R}$, and the set of points on the number line.

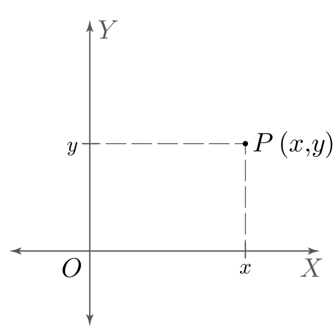
One-to-one correspondence means that for every point, there is exactly one real number (its coordinate) and for every real number, there is exactly one point (its graph).

- Suppose P is a point on the number line that is located x units to the right of O. Then x is called the **coordinate** of P, and P is called the **graph** of x.
- Suppose Q is a point on the number line that is located x units to the *left* of O. Its distance from O is still x, but it's not the same point as P. In this case, $-x$ is the coordinate of Q, and Q is the graph of $-x$.

The figure below shows several points and their coordinates.



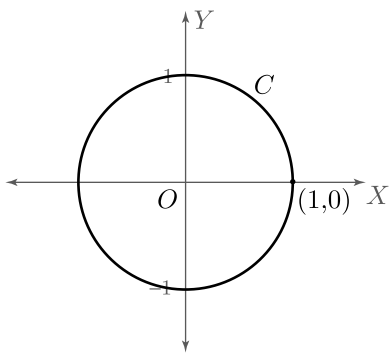
This idea of relating the set of all points on a line with the real numbers goes back to antiquity, but it was not until the 17th century that mathematicians (notably Descartes and Fermat) had a clear notion of how to coordinatize a plane: draw two perpendicular coordinatized lines (usually the scale is the same on each) that intersect at their common origin. These lines are called the ***x*-axis** and ***y*-axis**. You can now uniquely identify every point on the plane using an ordered pair of numbers. If the point P corresponds to the ordered pair (x, y), x and y are the **coordinates** of P.



In fact, the idea of coordinatizing the plane does not require that the two axes be perpendicular, only that each point lies on a unique pair of lines parallel to a given pair of axes.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. Because of the correspondence between $\mathbb{R}^2$ and points on a plane, you can think of $\mathbb{R}^2$ as the set of points on a coordinatized plane, so that statements like "the point $(5, 0)$ is the same distance from the point $(0, 0)$ as the point $(3, 4)$" make sense.

Identifying$\mathbb{R}^2$ with a plane provides a way to use algebra to describe geometric objects. Indeed, this is the central theme of analytic geometry.



Consider the circle C on the left. You can describe C geometrically by saying that C consists of all points in the plane that are 1 unit from O. However, you can also describe C algebraically in terms of the coordinates of the points that lie on C: C is the set of points (x, y) so that $x^2 + y^2 = 1$.

The connection between the geometric description ("C consists of all points . . . ") and the equation ("$x^2 + y^2 = 1$") is that *the equation is a* **point-tester** *for the geometric definition*. This means that you can test a point to see if it's on the circle by seeing if its coordinates satisfy the equation. For example,

⟵ Many people make statements like "C is the circle $x^2 + y^2 = 1$"; this statement is shorthand for "C is the circle whose equation is $x^2 + y^2 = 1$."

- $(1, 0)$ is on C because $1^2 + 0^2 = 1$
- $\left(\frac{1}{2}, \frac{1}{3}\right)$ is not on C because
$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 \neq 1$$
- $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ is on C because
$$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = 1$$

Think about how you get the equation of the circle in the first place: you take the geometric description—"all the points that are 1 unit from the origin"—and use the distance formula to translate that into algebra.

$$\begin{aligned} P = (x, y) \text{ is on } C &\Leftrightarrow \text{the distance from } P \text{ to } O \text{ is } 1 \\ &\Leftrightarrow \sqrt{x^2 + y^2} = 1 \quad \text{(the distance formula)} \\ &\Leftrightarrow x^2 + y^2 = 1 \end{aligned}$$

⟵ The symbol "⇔" means "the two statements are equivalent." You can read it quickly by saying "if and only if."

For You to Do

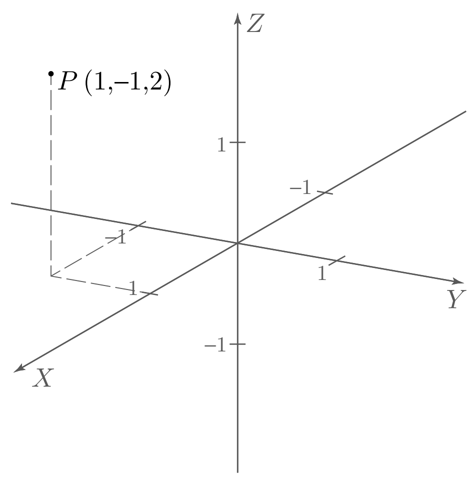
1. Find five points on the circle of radius 5 centered at $(3, 4)$. Find the equation of this circle.

2. **Take It Further.** Find five points on the sphere of radius 5 centered at $(3, 4, 2)$. Find the equation of this sphere.

Developing Habits of Mind

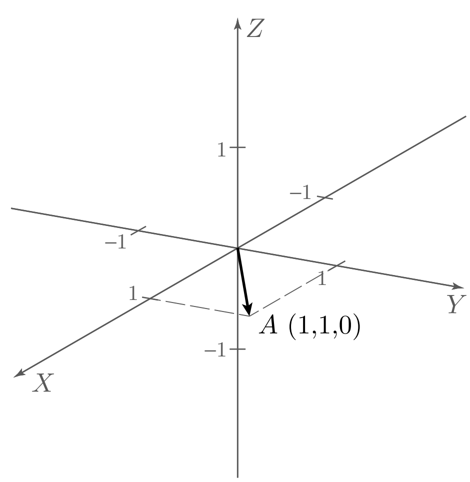
Explore multiple representations. All of plane geometry *could* be carried out using the algebra of $\mathbb{R}^2$ without any reference to diagrams or points on a plane. For example, you could define a line to be the set of pairs (x, y) that satisfy an equation of the form $ax + by = c$ for some real numbers a, b, and c. The fact that two distinct lines intersect in, at most, one point would then be a fact about the solution set of two equations in two unknowns. This would be silly when studying two- or three-dimensional geometry—the pictures help so much with understanding—but you will see shortly that characterizing geometric properties algebraically makes it possible to generalize many of the facts in elementary geometry to situations for which there is no physical model.

The method for coordinatizing three-dimensional space is similar. Choose three mutually perpendicular coordinatized lines (all with the same scale) intersecting at their origin. Then set up a one-to-one correspondence between points in space and ordered triples of numbers (x, y, z). In the following figure, point P has coordinates $x = 1$, $y = -1$, and $z = 2$.



The set of ordered triples of real numbers is denoted by $\mathbb{R}^3$, and the elements of $\mathbb{R}^3$ are spoken of as points. In the next figure, the line through $O = (0, 0, 0)$ and $A = (1, 1, 0)$ makes an angle of 45° with the x- and y-axes and an angle of 90° with the z-axis.

⟵ If you're not convinced, stay tuned . . . you'll revisit the notion of angles in $\mathbb{R}^3$ in Chapter 2.



Minds in Action Episode 1

Tony and Sasha are two students studying Linear Algebra. They are thinking about how to use the point-tester idea to describe objects in space.

TONY: What would the equation of the x–y plane in $\mathbb{R}^3$ be?

SASHA: Don't you remember using point-tester way back in Algebra 1 when we were finding equations of lines? First, think about some points on the x–y plane.

TONY: Well, $(0, 0, 0)$ is on that plane. So is $(1, 0, 0)$ and $(2, 3, 0)$. There are a lot, Sasha. How long do you want me to go on for?

SASHA: Until you see the pattern, of course! But this one's easy. In fact, all the points on the x–y plane have one thing in common: the z-coordinate is 0.

TONY: Yes! So that's easy. The equation would be $z = 0$. But isn't that the equation of a line?

SASHA: I guess it's not in $\mathbb{R}^3$. It has to describe a plane.

TONY: So what does the equation of a line look like in $\mathbb{R}^3$?

SASHA: Here, let's try an easy line, like the x-axis . . . Well, all the points would look like $(\text{something}, 0, 0)$. So the y-coordinate is always 0 and the z-coordinate is always 0. How do I say that in one equation?

TONY: I don't know . . . I guess the best we can do for now is to say the line is given by *two* equations: $y = 0$ and $z = 0$.

SASHA: Wait a second . . . what about $y^2 + z^2 = 0$?

TONY: Sasha, where do you get these ideas? It works, but that's not a linear equation, is it?

For You to Do

3. Find the equation of

a. the x–z plane
b. the plane parallel to the x–z plane that contains the point $(3, 1, 4)$

In the middle of the 19^{th} century, mathematicians began to realize that it was often convenient to speak of quadruples of numbers (x, y, z, w)as points of "four-dimensional" space. It seemed very natural to speak of $(1, 3, 2, 0)$ as being a point on the graph of $x+2y-z+w = 5$ rather than saying, "One solution to $x + 2y - z + w = 5$ is $x = 1, y = 3, z = 2$, and $w = 0$." Defining $\mathbb{R}^4$ as the set of all quadruples of real numbers, you can call its elements "points" in $\mathbb{R}^4$. Although there is no physical model for $\mathbb{R}^4$ (as there was for $\mathbb{R}^2$ and $\mathbb{R}^3$), you can borrow the geometric language used for $\mathbb{R}^2$ and speak of $O = (0, 0, 0, 0)$ as the origin in $\mathbb{R}^4$, $A = (1, 0, 0, 0)$ as a point on the x-axis in $\mathbb{R}^4$, and so on. This is, for now, just an analogy: $\mathbb{R}^4$ "is" the set of all ordered quadruples of numbers, and geometric statements about $\mathbb{R}^4$ are simple analogies with $\mathbb{R}^2$ and $\mathbb{R}^3$.

⟵ In 1884, E. A. Abbott wrote a book that captured what it might be like to visualize four dimensions. The book has been adapted in an animated film: see `flatlandthemovie.com`

Of course, there is no need to stop here. You can define $\mathbb{R}^5$ as the set of all ordered quintuples of numbers, and so on . . . .

Habits of Mind

Think like a mathematician. Like many mathematicians, after awhile you may develop a sense for picturing things in higher dimensions. This happens when the algebraic descriptions become identified with the geometric descriptions, deep in your mind.

Definition

If n is a positive integer, an **ordered n-tuple** is a sequence of n real numbers $(x_1, x_2, \ldots, x_n)$. The set of all ordered n-tuples is called ***n*-dimensional Euclidean space** and is denoted by $\mathbb{R}^n$.

An ordered n-tuple will be referred to as a **point** in $\mathbb{R}^n$.

Facts and Notation

- Capital letters (such as A, B, or P) are often used for points.
- If $A = (a_1, a_2, \ldots, a_n)$, the numbers $a_1, a_2, \ldots, a_n$ are called the **coordinates** of A.
- Two points $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ are **equal** if their corresponding coordinates are equal; that is, $A = B$ means $a_1 = b_1, a_2 = b_2, \ldots,$ and $a_n = b_n$.
- By analogy with $\mathbb{R}^2$, the origin of $\mathbb{R}^n$ is the point $O = (0, 0, 0, \ldots, 0)$.

⟵ Note that instead of writing $a_1 = b_1, a_2 = b_2, \ldots,$ and $a_n = b_n$, you can use shorthand notation "$a_i = b_i$ for each $i = 1, 2, \ldots, n$."

To extend a definition from geometry to $\mathbb{R}^n$, you must first characterize the geometric notions in terms of coordinates of points. You can accomplish this goal most easily by defining several operations on $\mathbb{R}^n$.

The first of these operations is addition. If you were asked to decide what $(2, 3) + (6, 1)$ should be, you might naturally say "$(8, 4)$, of course." It turns out that this definition is very useful: you add points by adding their corresponding coordinates.

Definition

If $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are points in $\mathbb{R}^n$, the **sum** of A and B, written $A + B$, is

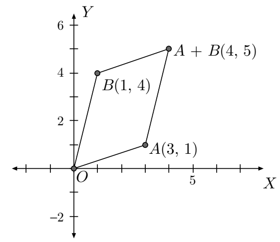
$$A + B = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

Developing Habits of Mind

Make strategic choices. This is a *definition*—$(2, 3) + (6, 1)$ equals $(8, 4)$, not because of any intrinsic reason. It isn't forced on you by the laws of physics or the basic rules of algebra, for example. Mathematicians have defined the sum in this way because it has many useful properties. One of the most useful is that there is a nice geometric interpretation for this method for adding in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 1

Consider the points $A = (3, 1)$, $B = (1, 4)$, and $A + B = (4, 5)$. If you plot these three points, you may not see anything interesting, but if you throw the origin into the figure, it looks as if O, A, B, and $A + B$ lie on the vertices of a parallelogram.



Example 1 suggests the following theorem.

Theorem 1.1 (The Parallelogram Rule)
If A and B are any *points in* $\mathbb{R}^2$*, then* O*,* A*,* $A + B$*, and* B *lie on the vertices of a parallelogram.*

You can refer to this parallelogram as "the parallelogram determined by A and B."

For You to Do

4. **a.** Show that $(0, 0)$, $(3, 1)$, $(1, 4)$, and $(4, 5)$ from Example 1 form the vertices of a parallelogram.
   b. While you're at it, explain why Theorem 1.1 must be true.

⟵
First generate some examples, then try to generalize them.

In linear algebra, it is customary to refer to real numbers (or the elements of any number system) as **scalars**. The second operation to define on $\mathbb{R}^n$ is the multiplication of a point by a scalar, and it is called *multiplication of a point by a scalar*.

⟵
In this book, the terms *scalar* and *number* will be used interchangeably. Scalar has a geometric interpretation—see below.

Definition
Let $A = (a_1, a_2, \ldots, a_n)$ be a point in $\mathbb{R}^n$ and suppose c is a scalar. The **scalar multiple** cA is

$$cA = (ca_1, ca_2, \ldots, ca_n).$$

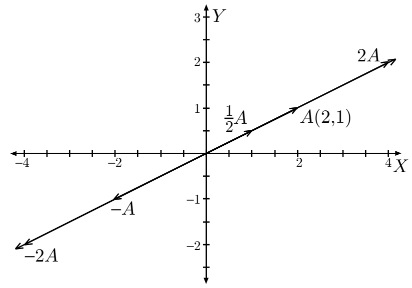
In other words, to multiply a point by a number, you simply multiply each of the point's coordinates by that number. So, $3(1, 4, 2, 0) = (3, 12, 6, 0)$.

Habits of Mind
Try proving that one of these statements is true. For example, show that $2A$ is collinear with O and A and is twice as far from O as A is.

Why are numbers called "scalars"? Scalar multiplication can be visualized in $\mathbb{R}^2$ or $\mathbb{R}^3$ as follows: if $A = (2, 1)$, then $2A = (4, 2)$, $\frac{1}{2}A = (1, \frac{1}{2})$, $-1A = (-2, -1)$, and $-2A = (-4, -2)$.

Habits of Mind
In previous courses, you saw that if you view $\mathbb{R}$ as a number line, multiplication by 3 stretches points by a factor of 3.

From this figure, you can see that if c is any real number, cA is collinear with O and A; cA is obtained from A by stretching or shrinking—so *scaling*—the distance from O to A by a factor of $|c|$. If $c > 0$, cA is in the same "direction" as A; multiplying by a negative reverses direction.



For You to Do

5. Let c be a real number, and let A be a point in $\mathbb{R}^2$.

a. Show that cA is collinear with O and A.

b. Show that if $c \geq 0$, cA is obtained from A by scaling the distance from O to A by a factor of $|c|$. If $c < 0$, cA is obtained from A by scaling the distance from O to A by a factor of $|c|$ and reversing direction.

So, now you have two operations on points: addition and scalar multiplication. How do the operations behave?

Theorem 1.2 (The Basic Rules of Arithmetic with Points)
Let

$$A = (a_1, a_2, \ldots, a_n)$$
$$B = (b_1, b_2, \ldots, b_n) \text{ and}$$
$$C = (c_1, c_2, \ldots, c_n)$$

be points in $\mathbb{R}^n$*, and let* d *and* e *be scalars. Then*

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $A + O = A$

(4) $A + (-1)A = O$

(5) $(d + e)A = dA + eA$

(6) $d(A + B) = dA + dB$

(7) $d(eA) = (de)A$

(8) $1A = A$

⟵ Because of property ((4)), $(-1)A$ is called the negative of A and is often written $-A$.

Proof. The proofs of these facts all use the same strategy: reduce the property in question to a statement about real numbers. To illustrate, here are proofs for ((1)) and ((7)). The proofs of the other facts are left as exercises.

$$
\begin{aligned}
((1)) \qquad A + B &= (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) \\
&= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) && \text{(definition of addition in } \mathbb{R}^n) \\
&= (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n) && \text{(commutativity of addition in } \mathbb{R}) \\
&= (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n) && \text{(definition of addition in } \mathbb{R}^n) \\
&= B + A
\end{aligned}
$$

$$
\begin{aligned}
((7)) \qquad d(eA) &= d\Big(e(a_1, a_2, \ldots, a_n)\Big) \\
&= d(ea_1, ea_2, \ldots, ea_n) && \text{(definition of scalar multiplication)} \\
&= \Big(d(ea_1), d(ea_2), \ldots, d(ea_n)\Big) && \text{(definition of scalar multiplication)} \\
&= \Big((de)a_1, (de)a_2, \ldots, (de)a_n\Big) && \text{(associativity of multiplication in } \mathbb{R}) \\
&= (de)(a_1, a_2, \ldots, a_n) && \text{(definition of scalar multiplication)} \\
&= (de)A
\end{aligned}
$$

■

Subtraction for points is defined by the equation

$$A - B = A + (-B)$$

⟵ . . . and "$-B$" means $(-1)B$.

Developing Habits of Mind

Use coordinates to prove statements about points. The strategy of reducing a statement about points to one about coordinates will be used throughout this book.

But how do you come up with valid statements about points in the first place? One way is to see if analogous statements are true in one dimension—with numbers. So, $2 + 3 = 3 + 2$ might give you a clue that $A + B = B + A$ for points. Once you have a clue, *try it with actual points*. Does $(7, 1) + (9, 8) = (9, 8) + (7, 1)$? Yes. And why? You might reason by writing things out and not simplifying until the end:

$$
\begin{aligned}
(7, 1) + (9, 8) &= (7 + 9, 1 + 8) \quad \text{and} \\
(9, 8) + (7, 1) &= (9 + 7, 8 + 1)
\end{aligned}
$$

Since $7 + 9 = 9 + 7$ and $1 + 8 = 8 + 1$, $(7, 1) + (9, 8) = (9, 8) + (7, 1)$. And this gives you an idea for how a proof in general will go.

⟵ This habit of "writing things out and not simplifying until the end" is an important algebraic strategy, often called *delayed evaluation*.

Example 2

Problem. Find A if A is in $\mathbb{R}^3$ and $2A + (-3, 4, 2) = (5, 2, 2)$.

Solution. Here are two different ways to find A.

1. Suppose $A = (a_1, a_2, a_3)$ and calculate as follows.

$$
\begin{aligned}
2(a_1, a_2, a_3) + (-3, 4, 2) &= (5, 2, 2) \\
(2a_1, 2a_2, 2a_3) + (-3, 4, 2) &= (5, 2, 2) \\
(2a_1 - 3, 2a_2 + 4, 2a_3 + 2) &= (5, 2, 2) \\
2a_1 - 3 = 5,\ 2a_2 + 4 = 2,\ 2a_3 + 3 &= 2 \\
a_1 = 4,\ a_2 = -1,\ a_3 = 0; \qquad A &= (4, -1, 0)
\end{aligned}
$$

Habits of Mind

Fill in the reasons.

2. Instead of calculating with coordinates, you can also use Theorem 1.2.

$$\begin{aligned} 2A + (-3,4,2) &= (5,2,2) \\ 2A &= (8,-2,0) \qquad \text{(subtract } (-3,4,2) \text{ from both sides)} \\ A &= (4,-1,0) \qquad \text{(multiply both sides by } \tfrac{1}{2}\text{)} \end{aligned}$$

Minds in Action Episode 2

Tony and Sasha are working on the following problem:

Find points A and B in $\mathbb{R}^2$, where $A + B = (3, 11)$ and $2A - B = (3, 1)$.

SASHA: In Example 2, we solved the equation with points just like any other equation. So, here we have two equations and two unknowns . . .

TONY: So we can use elimination. And look, it's easy—if we add both the equations together, the B's cancel out and we get $3A = (6, 12)$.

SASHA: So we divide both sides by 3 to get $A = (2, 4)$. We can plug that into the first equation . . .

TONY: . . . and subtract $(2, 4)$ from both sides to get $B = (1, 7)$.

SASHA: Smooth, Tony. I wonder how much harder it would be to use coordinates. What if we say $A = (a_1, a_2)$ and $B = (b_1, b_2)$. We can then work it through like the first part of Example 2.

TONY: Have fun with that, Sasha.

Habits of Mind

Make sure that Sasha and Tony's calculations are legal. Theorem 1.2 gives the basic rules.

Developing Habits of Mind

Find connections. After using Theorem 1.2 for a while to calculate with points and scalars, you might begin to feel like you did in Algebra 1 when you first practiced solving equations like $3x + 1 = 7$: you can forget the meaning of the letters and just proceed formally, applying the basic rules.

Example 3

Problem. Find scalars c_1 and c_2 so that

$$c_1(1, 4, -1) + c_2(3, -1, 2) = (-1, 9, -4)$$

Solution. Simplify the left-hand side to get

$$(c_1 + 3c_2, 4c_1 - c_2, -c_1 + 2c_2) = (-1, 9 - 4,), \text{ or}$$

$$\begin{aligned} c_1 + 3c_2 &= -1 \\ 4c_1 - c_2 &= 9 \\ -c_1 + 2c_2 &= -4 \end{aligned}$$

So, you are looking for a solution to this system of equations.

Solve the first two equations simultaneously to find the solution $c_1 = 2$, $c_2 = -1$. This solution works in the third equation also, so 2 and -1 are the desired scalars. Because $(-1, 9, -4)$ can be written as $2(1, 4, -1) + -1(3, -1, 2)$, $(-1, 9, -4)$ is a **linear combination** of $(1, 4, -1)$ and $(3, -1, 2)$.

⟵ In Chapter 3, you will study other methods for solving systems of linear equations.

Exercises

1. Let $A = (3, 1)$, $B = (2, -4)$, and $C = (1, 0)$. Calculate and plot the following:

a. $A + 3B$ **b.** $2A - C$
c. $A + B - 2C$ **d.** $-A + \frac{1}{2}B + 3C$
e. $\frac{1}{2}(A + B) + \frac{1}{2}(A - B)$

2. For each choice of U and V, find $U + V$ and $3U - 2V$.

a. $U = (4, -1, 2), V = (1, 3, -2)$
b. $U = (3, 0, 1, -2), V = (1, -1, 0, 1)$
c. $U = (3, 7, 0), V = (0, 0, 2)$
d. $U = (1, \frac{1}{2}, 3), V = 2U$

3. Let $A = (3, 1)$ and $B = (-2, 4)$. Calculate each result and plot your answers.

a. $A + B$ **b.** $A + 2B$ **c.** $A + 3B$
d. $A - B$ **e.** $A + \frac{1}{2}B$ **f.** $A + 7B$
g. $A - \frac{1}{3}B$ **h.** $A + \frac{5}{2}B$ **i.** $A - 4B$

4. Let $A = (5, -2)$ and $B = (2, 5)$. Calculate each result and plot your answers.

a. $A + B$ **b.** $A + 2B$ **c.** $2A + 3B$
d. $2A - 3B$ **e.** $\frac{1}{2}A + \frac{1}{2}B$ **f.** $\frac{1}{3}A + \frac{2}{3}B$
g. $\frac{1}{10}A + \frac{9}{10}B$ **h.** $-3A - 4B$ **i.** $A - 4B$

5. Let $A = (5, -2)$ and $B = (2, 5)$. Calculate each result and plot your answers.

a. $A + (B - A)$ **b.** $A + 2(B - A)$ **c.** $A + 3(B - A)$
d. $A - 3(B - A)$ **e.** $A + \frac{1}{2}(B - A)$ **f.** $A + \frac{2}{3}(B - A)$
g. $A + \frac{9}{10}(B - A)$ **h.** $A - 4(B - A)$ **i.** $A + 4(B - A)$

6. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Find an expression for the area of the parallelogram whose vertices are O, A, $A + B$, and B.

7. In $\mathbb{R}^3$, find the equation of each of the following:

a. the y–z plane
b. the plane through $(-3, 5, -1)$ parallel to the y–z plane
c. the plane through $(-3, 5, -1)$ parallel to the x–y plane
d. the sphere with center $(0, 0, 0)$ and radius 1
e. the sphere with center $(2, 3, 6)$ and radius 1

8. Find the point $A = (a_1, a_2, a_3, a_4, a_5)$ in $\mathbb{R}^5$ if $a_j = j^2$ for each $j = 1, 2, \ldots, 5$.

9. For each of the following equations, solve for A.

a. $3A - (4, 7) = (-1, -4)$
b. $2A + 3(2, -1, 3, 6) = 4A + (2, -1, 3, 2)$
c. $2A - (4, 6, 2) = O$
d. $5A - (-1, 7, 1) = 3A + 4(8, -1, 2)$

10. Find A and B if $A + B = (4, 8)$ and $A - B = (-2, -6)$.

11. For each of the following equations, find c_1 and c_2.

a. $c_1(2, 3, 9) + c_2(1, 2, 5) = (1, 0, 3)$
b. $c_1(2, 3, 9) + c_2(1, 2, 5) = (0, 1, 1)$

12. Show that there are no scalars c_1 and c_2 so that

$$c_1(4, 1, 2) + c_2(-8, -2, -4) = (3, 1, 2)$$

⟵ What does the set of all points of the form $c_1(4, 1, 2) + c_2(-8, -2, -4)$ look like in $\mathbb{R}^3$?

13. Find nonzero scalars c_1, c_2, and c_3 so that

$$c_1(1, 5, 1) + c_2(2, 0, 3) + c_3(3, 5, 4) = (0, 0, 0)$$

14. Show that if $c_1(3, 2) + c_2(4, 1) = (0, 0)$, then $c_1 = c_2 = 0$.

15. Prove (2), (3), and (4) in Theorem 1.2.

16. Prove (5), (6), and (8) in Theorem 1.2.

1.3 Vectors

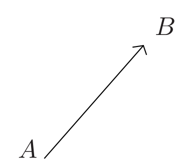
The real number system evolved in an attempt to measure physical quantities like length, area, and volume. Certain physical phenomena, however, cannot be characterized by a single real number. For example, there are two equally important pieces of information that specify the velocity of an object: the speed (or magnitude of the velocity) and the direction. You can represent velocity using a single object: a *vector*.

⟵ Unless, of course, the speed is 0.

In this lesson, you will learn how to

- test vectors for equivalence using the algebra of points
- prove simple geometric theorems with vector methods
- develop a level of comfort moving back and forth between points and vectors
- think of linear combinations geometrically

A vector is a directed line segment that is usually represented by drawing an arrow. The arrow has a length (or magnitude), and one end has an arrowhead that denotes the direction the arrow is pointing. In this figure, the two endpoints of the line segment are labeled A and B. If you know the two endpoints, you can completely describe the vector. This vector starts at A and ends at B, so it is denoted by $\overrightarrow{AB}$. The point A is called the **tail** (or **initial point**) of $\overrightarrow{AB}$, and the point B is called the **head** (or **terminal point**) of $\overrightarrow{AB}$.



⟵ Note that $\overrightarrow{AB}$ is not the same vector as $\overrightarrow{BA}$.

In fact, you can completely describe any vector by specifying just its tail and its head, so you do not have to rely on a drawing. The following definition works for any dimension.

Definition

If A and B are points in $\mathbb{R}^n$, the **vector** with tail A and head B is the ordered pair of points $[A, B]$. You can denote the vector with tail A and head B by $\overrightarrow{AB}$.

Facts and Notation

There's no real agreement about the definition of "vector." Many books insist that a vector must have its tail at the origin, calling vectors that don't start at the origin "located vectors" or "free vectors." While there are good reasons for making such fine distinctions, they are not necessary at the start. *This* book will soon concentrate on vectors that start at the origin too, but for now, think of a vector as an arrow or an ordered pair of points.

⟵ There's plenty of time for formalities later.

Developing Habits of Mind

Use algebra to extend geometric ideas. There are many ways to think about vectors. Physicists talk about quantities that have a "magnitude" and "direction" (like velocity, as opposed to speed). Football coaches draw arrows. Some people talk about "directed" line segments. Mathematics, as usual, makes all this fuzzy talk precise: a vector is nothing other than an *ordered pair of points*.

⟵ The gain in precision is accompanied by a loss of all these romantic images carried by the arrows and colorful language.

But the geometry is essential: a central theme in this book is to start with a geometric idea in $\mathbb{R}^2$ or $\mathbb{R}^3$, find a way to characterize it with algebra, and then use that algebra as the *definition* of the idea in higher dimensions. The details of how this theme is carried out will become clear over time. The next discussion gives an example.

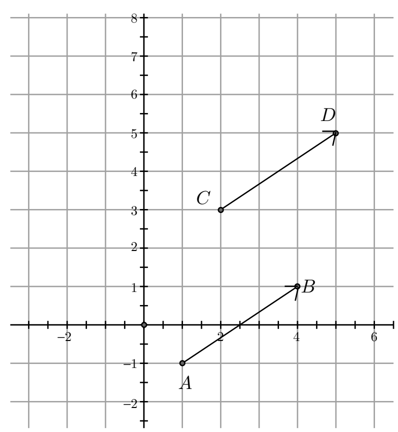
Habits of Mind

"The geometry" referred to here is the regular Euclidean plane geometry you studied in earlier courses. Later, you may study just how many of these ideas can be extended if you start with, say, geometry on a sphere.

In $\mathbb{R}^2$ or $\mathbb{R}^3$, two vectors are called **equivalent** if they have the same magnitude (length) and the same direction. For example, in the figure to the right,

$$A = (1, -1)$$
$$B = (4, 1)$$
$$C = (2, 3) \text{ and}$$
$$D = (5, 5)$$

$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$.



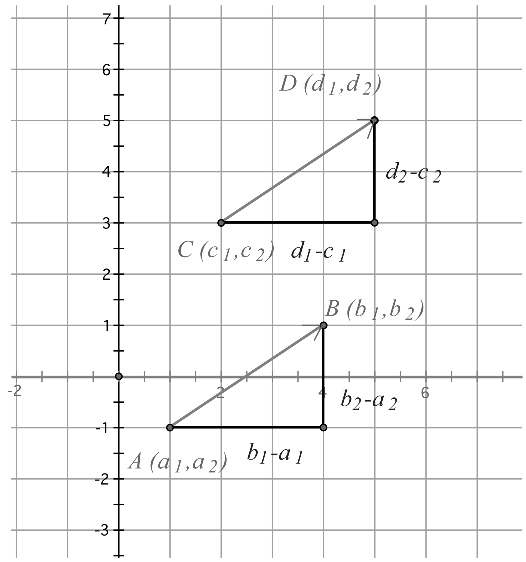
For You to Do

1. Show that vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same length.

What's Wrong Here?

2. Derman calculates the slope from A to B as $\frac{2}{3}$. But he also remembers that the slope from B to A is also $\frac{2}{3}$, so he thinks that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{BA}$. Can that be right?

So, there is this geometric idea of equivalent vectors. To define equivalence of vectors in $\mathbb{R}^n$ in a way that agrees with this notion of equivalence in $\mathbb{R}^2$, you need to characterize equivalent vectors in $\mathbb{R}^2$ without using words like "magnitude" or "direction." Suppose $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$, and B is to the right and above A in $\mathbb{R}^2$ as in the following figure.



To find a point $D = (d_1, d_2)$ so that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$, starting from C, move to the right a distance equal to $b_1 - a_1$, and then move up a distance equal to $b_2 - a_2$. In other words, $d_1 = c_1 + (b_1 - a_1)$ and $d_2 = c_2 + (b_2 - a_2)$. Therefore, $d_1 - c_1 = b_1 - a_1$ and $d_2 - c_2 = b_2 - a_2$.

This can be written as $(d_1 - c_1, d_2 - c_2) = (b_1 - a_1, b_2 - a_2)$, or, using the algebra of points,

$$D - C = B - A$$

You can call this the "head minus tail test" in $\mathbb{R}^2$.

Habits of Mind

What are the slopes of $\overrightarrow{AB}$ and $\overrightarrow{CD}$?

For You to Do

3. In the figure above, show that if $D - C = B - A$, the distance from A to B is the same as the distance from C to D and that the slope from A to B is the same as the slope from C to D.

⟵ In the *CME Project* series, the slope from A to B is written as $m(A, B)$.

Theorem 1.3 (Head Minus Tail Test)
In $\mathbb{R}^2$, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if and only if

$$D - C = B - A$$

The discussion leading up to Theorem 1.3 makes its result seem plausible, but there are other details to check.

1. The preceding argument for finding point D depends on a particular orientation of the two vectors—B is to the right and above A. A careful proof would have to account for other cases.
2. That argument shows that if $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$, then $B - A = D - C$. A careful proof would also show the converse: if $B - A = D - C$, then $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$.

Both of these details can be handled with some careful analytic geometry.

It also can be shown (using analytic geometry in three dimensions) that the Head Minus Tail (HmT) Test works in $\mathbb{R}^3$. Since this characterization of equivalence makes no use of geometric language, it makes sense in $\mathbb{R}^n$.

⟵ Much more attention will be given to the geometry of $\mathbb{R}^3$ in the next chapter.

Definition

If A, B, C, and D are points in $\mathbb{R}^n$, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equivalent** if $B - A = D - C$.

Habits of Mind

Use algebra to extend geometric ideas. This definition of "equivalent" uses the algebra you developed in Theorem 1.3 and extends that algebra to any dimension.

Example 1

Problem. Is $\overrightarrow{\left(\frac{1}{2},3\right)\left(3,5\right)}$ equivalent to $\overrightarrow{\left(\frac{17}{8},0\right)\left(\frac{35}{8},2\right)}$?

Habits of Mind

Draw a picture.

Solution. You could check slopes and (directed) distances, but both of those are checked in the HmT Test. For the first vector, HmT yields $\left(\frac{5}{2},2\right)$; for the second, you get $\left(\frac{9}{4},2\right)$. So the vectors are not equivalent.

Example 2

Problem. In $\mathbb{R}^4$, if $X = (-1,2,3,1), Y = (1,-2,5,4)$, and $Z = (3,1,1,0)$, find W so that $\overrightarrow{XY}$ is equivalent to $\overrightarrow{ZW}$.

Solution. By *definition*, this means $W - Z = Y - X$ or

$$W = Z + Y - X = (5,-3,3,3)$$

Developing Habits of Mind

Use algebra to extend geometric ideas. The process that led to the definition of equivalent vectors in $\mathbb{R}^n$ is important.

- First, equivalent vectors in $\mathbb{R}^2$ are defined using geometric ideas.
- Next, equivalent vectors in $\mathbb{R}^2$ are characterized by an equation involving only the operation of subtraction of points.
- Finally, this *equation* is used as the *definition* of equivalent vectors in $\mathbb{R}^n$.

⟵ This process will be called the *extension program* from now on.

This theme will be used throughout the book, and it will allow you to generalize many familiar geometric notions from the plane (and in the next chapter, from three-dimensional space) to $\mathbb{R}^n$.

For You to Do

4. Show that in $\mathbb{R}^2$ every vector is equivalent to a vector whose tail is at O.

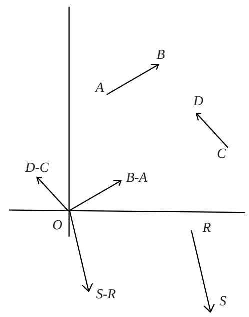
The same result—every vector is equivalent to a vector whose tail is at O—is true in $\mathbb{R}^n$, and the proof may seem surprising.

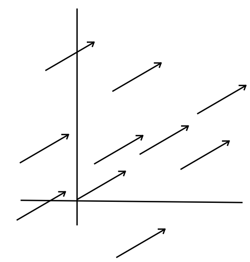
Theorem 1.4

Every vector in $\mathbb{R}^n$ is equivalent to a vector whose tail is at O. In fact, $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B-A)}$.

Proof. $B - A = (B - A) - O$. ■

The following figure illustrates Theorem 1.4 for several vectors.





Facts and Notation

The vectors in $\mathbb{R}^n$ break up into "classes": two vectors belong to the same "class" if and only if they are equivalent. Every nonzero point in $\mathbb{R}^n$ determines one of these classes. That is, the point A determines the class of vectors equivalent to $\overrightarrow{OA}$. Theorem 1.4 shows that every class of vectors is obtained in this way. Furthermore, you can show (see Exercise 5) that if $\overrightarrow{OA}$ is equivalent to $\overrightarrow{OB}$, then $A = B$.

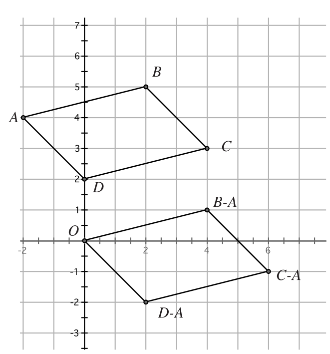
Because of this, the following convention will be in force for the rest of this book: from now on, an ordered n-tuple $A \neq O$ will stand for *either* a point in $\mathbb{R}^n$ or the vector $\overrightarrow{OA}$. You can also consider O as a vector (the **zero vector**). The context will always make it clear whether an element in $\mathbb{R}^n$ is considered a point or a vector.

Example 3

Problem. Show that the points $A = (-2, 4)$, $B = (2, 5)$, $C = (4, 3)$, and $D = (0, 2)$ lie on the vertices of a parallelogram.

Solution Method 1. Translate the quadrilateral to the origin; that is, slide the parallelogram so that one of the vertices (say, A) lands at the origin, and translate the other three points similarly.

$\longleftarrow$ A **translation** is a transformation that slides a figure without changing its size, its shape, or its orientation. If A is a point, subtracting A from each vertex of a polygon translates that polygon (why?).



Since $A - A = O$, you can translate the other three points by subtracting A. More precisely, $\overrightarrow{AB}$ is equivalent to $B - A = (4, 1)$, $\overrightarrow{AC}$ is equivalent to $C - A = (6, -1)$, and $\overrightarrow{AD}$ is equivalent to $D - A = (2, -2)$. Since $(4, 1) + (2, -2) = (6, -1)$, $C - A = (B - A) + (D - A)$, and you can say that, by the Parallelogram Rule (Theorem 1.1 from Lesson 1.2), $O, B - A, C - A$, and $D - A$ lie on the vertices of a parallelogram. Since the translation affects only the position of the figure, A, B, C, and D must also lie on the vertices of a parallelogram.

Solution Method 2. $\overrightarrow{AB}$ is equivalent to $\overrightarrow{DC}$, since $B - A = C - D = (4, 1)$. Since they are equivalent, you know they have the same length (magnitude), so the line segments are congruent. They also have the same direction, so the line segments have the same slope and are thus parallel.

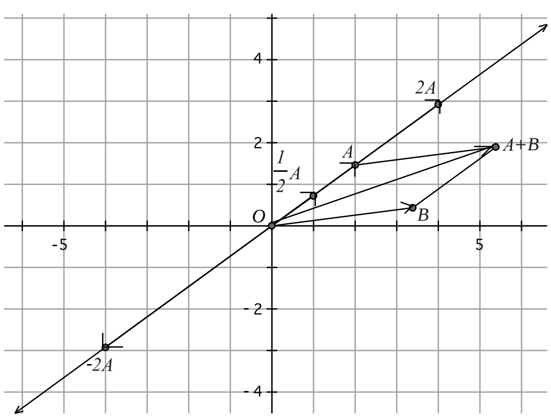
⟵ A convex quadrilateral is a parallelogram if *one* pair of opposite sides is both parallel and congruent, so there's no need to show that $\overrightarrow{BC}$ is equivalent to $\overrightarrow{AD}$.

For You to Do

5. Derman tried to show that he had a parallelogram if $A = (2, 1)$, $B = (4, 2)$, $C = (6, 3)$, and $D = (10, 5)$, and he ended up scratching his head. Help him figure out why these points do not form a parallelogram.

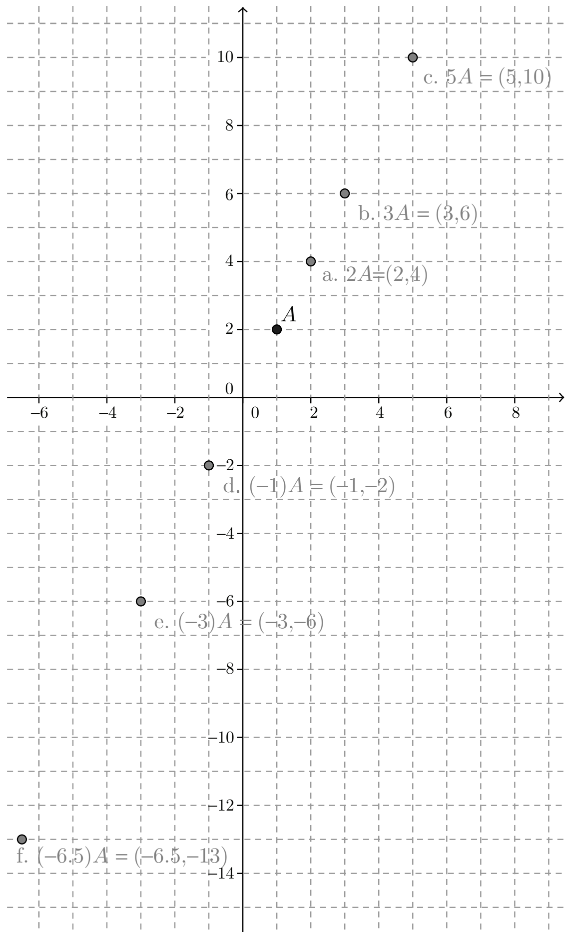
Developing Habits of Mind

Use vectors to describe geometric ideas. If points in $\mathbb{R}^2$ and $\mathbb{R}^3$ are viewed as vectors, the geometric description of addition and scalar multiplication is much easier.



If A and B are vectors in $\mathbb{R}^2$, $A + B$ is the diagonal of the parallelogram whose sides are A and B. Multiplying A by c yields a vector whose length is the length of A multiplied by $|c|$; cA has the same direction as A if $c > 0$; cA has the opposite direction of A if $c < 0$.

In Exercise 1 from Lesson 1.1, you plotted several points that were scalar multiples of a point $A = (1, 2)$. What you might have noticed was that all of the resulting points ended up on the same line:

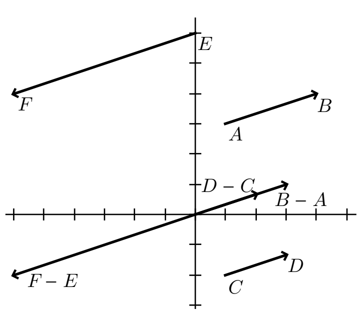


$\longleftarrow$ In Exercise 6, you'll show that if P and Q are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and that if O, P, and Q are collinear, then $Q = cP$ for some real number c. In fact, the set of points collinear with the points O and P is the collection of multiples cP of P.

The fact that scalar multiples of points are collinear hints at an algebraic way to characterize parallel vectors in $\mathbb{R}^2$. In the following figure, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction, $\overrightarrow{AB}$ and $\overrightarrow{EF}$ have opposite directions, and all three vectors appear to be parallel.

Remember

A point A corresponds to the vector $\overrightarrow{OA}$.



If you construct equivalent vectors starting at O, vectors $B - A$, $D - C$, and $F - E$ are all scalar multiples of each other. More precisely, $D - C = k(B - A)$ for some $k > 0$, and $F - E = k(B - A)$ for some $k < 0$. This statement was developed in $\mathbb{R}^2$, but it makes sense in $\mathbb{R}^n$ for any n.

Definition

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in $\mathbb{R}^n$ are said to be **parallel** if there is a nonzero real number k so that

$$B - A = k(D - C)$$

- If $k > 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the **same direction**.
- If $k < 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have **opposite directions**.

The zero vector O is parallel to every vector (with no conclusion about same or opposite direction).

Habits of Mind

Use algebra to extend geometric ideas. The definition takes an algebraic characterization—$B - A = k(D - C)$—of a geometric property—$\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$—and makes it the *definition* of the geometric property in $\mathbb{R}^n$.

For You to Do

6. Let $A = (3, -1, 2, 4), B = (1, 2, 0, 1), C = (3, 2, -3, 5)$, and $D = (7, -4, 1, 11)$. Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Minds in Action Episode 3

Tony and Sasha are working on the following problem.

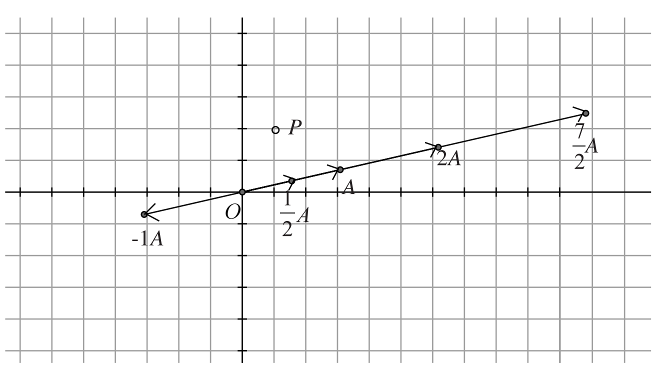
Problem. Suppose that $P = (2, 4), A = (3, 1)$, and S is the set of points Q so that $\overrightarrow{PQ}$ is parallel to A. Draw a picture of S and find an equation for it.

TONY: From the definition, if $\overrightarrow{PQ}$ is parallel to A, then there is a real number k so that $Q - P = kA$. Okay, now what?

SASHA: Well, let's rewrite that as

$$Q = P + kA$$

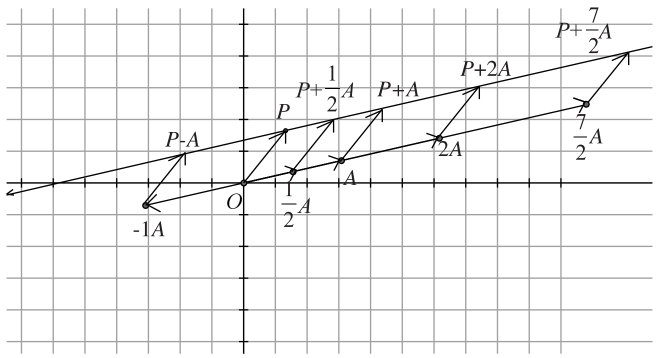
I can draw what kA looks like for all real values of k: it's a line through the origin and the point A.



TONY: Great, but what about the P?

SASHA: Well, we have to add P to each of the multiples.

TONY: That's a lot of parallelograms! All right, let's try it.



Ah, I see it. That works . . . it's a line that goes through P and is parallel to A. So is $Q = P + kA$ the equation of the line?

SASHA: Well, it's not *the* equation for the line, but I guess it's *an* equation for the line. But it's different than ones we've used before, because it has vectors in it and not coordinates. And, there's this k in there, standing for a real number.

TONY: There's probably a special name for it, then.

As Tony and Sasha found out, the graph of $Q = P + kA$ is a straight line through P in the direction of A. An equation of the form

$$X = P + kA$$

is called a **vector equation of a line**. It works as a point-tester, too, in the sense that a point X is on this line if and only if there is a number k so that $X = P + kA$—although you'll have to do some algebra to see if such a k exists in specific cases. But it also works as a *point-generator* (see the Developing Habits of Mind below).

⟵ Context clues: Tony and Sasha are thinking of P as a point and A as a vector. Why?

⟵ The equation $X = P + kA$ is also called a **parametric equation** for the line, when you want to emphasize the role of the "parameter" k.

To find a more familiar coordinate equation for S, replace the vector letters with coordinates. If $X = (x, y)$, you can say

$$(x, y) = (2, 4) + k(3, 1)$$

and thus

$$x = 2 + 3k$$
$$y = 4 + k$$

⟵ In the next chapter, you'll become very familiar with this vector equation, so you'll have less need to move to the linear coordinate equation.

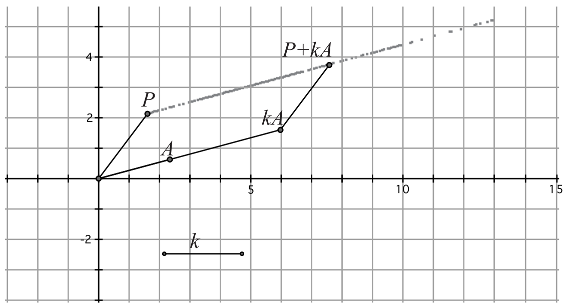
Multiply the second equation by 3 and subtract from the first to obtain

$$x - 3y = -10$$

This kind of equation probably looks more like equations of lines than you are used to, so it is pretty clear that S is in fact a line.

Developing Habits of Mind

Use the vector equation as a point-generator. You can use the equation $X = P + kA$ to generate points: consider a "slider" of length k that you can control with your mouse. As you change the length of the slider, A gets scaled by k and added to P. The varying $P + kA$ traces out ℓ.



Every value of k generates a point on the line, so the equation $X = P + kA$ is a kind of "function machine" that takes in numbers k and produces points on the line through P in the direction of A.

⟵ You can create a drawing like this with geometry software.

For You to Do

7. Tony and Sasha worked on multiples of a single vector in $\mathbb{R}^2$. What if, in the equation $X = P + kA$, X, P, and A are in $\mathbb{R}^3$? Would the equation still describe a line? Why or why not?

Minds in Action Episode 4

Tony and Sasha are working on the following problem.

Problem. In $\mathbb{R}^3$, let $A = (2, 3, 9)$ and $B = (1, 2, 5)$. What do all the linear combinations of A and B describe?

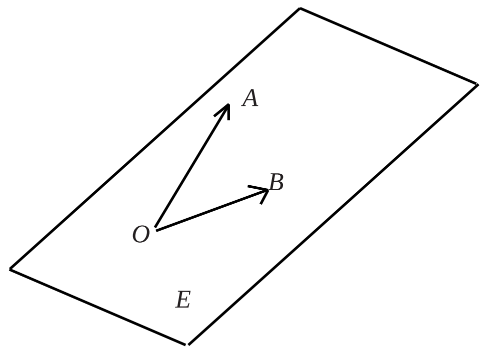
TONY: I'm on it, Sasha. "Linear combination" means we multiply A by something, and B by something, and then add them together. In other words, we want something that looks like $k_1A + k_2B$. Should I start plugging away?

SASHA: Well, let's think about it for a second. We can find three points easily enough: A, B, and O.

TONY: Oh. O?

SASHA: Yes, O, because $O = 0A + 0B$. As I was saying, we have three points: A, B, and O. Those three points aren't on the same line, so they determine a plane. Let's call that plane E.

⟵ How does Sasha know that O, A, and B are not collinear? What would happen if they were?



TONY: But I can pick anything for k_1 and anything for k_2, so won't that give us infinitely many points? What does that determine?

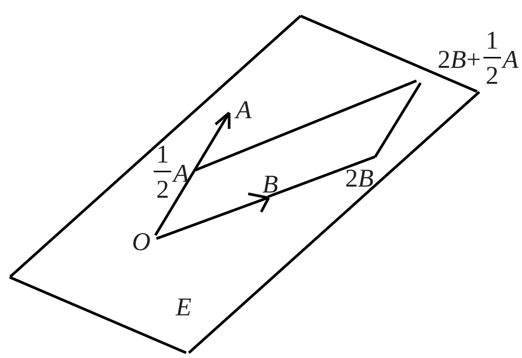
SASHA: Hold on. Say both k_1 and k_2 are 1. Then $A+B$ is another point. But we already know that completes a parallelogram, right? And if that's the case, $A+B$ should be on the same plane as the other three points, otherwise, it wouldn't make a parallelogram, but some weird twisted 3D shape.

TONY: I gotcha. That makes sense.

SASHA: And look at this. $2B$ would have to be on the same plane, too, right? I mean, it's on the line $\overleftrightarrow{OB}$, and if O and B are both on the plane E, all of the line $\overleftrightarrow{OB}$ has to be on E.

TONY: Same deal with multiples of A, like $\frac{1}{2}A$ has to be on E, too.

SASHA: And by the Parallelogram Rule again, their sum, $\frac{1}{2}A+2B$, has to be, too.



TONY: Then all these other points will have to stay on the plane E, right? Because any point k_1A will be on the line $\overleftrightarrow{OA}$, so it's on E. And any point k_2B will be on the line $\overleftrightarrow{OB}$, so it's on E, too. And the sum of any of those two points is part of a parallelogram where we know three of the points are on one plane, so the fourth has to be too.

SASHA: Brilliant, Tony! So an equation for E could be $X = k_1A + k_2B$, or, better yet, $X = k_1(2,3,9) + k_2(1,2,5)$. Wait . . . uh oh . . .

⟵ E is called the plane **spanned** by A and B.

TONY: What now?

SASHA: Well, we showed that any linear combination of A and B must be on E. But we *didn't* show it the other way . . . must every point on E be a linear combination of A and B?

TONY: It's not the same thing?

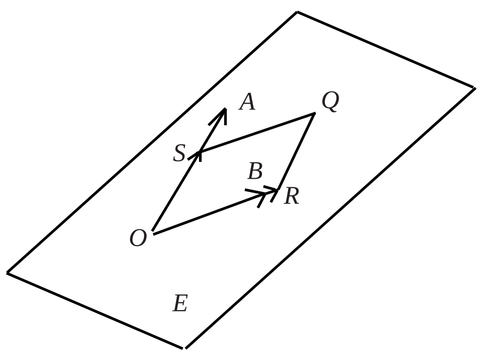
SASHA: No, we have to make sure that taking all the linear combinations doesn't leave holes in the plane.

They both sit quietly for a while, thinking.

I think I got it. Say Q is a point on E. I can draw a line through Q that's parallel to A—that line will be on E. I can also draw a line through Q parallel to B, also on E.

TONY: Good job, Sasha. The line through Q parallel to A will intersect $\overleftrightarrow{OB}$ somewhere. And the line through Q parallel to B will intersect $\overleftrightarrow{OA}$ somewhere, too. And that makes a parallelogram.

SASHA: That's what I was thinking. Say S is on $\overleftrightarrow{OA}$ and $\overrightarrow{SQ}$ is parallel to B, and say R is on $\overleftrightarrow{OB}$ and $\overrightarrow{RQ}$ is parallel to A. Here, look at my picture.



TONY: Yep. So $Q = S + R$, by the Parallelogram Rule. But since S is on $\overleftrightarrow{OA}$, it has to equal $k_1 A$ for some k_1, and R being on $\overleftrightarrow{OB}$ means it has to equal $k_2 B$ for some k_2. So, $Q = k_1 A + k_2 B$. And we're done.

SASHA: Great work, Tony.

TONY: Awesome . . . I think we've got a vector equation of a plane: $X = k_1 A + k_2 B$.

SASHA: Maybe, but something's missing, I think.

For Discussion

8. **a.** Suppose X, A, and B are vectors in $\mathbb{R}^2$, and A and B are *not* parallel. What does $X = k_1 A + k_2 B$ describe?

b. Suppose X, A, and B are vectors in $\mathbb{R}^4$, and A and B are *not* parallel. What does $X = k_1 A + k_2 B$ describe?

9. Sasha thinks something is missing from Tony's "vector equation of a plane." What do you think she means?

So, the point-tester for the plane E is "a point Q is on the plane E if and only if Q is a linear combination of A and B." This point-tester leads to the vector equation

$$X = c_1 A + c_2 B$$

You can use this equation to test any point X in $\mathbb{R}^3$ to see if it is on the plane E.

⟵ $X = c_1A + c_2B$ is also a point-generator. Why?

Example 4

Problem. Is $U = (3, 5, 14)$ on Sasha's plane E (from Episode 4)?

Solution. You might spot that $U = A + B$, so U is a linear combination of A and B, and hence it's on E. If you didn't spot that, you could set up the equations

$$U = c_1 A + c_2 B \quad \text{or}$$
$$(3, 5, 14) = c_1(2, 3, 9) + c_2(1, 2, 5)$$

Now look at it coordinate by coordinate:

$$3 = 2c_1 + c_2$$
$$5 = 3c_1 + 2c_2$$
$$14 = 9c_1 + 5c_2$$

Solve the first two equations for c_1 and c_2; $(1, 1)$ works. And $(1, 1)$ works in the last equation, too, so $c_1 = 1$ and $c_2 = 1$ is a solution. Hence, $U = 1A + 1B$, so U is on E.

Habits of Mind

It's a good idea to train yourself to check if a point is a linear combination of some other points by playing around with the numbers in your head.

For You to Do

10. Check to see if the following points are on E:
a. $(3, 5, 13)$ **b.** $(5, 8, 23)$ **c.** $(10, 16, 46)$

Example 5

Problem. Tony and Sasha showed that the plane E spanned by $A = (2, 3, 9)$ and $B = (1, 2, 5)$ has a vector equation

$$X = k_1 A + k_2 B$$

Find a coordinate equation for E.

Solution. A coordinate equation is just a point-tester for E whose test is carried out by calculating with coordinates and not vectors. Start out with the vector equation

$$X = k_1 A + k_2 B$$

and substitute $A = (2, 3, 9), B = (1, 2, 5),$ and $X = (x, y, z)$ to get

$$(x, y, z) = (2k_1 + k_2, 3k_1 + 2k_2, 9k_1 + 5k_2)$$

So,

$$x = 2k_1 + k_2$$
$$y = 3k_1 + 2k_2$$
$$z = 9k_1 + 5k_2$$

You want a relation between x, y, and z without using any of the k's. One way to start it is to eliminate k_2 from two pairs of equations:

1. Multiply the first equation by 2 and subtract the second from the result to get

$$2x - y = k_1$$

2. Multiply the second equation by 5 and the third equation by 2 and then subtract to get

$$5y - 2z = -3k_1$$

Now substitute the left-hand side of the equation into the right-hand side of the last equation to get

$$5y - 2z = -3(2x - y)$$

This equation simplifies to

$$6x + 2y - 2z = 0$$

or

$$3x + y - z = 0 \qquad (1)$$

This is a coordinate equation for E.

The derivation of the coordinate equation in this example should feel like the calculation shown in Example 4. Compare the system from Example 4 with the system from Example 5. To *prove* that the last equation is a coordinate equation for E, you'd have to show that a point (x, y, z) is on E if and only if (x, y, z) satisfies it. There are some details to be filled in (what are they?).

In the next chapter, you'll develop a much more efficient way to find coordinate equations for planes that will build on the technique developed in this example.

Exercises

1. For each set of vectors, determine whether the pairs of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent, parallel in the same direction, or parallel in the opposite direction.

⟵ Look for shortcuts. For instance, notice how solutions to earlier problems can help with later ones.

a. $A = (3, 1)$, $B = (4, 2)$, $C = (-1, 4)$, $D = (0, 5)$
b. $A = (3, 1)$, $B = (4, 2)$, $C = (0, 5)$, $D = (-1, 4)$
c. $A = (3, 1, 5)$, $B = (-4, 1, 3)$, $C = (0, 1, 0)$, $D = (14, 1, 4)$
d. $A = (-4, 1, 3)$, $B = (3, 1, 5)$, $C = (0, 1, 0)$, $D = (14, 1, 4)$
e. $A = (1, 3)$, $B = (4, 1)$, $C = (-2, 3)$, $D = (13, -7)$
f. $A = (3, 4)$, $B = (5, 6)$, $C = B - A$, $D = O$

g. $A = O$, $B = (4, 7)$, $C = (5, 2)$, $D = B + C$
h. $A = (-1, 2, 1, 5)$, $B = (0, 1, 3, 0)$, $C = (-2, 3, 2, 1)$, $D = (-1, 2, 4, -4)$
i. $A = (-1, 2, 1, 5)$, $B = (-2, 3, 2, 1)$, $C = (0, 1, 3, 0)$, $D = (-1, 2, 4, -4)$

2. For each set of vectors from Exercise 1, parts **a**–**g**, sketch the points A, B, C, D, $B - A$, and $D - C$. Use a separate coordinate system for each set.

3. Find a point P if $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$, where

$$A = (2, -1, 4), B = (3, 2, 1), \text{ and } Q = (1, -1, 6)$$

4. In $\mathbb{R}^4$, suppose $A = (3, 1, -1, 4)$, $B = (1, 3, 2, 0)$, and $C = (1, 1, -1, 3)$. If $D = (-3, a, b, c)$, find a, b, and c so that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

5. In $\mathbb{R}^n$, show that if $\overrightarrow{AB}$ is equivalent to $\overrightarrow{AC}$, then $B = C$.

6. In $\mathbb{R}^2$, show that if A, B, and O are collinear, then $B = cA$ for some number c.

7. Suppose $A = (1, 2, 3)$ and $B = (4, 5, 6)$. Is each point a linear combination of A and B? If so, give the coefficients. If not, explain why.

⟵ Try to do this problem in your head. If you get stuck, write down equations.

a. $(5, 7, 9)$ **b.** $(3, 3, 3)$ **c.** $(-5, -7, -9)$
d. $(10, 14, 18)$ **e.** $(8, 10, 12)$ **f.** $(7, 8, 9)$
g. $(7, 8, 10)$ **h.** $(1, 2, 3)$ **i.** $(1, 2, 4)$

8. Some people, especially physicists, talk about adding vectors "head to toe" in the following way: to find $\overrightarrow{AB} + \overrightarrow{CD}$, move $\overrightarrow{CD}$ to an equivalent vector starting at B, say $\overrightarrow{BQ}$. Then

$$\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AQ}$$

a. Draw a picture of how this works.
b. Show that $\overrightarrow{AQ}$ is equivalent to $(B - A) + (D - C)$.

9. In $\mathbb{R}^n$, if $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$, show that $\overrightarrow{AC}$ is equivalent to $\overrightarrow{BD}$. Illustrate geometrically in $\mathbb{R}^2$ or $\mathbb{R}^3$.

10. In $\mathbb{R}^2$, let ℓ be the line whose equation is $5x + 4y + 20 = 0$. If $P = (-4, 0)$ and $A = (-4, 5)$, show that ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A.

11. Let $P = (3, 0)$ and $A = (1, 5)$. If ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A, find a vector equation and a coordinate equation for ℓ.

12. In $\mathbb{R}^3$, let $A = (1, 0, 2)$ and $B = (0, 1, 3)$.

a. Find a vector equation and a coordinate equation for the plane spanned by A and B.

b. **Take It Further.** Find a vector equation and a coordinate equation for the plane parallel to the one you found in part **a** that passes through the point $C = (1, 1, 1)$.

13. Show that the following definition for the midpoint of a vector in $\mathbb{R}^n$ agrees with the usual midpoint formula in $\mathbb{R}^2$: the **midpoint** of $\overrightarrow{AB}$ is the point $\frac{1}{2}(A + B)$.

14. In $\mathbb{R}^4$, let $A = (-3, 1, 2, 4)$, $B = (5, 3, 6, -2)$, and $C = (1, 1, -2, 0)$. If M is the midpoint (see Exercise 13) of $\overrightarrow{AB}$ and N is the midpoint of $\overrightarrow{BC}$, show that $\overrightarrow{MN}$ is parallel to $\overrightarrow{AC}$. This exhibits a generalization of what fact in plane geometry?

15. If A, B, and C are points in $\mathbb{R}^n$, M is the midpoint (see Exercise 13) of $\overrightarrow{AB}$, and N is the midpoint of $\overrightarrow{BC}$,

a. show that $M - A = B - M$
b. show that $M - N = \frac{1}{2}(A - C)$
c. prove that $\overrightarrow{MN}$ is parallel to $\overrightarrow{AC}$

1.4 Length

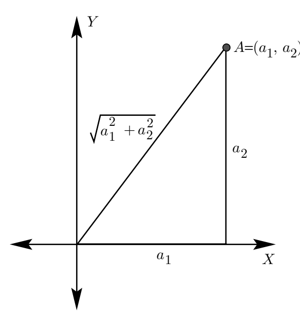
In the previous lesson, you saw that the two key attributes of a vector are its *magnitude* and *direction*, but you didn't spend much time on either one. This lesson focuses on magnitude (length).

⟵ You'll explore direction in the next chapter.

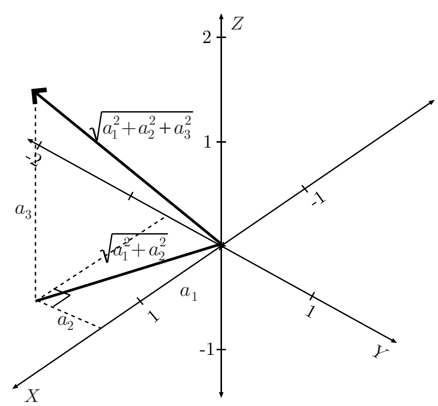
In this lesson, you will learn how to

- calculate the length of a vector and apply the algebraic properties described in the theorems
- give geometric interpretations of algebraic results that involve length
- understand how the extension program is used to define length in higher dimensions
- identify a unit vector

If $A = (a_1, a_2)$ is a vector in $\mathbb{R}^2$, the length of A can be calculated as the distance between the origin O and A using the distance formula (which is derived from the Pythagorean Theorem): $\sqrt{a_1^2 + a_2^2}$. So the length of $(3, 4)$ is 5 and the length of $(5, 1)$ is $\sqrt{26}$.



You can find a similar formula in $\mathbb{R}^3$.



If $A = (a_1, a_2, a_3)$, repeated applications of the Pythagorean Theorem shows that

$$\begin{aligned}\text{the length of } A &= \sqrt{\left(\sqrt{a_1^2 + a_2^2}\right)^2 + a_3^2} \\ &= \sqrt{a_1^2 + a_2^2 + a_3^2}\end{aligned}$$

You can define length in $\mathbb{R}^n$ by continuing the process that used the Pythagorean Theorem to go from two to three dimensions.

Definition

Let $A = (a_1, a_2, \ldots, a_n)$ be a vector in $\mathbb{R}^n$. The **length** of A, written $\|A\|$, is given by the formula

$$\|A\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Notice that while A is a vector, $\|A\|$ is a **number**.

Remember

If a is a positive real number, $\sqrt{a}$ is defined as the *positive* root of the equation $x^2 = a$.

For You to Do

1. **a.** Find the length of $(5, 3, 1)$. **b.** Find $\|(9, 3, 3, 1)\|$.

There are some fundamental properties of length that are inspired by geometry and proved by algebra.

Theorem 1.5

Let A and B be vectors in $\mathbb{R}^n$ and let c be a real number. Then

(1) $\|A\| \geq 0$, *and* $\|A\| = 0$ *if and only if* $A = O$

(2) $\|cA\| = |c|\,\|A\|$

(3) $\|A + B\| \leq \|A\| + \|B\|$

Proof. Here are the proofs of parts ((1)) and ((2)); the proof of ((3)) will be given in the next chapter.

⟵ Or, you can try to prove it yourself, right now.

(1) Since $\|A\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$, $\|A\|$ is the square root of the sum of squares, hence it is nonnegative. And $\|A\| = 0$ if and only if $\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = 0$. But for the sum of a set of nonnegative numbers to equal 0, each number must equal 0, and thus all the coordinates $a_1, a_2, \ldots, a_n$ must be 0. In other words, $A = O$.

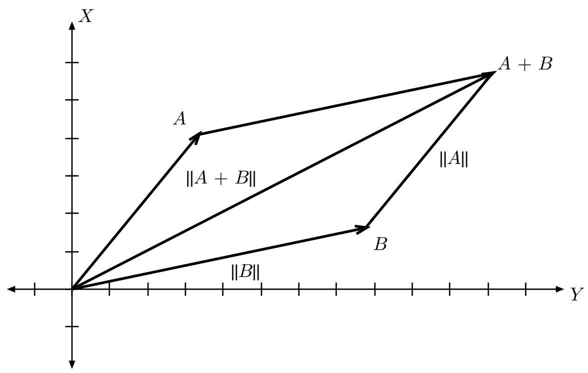
(2) If $A = (a_1, a_2, \ldots, a_n)$, then $cA = (ca_1, ca_2, \ldots, ca_n)$. So

$$\begin{aligned}\|cA\| &= \sqrt{(ca_1)^2 + (ca_2)^2 + \cdots + (ca_n)^2} \\ &= \sqrt{c^2a_1^2 + c^2a_2^2 + \cdots + c^2a_n^2} \\ &= \sqrt{c^2(a_1^2 + a_2^2 + \cdots + a_n^2)} \\ &= \sqrt{c^2} \cdot \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \\ &= |c|\,\|A\|\end{aligned}$$

■

Part ((3)) of Theorem 1.5 is known as the **Triangle Inequality** because, in $\mathbb{R}^2$, it says that if you form a triangle with vertices O, A, and $A+B$, the length of $A+B$ is less than or equal to the sum of the lengths of the other two sides:

⟵ You'll see a proof of the Triangle Inequality in $\mathbb{R}^n$ in Chapter 2.



Unit Vectors

For every nonzero vector A in $\mathbb{R}^2$ or $\mathbb{R}^3$, there is a vector with the same direction as A with length 1. If $A = (3, 4)$, this vector is $\left(\frac{3}{5}, \frac{4}{5}\right)$.

This same phenomenon occurs in $\mathbb{R}^n$, and, as usual, the proof is algebraic. Suppose A is a nonzero vector in $\mathbb{R}^n$, and suppose $\|A\| = k$ (so that $k > 0$). The vector $\frac{1}{k}A$ is in the same direction as A, and

$$\begin{aligned}\left\|\tfrac{1}{k}A\right\| &= \left|\tfrac{1}{k}\right|\,\|A\| \\ &= \tfrac{1}{k}\,\|A\| \\ &= \tfrac{1}{k}k = 1\end{aligned}$$

⟵ Where is Theorem 1.5 used here?

That is, by dividing each component of A by $\|A\|$, you get a vector in the same direction as A but with length 1.

Theorem 1.6

Let A be a nonzero vector in $\mathbb{R}^n$. There is a vector in the same direction as A with length 1; in fact, this vector is $\frac{1}{\|A\|}A$.

The vector $\frac{1}{\|A\|}A$ is called the **unit vector** in the direction of A.

⟵ Can there be more than one unit vector in any given direction?

Example 1

Problem. In $\mathbb{R}^4$, find the unit vector in the direction of $(5, 10, 6, 8)$.

Solution.

$$\|(5, 10, 6, 8)\| = 15$$

so the unit vector in the direction of $(5, 10, 6, 8)$ is $\frac{1}{15}(5, 10, 6, 8)$ or

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{8}{15}\right)$$

For You to Do

2. **a.** If $A \neq O$ is a vector in $\mathbb{R}^n$ and $c > 0$ is a number, show that the unit vector in the direction of cA is the same as the unit vector in the direction of A.
b. What if $c < 0$?
c. What if $A = O$?

Distance

In $\mathbb{R}^2$, the distance between A and B is the length of the vector $\overrightarrow{AB}$, which is the same as $\|B - A\|$. You can use this characterization of distance as a *definition* in $\mathbb{R}^n$.

Definition

The **distance between two points** A and B in $\mathbb{R}^n$, written $d(A, B)$, is defined by the equation

$$d(A, B) = \|B - A\|$$

⟵ The extension program again.

Example 2

Problem. In $\mathbb{R}^4$, show that the triangle with the following vertices is isosceles.

$$\begin{aligned} A &= (476, -306, -932, 1117) \\ B &= (-1060, -690, 220, -995) \\ C &= (140, -210, 580, 205) \end{aligned}$$

Solution. Compute the lengths of the three sides.

⟵ A calculator will help with the arithmetic.

$$\begin{aligned} d(A, B) &= \|B - A\| = \|(-1536, -384, 1152, -2112)\| = 2880 \\ d(B, C) &= \|C - B\| = \|(1200, 480, 360, 1200)\| = 1800 \\ d(A, C) &= \|C - A\| = \|(-336, 96, 1512, -912)\| = 1800 \end{aligned}$$

Since $d(B, C) = d(A, C)$, the triangle is isosceles.

The next theorem gives some important properties of the distance function. These properties generalize from the geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$. To prove them in $\mathbb{R}^n$, use Theorem 1.5.

Theorem 1.7

Let A, B, and C be points in $\mathbb{R}^n$.

(1) $d(A,B) \geq 0$, *and* $d(A,B) = 0$ *if and only if* $A = B$.

(2) $d(A,B) = d(B,A)$.

(3) $d(A,C) \leq d(A,B) + d(B,C)$.

⟵ You should illustrate each of these properties in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a sketch.

Exercises

1. Find $\|A\|$ for each of the following:

a. $A = (3,6)$ **b.** $A = (4,3,0)$ **c.** $A = (-1,3,4,1)$
d. $A = (1,0,1,0)$ **e.** $A = (4,-1,3,5)$ **f.** $A = (0,0,0,1)$

2. Let $A = (3,-1,4)$ and $B = (4,2,-1)$. Find

a. $\|A+B\|$ **b.** $\|A-B\|$
c. $\|2A+2B\|$ **d.** $\|A-2B\|^2$

3. Find the unit vector in the direction of A if

a. $A = (1,1)$ **b.** $A = (6,8)$ **c.** $A = (440,-539,330,598)$
d. $A = (1,3,0,-1)$ **e.** $A = (1,0,0)$ **f.** $A = (3,4,5)$

4. **Write About It.** Prove that, in $\mathbb{R}^2$ and $\mathbb{R}^3$, the distance between two points A and B is the length of $B - A$. Give a formula for the distance between $A = (a_1,a_2,a_3)$ and $B = (b_1,b_2,b_3)$.

5. In each exercise, find $d(A,B)$.

a. $A = (3,1), B = (4,2)$
b. $A = (1,0,1), B = (0,1,0)$
c. $A = (1,3,2), B = (4,1,3)$
d. $A = (1,3,-1,4), B = (2,1,3,8)$

6. Find the length of the sides of $\triangle ABC$ if

$$A = (-120,-1680,-115,465)$$
$$B = (680,1120,485,865)$$
$$C = (240,1659,155,267)$$

⟵ Once again, a calculator is useful for these problems.

7. Which triangles are isosceles?

a. $\triangle PQR$, where

$$P = (-1791,11089,-279)$$
$$Q = (5954,16991,7835)$$
$$R = (1234,-1209,235)$$

b. $\triangle MNP$, where

$$M = (-120, -1680, -115, 465)$$
$$N = (680, 1120, 485, 865)$$
$$P = (240, 1659, 155, 267)$$

c. $\triangle ABC$, where

$$A = (1, 1, 3, -2)$$
$$B = (-2, 4, 6, 4)$$
$$C = (5, 3, 3, 2)$$

8. Prove Theorem 1.7.

9. If A and B are points in $\mathbb{R}^n$ and M is the midpoint of $\overrightarrow{AB}$ (see Exercise 13 from Lesson 1.3), show that

$$d(A, M) = d(M, B)$$

10. Let $A = (1, 2)$, $B = (13, 4)$, and $C = (7, 10)$. Suppose M is the midpoint of $\overrightarrow{AB}$ and N is the midpoint of $\overrightarrow{BC}$. Show that $d(M, N) = \frac{1}{2}d(A, C)$.

11. Let $A = (1, -2, 2)$ and $B = (7, -5, 4)$. If M is the midpoint of $\overrightarrow{OA}$ and N is the midpoint of $\overrightarrow{OB}$, show that $d(M, N) = \frac{1}{2}d(A, B)$.

12. If A, B, and C are distinct points in $\mathbb{R}^n$, M is the midpoint of $\overrightarrow{AB}$, and N is the midpoint of $\overrightarrow{BC}$, show that

$$d(M, N) = \tfrac{1}{2}d(A, C)$$

What fact from plane geometry does this generalize?

13. The centroid of a triangle is the point where its three medians meet.

← A median of a triangle is a line segment from a vertex to the midpoint of the opposite side.

- **a.** Find the centroid of the triangle whose vertices are O, $A = (3, 1)$, and $B = (6, 5)$.
- **b.** Show that this centroid is $\frac{A+B}{3}$.
- **c.** Find the centroid of the triangle whose vertices are $Q = (4, 1)$, $Q + A = (7, 2)$, and $Q + B = (10, 6)$.
- **d.** **Take It Further.** Show that in $\mathbb{R}^2$ the centroid of a triangle whose vertices are M, N, and R is $\frac{M+N+R}{3}$.

14.
- **a.** Pick three cities, say Boston, New York, and Cleveland. Approximately where is the centroid of the triangle that has your three cities as vertices?
- **b.** What is a reasonable definition of the "population center" for three cities?
- **c.** Find the population center for your three cities using your definition.

15. Show that the points $(5, 7, 4)$, $(7, 3, 2)$, and $(3, 7, -2)$ all lie on a sphere with center $(1, 3, 2)$.

16. If A and B are nonzero vectors in $\mathbb{R}^n$ and $a = \|A\|$ and $b = \|B\|$, show that $\|bA\| = \|aB\|$.

17. Suppose A and B are nonzero vectors in $\mathbb{R}^2$. Let $U = \|B\|\, A$ and $V = \|A\|\, B$. Show that the parallelogram determined by U and V is a rhombus.

⟵ Draw a picture.

Chapter 1 Mathematical Reflections

These problems will help you summarize what you have learned in this chapter:

1. For each of the following equations, solve for A.

a. $2A + (1, 6) = (-3, 4)$
b. $-A - (5, 3, -4) = O$
c. $4A + (1, -3, -2) = 2A - (0, -5, 2)$

2. For each set of vectors, determine whether the pairs of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent, parallel in the same direction, or parallel in the opposite direction.

a. $A = (2, 5)$, $B = (6, 3)$, $C = (-5, 6)$, $D = (-1, 4)$
b. $A = (2, 5)$, $B = (6, 3)$, $C = (-5, 6)$, $D = (3, 2)$
c. $A = (-4, 2, 0)$, $B = (7, -1, 9)$, $C = (0, 0, 2)$, $D = (-11, 3, -7)$
d. $A = (2, 9, 5, -7)$, $B = (4, -3, 8, 0)$, $C = (-1, -9, 0, 5)$, $D = (1, -21, 3, 12)$

3. The vector equation of line ℓ is $X = (3, 5) + k(4, 2)$.

a. Find X if $k = 2$.
b. Find another point on line ℓ.
c. Find a coordinate equation for line ℓ. Verify that the points from parts **a** and **b** are on line ℓ using the coordinate equation.

4. Let $A = (4, 2)$ and $B = (6, -8)$. Find

a. $\|A\|$ **b.** $\|B\|$ **c.** $\|A + B\|$ **d.** $d(A, B)$

5. Let$A = (4, 2)$ and $B = (6, -8)$. O, A, B, and $A + B$ form the vertices of a parallelogram. What additional information about the parallelogram do $\|A + B\|$ and $d(A, B)$ give you?

6. How can you describe adding and scaling vectors in geometric terms?

7. How can you use vectors to describe lines in space?

8. Let $A = (3, 2)$ and $B = (-1, 4)$.

a. Calculate and graph the following: $A + B$, $2A$, $-3B$, $2A - 3B$.
b. Calculate $\|2A - 3B\|$.

Vocabulary

In this chapter, you saw these terms and symbols for the first time. Make sure you understand what each one means, and how it is used.

- coordinates
- direction
- equivalent vectors
- extension program
- initial point (tail)
- length $\|X\|$
- linear combination
- magnitude
- n-dimensional Euclidean space
- opposite direction
- ordered n-tuple
- point
- same direction
- scalar multiple
- spanned
- terminal point (head)
- unit vector
- vector
- vector equation
- zero vector

$\longleftarrow$
A and B are the same as in Exercise 4.

Chapter 1 Review

In Lesson 1.2, you learned to

- locate points in space and describe objects with equations
- use the algebra of points to calculate, solve equations, and transform expressions, all in $\mathbb{R}^n$
- understand the geometric interpretations of adding and scaling

The following problems will help you check your understanding.

1. Given $A = (2, 3)$, $B = (4, -3)$, and $C = (-5, -4)$. Calculate and plot the following:

a. $A + B$ **b.** $A + 2B$ **c.** $A + 3B$
d. $2 \cdot (A + B)$ **e.** $A + B + C$ **f.** $A + B - C$

2. In $\mathbb{R}^3$, find the equation of each of the following:

a. the x–y plane **b.** the x–z plane
c. the plane through $(-2, 3, 4)$ parallel to the x–y plane
d. the plane through $(-2, 3, 4)$ parallel to the x–z plane

3. For each of the following equations, solve for A.

a. $4A - (-4, 9) = (2, -5)$
b. $A + (-1, -7, 8) = 3A - (-11, 1, -8)$
c. $(1, 15, 2, -5) - 5A = 2(3, 0, -4, 10)$

4. For each of the following equations, find c_1 and c_2.

a. $c_1(2, -5, 3) + c_2(4, 1, 8) = (0, -11, -2)$
b. $c_1(2, -5, 3) + c_2(4, 1, 8) = (4, 12, 10)$

In Lesson 1.3, you learned to

- test vectors for equivalence using the algebra of points
- prove simple geometric theorems with vector methods
- think of linear combinations geometrically

The following problems will help you check your understanding.

5. For each set of points A, B, and Q, find a point P if $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$.

a. $A = (-2, 8), B = (3, -1),$ and $Q = (2, 5)$
b. $A = (3, 5, 7), B = (-1, -2, 4),$ and $Q = (-6, 8, 3)$

6. In $\mathbb{R}^3$, suppose $A = (2, -2, 1)$, $B = (3, 4, -2)$, and $C = (5, -1, 6)$. If $D = (3, a, b)$, find a and b so that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$. Are the vectors parallel in the same direction or in the opposite direction? How do you know?

7. Let $P = (2, -1)$ and $A = (3, 4)$. If ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A, find a vector equation and a coordinate equation for ℓ.

8. In $\mathbb{R}^3$, let $A = (3, -2, 1)$ and $B = (2, 4, 0)$.

a. Is $(14, 12, 2)$ a linear combination of A and B? If so, give the coefficients. If not, explain why not.
b. Is $(14, 12, 1)$ a linear combination of A and B? If so, give the coefficients. If not, explain why not.
c. Find a vector equation and a coordinate equation for the plane spanned by A and B.

In Lesson 1.4, you learned to

- calculate length and distance and apply the algebraic properties described in the theorems
- understand how the extension program is used to define length in higher dimensions
- identify a unit vector

The following problems will help you check your understanding:

9. Let $A = (-4, -3)$ and $B = (2, 2)$. Find

a. $\|A\|$ **b.** $\|B\|$ **c.** $\|A + B\|$
d. $\|2A\|$ **e.** $\|A - B\|$ **f.** $\|B - A\|$

10. Find the unit vector in the direction of A if

a. $A = (5, -12)$ **b.** $A = (-3, -3)$
c. $A = (2, 1, -3)$ **d.** $A = (1, 1, 1, 1)$

11. Find $d(A, B)$ if

a. $A = (-2, 4), B = (0, 5)$
b. $A = (1, 1, 1), B = (2, 2, 2)$
c. $A = (0, 3, 1, 2), B = (-2, 1, -3, 0)$

12. Find the perimeter of $\triangle ABC$ if $A = (2, 3)$, $B = (5, 9)$, and $C = (8, 0)$.

Chapter 1 Test

Multiple Choice

1. Let $A = (3, -6)$ and $B = (-4, 5)$. Which is equivalent to $A + 2B$?

A. $(-5, 4)$ **B.** $(-2, -2)$ **C.** $(-1, -1)$ **D.** $(2, -7)$

2. Which is the equation of the plane through $(1, -2, 6)$ parallel to the x–y plane?

A. $x = 1$
B. $y = -2$
C. $z = 6$
D. $x + y + z = 5$

3. Let $A = (4, 0)$, $B = (-2, -1)$, and $P = (-5, -3)$. If $\overrightarrow{AB}$ is equivalent to $\overrightarrow{PQ}$, which are the coordinates of Q?

A. $(-11, -4)$ **B.** $(-1, 2)$ **C.** $(1, -2)$ **D.** $(11, 4)$

4. In $\mathbb{R}^3$, suppose $A = (1, -4, 2)$, $B = (-5, 7, 3)$, $P = (2, -2, 3)$, and $Q = (-10, 20, 5)$. Which of these statements is true?

A. $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equivalent.
B. $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are parallel in the same direction.
C. $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are parallel in the opposite direction.
D. None of the above.

5. Let $P = (4, -7)$ and $A = (5, 2)$. If ℓ is the set of all points Q so that $\overrightarrow{PQ}$ is parallel to A, which is a coordinate equation for ℓ?

A. $2x - 5y - 43 = 0$
B. $2x + 5y + 27 = 0$
C. $4x - 7y - 6 = 0$
D. $4x + 7y + 33 = 0$

6. Let $A = (2, -4, 7)$ and $B = (-2, 1, 5)$. What is $\|A - B\|$?

A. $\sqrt{11}$ **B.** $\sqrt{13}$ **C.** $3\sqrt{5}$ **D.** $3\sqrt{17}$

Open Response

7. Solve each of the following equations for A.

a. $2A + (6, -1) = (8, 4)$
b. $3A - (4, 6, -2) = -A + 2(0, 5, 4)$

8. For each equation, find scalars c_1 and c_2. If it is not possible, explain why.

a. $c_1(1, -2, -1) + c_2(2, -3, 4) = (1, -3, -7)$
b. $c_1(1, -3, 4) + c_2(2, -6, 8) = (5, -1, 2)$

9. Let $A = (2, 5)$, $B = (4, 9)$, $C = (10, 11)$, and $D = (8, 7)$.

a. Translate the quadrilateral $ABCD$ so that A is at the origin.
b. Use the Parallelogram Rule to show that A, B, C, and D lie on the vertices of a parallelogram.
c. Sketch both parallelograms.

10. In R^3, let $A = (-2, 0, 3)$ and $B = (1, 4, 2)$. Find a vector equation and a coordinate equation for the plane spanned by A and B.

11. In $\mathbb{R}^4$, find the unit vector in the direction of A if $A = (-5, 1, -7, 5)$.

12. In R^3, let $A = (-4, 1, 5)$, $B = (2, 3, 4)$, and $C = (2, -1, 6)$. Show that $\triangle ABC$ is isosceles.

CHAPTER

2 Vector Geometry

The algebra of points and vectors that you learned about in Chapter 1 gives you the basic tools with which you can implement the extension program.

- Take a familiar geometric idea in two and three dimensions.
- Find a way to describe it algebraically with vectors.
- Use the algebra as the definition of the idea in higher dimensions.

In this chapter, you'll use vectors to describe and extend ideas like perpendicularity and angle. You'll also learn to describe lines (in $\mathbb{R}^2$ and $\mathbb{R}^3$) and planes (in $\mathbb{R}^3$) with *vector* equations. These vector equations are often much more useful than the coordinate equations you learned about in other courses, and they allow you to extend the ideas of lines and planes to higher dimensions. Along the way, you'll encounter some simple ways to calculate area and volume using vector methods.

By the end of this chapter, you will be able to answer questions like these:

1. How can you determine whether two vectors (of any dimension) are orthogonal?
2. How can you find a vector orthogonal to two given vectors in $\mathbb{R}^3$?
3. Let $A = (2, -1, 3)$, $B = (1, 1, 2)$, and $C = (2, 0, 5)$. What is an equation of the hyperplane E containing A, B, and C?

You will build good habits and skills for ways to

- use algebra to extend geometric ideas
- use vectors to prove facts about numbers
- generalize from numerical examples
- use different forms for different purposes

Vocabulary and Notation

- angle (between two vectors)
- component
- cross product
- determinant
- direction vector of a line
- dot product
- hyperplane
- initial point
- lemma
- linear equation
- normal
- orthogonal
- projection
- right-hand rule
- standard basis vectors
- vector equation of a line

2.1 Getting Started

Exercises

1. Suppose $A = (3, 1)$. Find an equation for each of the following lines:

a. the line through the origin perpendicular to A
b. the line through $P = (-3, 2)$ perpendicular to A
c. the line through $P = (3, -2)$ perpendicular to A
d. the line through $P = (6, -4)$ perpendicular to A
e. the line through $P = (0, 6)$ perpendicular to A
f. the line through $P = (0, 6)$ that's parallel to A

$\longleftarrow$ Think of A as a vector here. Write your equations in the form $ax + by = c$.

$\longleftarrow$ There's a point to these problems. Look for patterns.

2. Find a nonzero vector Q perpendicular to

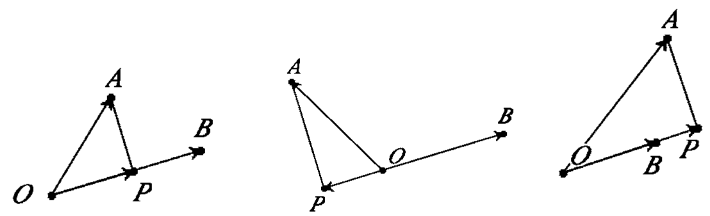
a. $A = (5, 1)$
b. $A = (3, 2)$
c. $A = (-2, 10)$
d. $A = (6, 4)$
e. $A = (a, b)$
f. both $A = (5, 0, 0)$ and $B = (0, 0, -3)$

Remember

Unless you are told otherwise, a vector starts at the origin.

Suppose A and B are vectors. If you drop a perpendicular from the head of A to the line along B, it will hit that line at a point P that's called the *projection* of A on B.

$\longleftarrow$ If $B = O$, the convention is that the projection of A on B is O.



3. Find the projection of A on B if

a. $A = (2, 9)$, $B = (10, 0)$
b. $A = (2, 9)$, $B = (-10, 0)$
c. $A = (2, 9)$, $B = (0, 6)$
d. $A = (2, 9)$, $B = (6, 4)$
e. $A = (2, 9)$, $B = (12, 8)$
f. $A = (2, 9)$, $B = (-6, -4)$
g. $A = (2, 9)$, $B = (4, 8)$
h. $A = (-8, 4)$, $B = (4, 8)$

4. **Write About It.** Given vectors A and B, describe a method for finding the projection of A on B.

5. Find the angle between each pair of vectors.

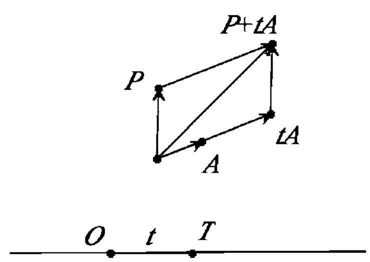
a. $A = (5, 5), B = (10, 0)$
b. $A = (5, 5), B = (-10, 0)$
c. $A = (1, \sqrt{3}), B = (0, 6)$
d. $A = (2, 9), B = (6, 4)$
e. $A = (2, 9), B = (12, 8)$
f. $A = (2, 9), B = (-6, -4)$
g. $A = (2, 9), B = (4, 8)$
h. $A = (-8, 4), B = (4, 8)$

6. Find the shortest distance from point P to line ℓ.

a. $P = (5, 6)$, ℓ is the x-axis in $\mathbb{R}^2$
b. $P = (5, 6, 7)$, ℓ is the y-axis in $\mathbb{R}^3$
c. $P = (5, 0)$, ℓ is the graph of $x = y$ in $\mathbb{R}^2$
d. $P = (5, 0)$, ℓ is the graph of $x = -y$ in $\mathbb{R}^2$
e. $P = (8, -24)$, ℓ is the graph of $7x + 4y = 25$ in $\mathbb{R}^2$
f. $P = (13, -49)$, ℓ is the graph of $7x + 4y = 25$ in $\mathbb{R}^2$
g. $P = (-387, 651)$, ℓ is the graph of $7x + 4y = 25$ in $\mathbb{R}^2$
h. $P = (3, 1)$, ℓ is the graph of $7x + 4y = 25$ in $\mathbb{R}^2$
i. $P = (0, 0)$, ℓ is the graph of $7x + 4y = 25$ in $\mathbb{R}^2$

7. **Write About It.** Given a point P and the equation for a line ℓ, describe a method for finding the distance from P to ℓ.

8. Suppose you had a sketch like this:



Here, P and A are fixed vectors. (You may instead think of P as a point and A as a vector.) O is a fixed point—the origin of a coordinate system. T has coordinates $(t, 0)$. The number t is used as a scale factor to construct tA and then $P + tA$.

Find a coordinate equation for the path of $P + tA$ as t ranges over $\mathbb{R}$ if

a. $P = (3, 5)$ and $A = (6, 1)$
b. $P = (5, -7)$ and $A = (6, 1)$
c. $P = (6, 10)$ and $A = (6, 1)$
d. $P = (0, 0)$ and $A = (6, 1)$
e. $P = (5, -7)$ and $A = (12, 2)$
f. $P = (5, -7)$ and $A = (-12, -2)$
g. $P = (p_1, p_2)$ and $A = (a_1, a_2)$

2.2 Dot Product

In Chapter 1, you explored addition of vectors and multiplication of a vector by a scalar. Dot product is another operation on vectors whose calculation may look familiar to you.

In this lesson, you will learn how to

- find the dot product of two vectors of any dimension
- determine whether two vectors are orthogonal
- use the basic properties of dot product to prove statements and solve problems

Much of the study of geometry involves lengths and angles in two dimensions. You can extend these ideas to higher dimensions by looking at them algebraically. To do so, characterize these ideas in terms of vectors. Luckily, much of the groundwork for this process has been established in analytic geometry and in trigonometry.

⟵ For instance, you spent a good deal of time proving congruence by comparing equal lengths and angle measures, and you proved similarity by comparing proportional lengths and congruent angles.

Developing Habits of Mind

Use the Pythagorean Theorem. How can you tell if two vectors are perpendicular? In $\mathbb{R}^2$, you can use slope: two lines in $\mathbb{R}^2$ are perpendicular if their slopes are negative reciprocals.In $\mathbb{R}^2$, a line is determined by its slope and a point on it.

⟵ Except that this doesn't work for horizontal and vertical lines.

Unfortunately, the idea of slope isn't quite so simple in $\mathbb{R}^3$. Sure, you can come up with ways to describe a line in $\mathbb{R}^3$ by its "steepness,"but there isn't a single number that would uniquely characterize the line.

⟵ Take two pencils. Hold one vertically and the other angled out with the erasers touching. Rotate the two without changing the angle between the two pencils. The slope appears to change as you rotate it.

Fortunately, you already know another way to test for perpendicularity: use the converse of the Pythagorean Theorem. If the side-lengths of a triangle are a, b, and c, and if $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

The Pythagorean Theorem assumes that the triangle lies in a two-dimensional plane. Here, you have two vectors that share the same tail point. So whether those vectors are in $\mathbb{R}^2$ or $\mathbb{R}^3$, there must be a plane that contains them both. So the theorem works in three dimensions as well.

⟵ You may have heard this stated as "three points determine a plane." You'll see later that another variation is "two vectors determine a plane."

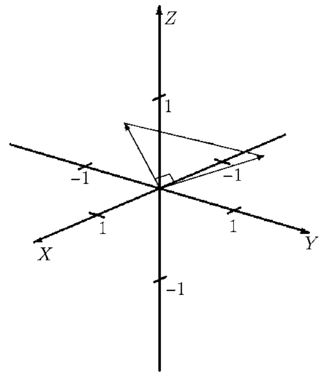
In $\mathbb{R}^3$, two vectors $A = (a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ will be perpendicular if and only if

$$\|A\|^2 + \|B\|^2 = \|A - B\|^2$$

⟵ Why $\|A - B\|$? This is $d(B, A)$.

Using the definitions of length and distance (see Lesson 1.4), this can be stated as

$$a_1^2 + a_2^2 + a_3^2 \;+\; b_1^2 + b_2^2 + b_3^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$



When you expand the right-hand side, you will see that $a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2$ ends up on both sides of the equation. Subtract that from both sides, and you get

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = 0$$

Divide both sides by -2, and the equation simplifies to

$$a_1b_1 + a_2b_2 + a_3b_3 = 0$$

All the steps in these calculations are reversible. So, $A \perp B$ if and only if the sum of the products of the corresponding coordinates is 0.

⟵ Make sure you check that all the steps are reversible. Start from "the sum of the products of the corresponding coordinates is 0" and work back to the statement about equal lengths.

For You to Do

1. Show that two vectors in $\mathbb{R}^2$, say $A = (a_1, a_2)$ and $B(b_1, b_2)$, are perpendicular if and only if

$$a_1b_1 + a_2b_2 = 0$$

So, now you have an algebraic description of what it takes for two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ to be perpendicular: the sum of the products of the corresponding coordinates has to equal 0. Because this sum is such a useful computation—not only in $\mathbb{R}^2$ and $\mathbb{R}^3$, but in any dimension—it has a name: dot product.

Definition

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be points in $\mathbb{R}^n$. The **dot product** of A and B, written $A \cdot B$, is defined by the formula

$$A \cdot B = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

Note carefully that while A and B are *vectors*, $A \cdot B$ is a *number*.

For You to Do

2. Let $A = (3, -1, 2, 4)$ and $B = (1, 5, -1, 6)$. Find $A \cdot B$.

So, now you can say that vectors A and B in $\mathbb{R}^2$ or $\mathbb{R}^3$ are perpendicular if and only if $A \cdot B = 0$. One more refinement:

Facts and Notation

While it's traditional to use the word "perpendicular" when talking about *lines* that meet at a right angle, it is more common to use the word **orthogonal** when talking about two *vectors*.

⟵ "Perpendicular" is from Latin and "orthogonal" is from Greek.

So, two vectors A and B in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if their dot product, $A \cdot B$, equals 0. Because the dot product is defined for vectors in any dimension, you can use the extension program to define orthogonal vectors in *any* dimension.

Remember

The extension program: Take a familiar geometric idea in two and three dimensions, find a way to describe it with vectors, and then use the algebra as the definition of the idea in higher dimensions.

Habits of Mind

The origin O is orthogonal to every vector. Why?

Definition

Two vectors A and B in $\mathbb{R}^n$ are said to be **orthogonal** if and only if their dot product is 0. In symbols,

$$A \perp B \Leftrightarrow A \cdot B = 0$$

For Discussion

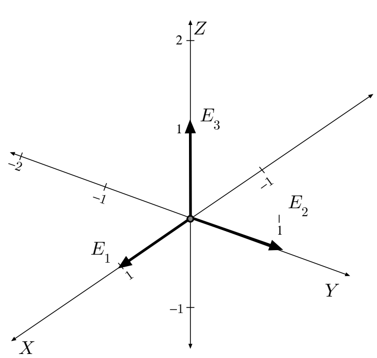
3. Why is the above definition a definition rather than a theorem?

Example 1

In $\mathbb{R}^4$, the vectors $A = (-1, 3, 2, 1), B = (1, 1, -1, 0)$, and $C = (6, -2, 4, 4)$ are **mutually orthogonal**; that is, $A \cdot B = 0$, $A \cdot C = 0$, and $B \cdot C = 0$.

For You to Do

In $\mathbb{R}^3$, the vectors $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ are mutually orthogonal.



4. Show that the dot product of any two of these vectors is 0.

5. In fact, in $\mathbb{R}^n$, let E_i be the vector whose i^{th} coordinate is 1 and whose remaining coordinates are all 0. So, $E_1 = (1, 0, \ldots, 0), E_2 = (0, 1, 0, \ldots, 0), \ldots, E_n = (0, 0, \ldots, 0, 1)$.

a. Show that $E_1, E_2, \ldots, E_n$ form a set of n mutually orthogonal vectors.

b. If $A = (a_1, a_2, \ldots, a_n)$ is an arbitrary vector in $\mathbb{R}^n$, show that $A \cdot E_i = a_i$, the i^{th} coordinate of A.

c. Use parts **a** and **b** to prove that A is orthogonal to E_i if and only if its i^{th} coordinate is zero.

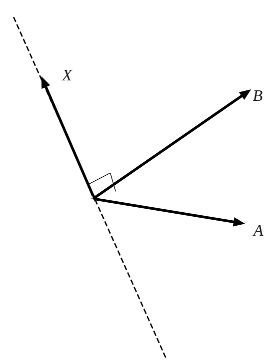
⟵ It follows that the only vector orthogonal to all of the E_i is O.

Minds in Action Episode 5

Tony and Sasha are trying to describe all the vectors that are orthogonal to both $A = (1, 1, -1)$ and $B = (-1, -2, 4)$.

Tony: The vectors A and B determine a plane, and the vectors we are looking for are those vectors X that are perpendicular to this plane. That's got to be a line.

Tony draws on the board.



Sasha: So, if we find one vector X that's orthogonal to A and B, we can just take all multiples of it. Let's see what the algebra tells us.

⟵ Notice how Sasha says "orthogonal" instead of "perpendicular."

Tony: If $X = (x, y, z)$ is orthogonal to both A and B, then $A \cdot X = 0$ and $B \cdot X = 0$. Writing this out, we have the system of two equations in three unknowns.

$$\begin{aligned} x + y - z &= 0 \\ -x - 2y + 4z &= 0 \end{aligned}$$

Now what?

Sasha: Let's just see what the algebra tells us.

Sasha starts writing on the board.

Solve the first equation for x: $x = -y + z$; substitute this for x in the second equation and simplify. We get $y = 3z$. Since $x = -y + z$ and $y = 3z$, we have $x = -2z$. That is, any vector $X = (x, y, z)$ where $x = -2z$ and $y = 3z$ will be orthogonal to both A and B. For example, letting $z = 1$, we have $x = (-2, 3, 1)$ as a solution. The general solution is $(-2z, 3z, z)$, where z can be anything it likes.

Tony: Looks messy.

Sasha: Hey! No, it's very simple: $(-2z, 3z, z) = z(-2, 3, 1)$. So the set of all vectors X orthogonal to both A and B is the set of all multiples of $(-2, 3, 1)$. Got it?

Tony: That's a line through the origin. Ohh . . . that makes sense—look at my picture.

For You to Do

6. Describe all the vectors that are orthogonal to both $C = (2, -1, 1)$ and $D = (-1, 3, 0)$.

Example 2

Problem. Characterize the set of all vectors X in $\mathbb{R}^3$ that are orthogonal to $A = (1, 3, 0)$, $B = (1, 4, 1)$, and $C = (3, 10, 2)$.

Solution. Let $X = (x, y, z)$ be a solution to the problem. Then $A \cdot X = 0$, $B \cdot X = 0$, and $C \cdot X = 0$. Writing this out, you get the system:

$$\begin{aligned} x + 3y &= 0 \\ x + 4y + z &= 0 \\ 3x + 10y + 2z &= 0 \end{aligned}$$

Solving this system, you obtain $x = y = z = 0$, so $X = O$.

⟵ If you draw a picture, it seems that the only vector in $\mathbb{R}^3$ that is orthogonal to three given vectors is O. But the algebra lets you know for sure.

Example 3

Problem. Characterize the set of vectors X that are orthogonal to $A = (1, 1, -1)$, $B = (-1, -2, 4)$, and $C = (1, 0, 2)$.

Solution. Again, you may expect the only solution to be O. Use algebra to make sure. Let $X = (x, y, z)$, and the system of equations becomes

$$\begin{aligned} x + y - z &= 0 \\ -x - 2y + 4z &= 0 \\ x + 2z &= 0 \end{aligned}$$

The third equation is twice the first equation plus the second equation. So the last equation is unnecessary and this system is equivalent to (that is, has the same solutions as) the following system:

$$\begin{aligned} x + y - z &= 0 \\ -x - 2y + 4z &= 0 \end{aligned}$$

Sasha solved this system of equations in Episode 5: it is satisfied by any multiple of $(-2, 3, 1)$, so any vector of the form $k(-2, 3, 1)$ is orthogonal to A and B, which is a line. But, since the two systems have the same solutions, that line is also orthogonal to C.

⟵ Show that any vector of the form $k(-2, 3, 1)$ is orthogonal to C.

For Discussion

7. In Example 3, you saw that the line formed by multiples of $(-2, 3, 1)$ is orthogonal to three vectors. How is this possible geometrically?

⟵ Draw a picture.

Example 4

Problem. Suppose $A = (3, 1)$ and $B = (5, 2)$. Find a vector X in $\mathbb{R}^2$ such that $A \cdot X = 4$ and $B \cdot X = 2$.

Solution. Let $X = (x, y)$. Then $A \cdot X = 3x + y$ and $B \cdot X = 5x + 2y$. So, the *vector* equations $A \cdot X = 4$ and $B \cdot X = 2$ can be written as a system of two equations in two unknowns:

$$\begin{aligned} 3x + y &= 4 \\ 5x + 2y &= 2 \end{aligned}$$

Solve this system to get $X = (6, -14)$.

The dot product is a new kind of operation: it takes two *vectors* and produces a *number*. Still, it has some familiar-looking algebraic properties that allow you to calculate with it.

Theorem 2.1 (The Basic Rules of Dot Product)
Let $A = (a_1, a_2, \ldots, a_n), B = (b_1, b_2, \ldots, b_n)$, and $C = (c_1, c_2, \ldots, c_n)$ be vectors in $\mathbb{R}^n$, and let k be a real number. Then

(1) $A \cdot B = B \cdot A$

(2) $A \cdot (B + C) = A \cdot B + A \cdot C$

(3) $A \cdot kB = kA \cdot B = k(A \cdot B)$

(4) $A \cdot A \geq 0$, *and* $A \cdot A = 0$ *if and only if* $A = O$

Proof. Here are the proofs of (1), (3), and (4). The proof of (2) is left as an exercise.

(1)
$$\begin{aligned} A \cdot B &= (a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &= b_1 a_1 + b_2 a_2 + \cdots + b_n a_n \\ &= (b_1, b_2, \ldots, b_n) \cdot (a_1, a_2, \ldots, a_n) \\ &= B \cdot A \end{aligned}$$

(3)
$$\begin{aligned} A \cdot kB &= (a_1, a_2, \ldots, a_n) \cdot k(b_1, b_2, \ldots, b_n) \\ &= (a_1, a_2, \ldots, a_n) \cdot (kb_1, kb_2, \ldots, kb_n) \\ &= a_1(kb_1) + a_2(kb_2) + \cdots + a_n(kb_n) \\ &= k(a_1 b_1) + k(a_2 b_2) + \cdots + k(a_n b_n) \\ &= k(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \\ &= k(A \cdot B) \end{aligned}$$

The proof that $kA \cdot B = k(A \cdot B)$ is exactly the same.

(4)
$$\begin{aligned} A \cdot A &= (a_1, a_2, \ldots, a_n) \cdot (a_1, a_2, \ldots, a_n) \\ &= a_1^2 + a_2^2 + \cdots + a_n^2 \end{aligned}$$

Now, the sum of squares of real numbers is nonnegative, and such a sum is 0 if and only if each $a_i = 0$. ■

Habits of Mind

Note that in the equation $kA \cdot B = k(A \cdot B)$, the insertion of parentheses changes the object that is being multiplied by k. On the left side, you are multiplying k by A, a vector in $\mathbb{R}^n$; on the right side, you are multiplying k by $A \cdot B$, a real number.

In the proof of part (4), you see the equation

$$A \cdot A = a_1^2 + a_2^2 + \cdots + a_n^2$$

The right-hand side of that equation should look familiar—you saw it in the definition of the length of a vector,

$$\|A\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

So you can substitute $A \cdot A$ for $a_1^2 + a_2^2 + \cdots + a_n^2$ to get a more efficient way to write the length of a vector.

Theorem 2.2

If A is a vector in $\mathbb{R}^n$, $\|A\| = \sqrt{A \cdot A}$.

Example 5

Problem. Show that if A and B are vectors in $\mathbb{R}^n$,

$$(A+B)\cdot(A+B) = A\cdot A + 2(A\cdot B) + B\cdot B$$

Solution. The proof of this fact is exactly the same as the proof from elementary algebra that established the identity $(a+b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned}
(A+B)\cdot(A+B) &= (A+B)\cdot A + (A+B)\cdot B \quad \text{(Theorem 2.1 (2))}\\
&= A\cdot(A+B) + B\cdot(A+B) \quad \text{(Theorem 2.1 (1))}\\
&= A\cdot A + A\cdot B + B\cdot A + B\cdot B \quad \text{(Theorem 2.1 (2))}\\
&= A\cdot A + A\cdot B + A\cdot B + B\cdot B \quad \text{(Theorem 2.1 (1))}\\
&= A\cdot A + 2(A\cdot B) + B\cdot B
\end{aligned}$$

Example 6

Problem. Let $A = (1, 4, 0, 1)$. For what values of c is $cA \cdot cA = 72$?

Solution. Using part (3) of Theorem 2.1, you have

$$cA\cdot cA = c(A\cdot cA) = c^2(A\cdot A)$$

Since $A \cdot A = 18$, this becomes $18c^2 = 72$, so $c = \pm 2$.

Example 7

Problem. Let A and B be vectors in $\mathbb{R}^n$, with $B \neq O$. Show that

$$\left(A - \left(\tfrac{A\cdot B}{B\cdot B}\right) B\right)\cdot B = 0$$

⟵ This example will be important in the next lesson.

Solution. An equation like this can be confusing, since it mixes operations between vectors and numbers. It may help to first read through the equation to check that the operations are working on the right kind of input.

- A and B are vectors.
- $A \cdot B$ and $B \cdot B$ (each the dot product of two vectors) are both numbers.
- Thus, $\frac{A \cdot B}{B \cdot B}$ (the quotient of two numbers) is also a number.
- $\left(\frac{A \cdot B}{B \cdot B}\right) B$ (a scalar multiple of a vector) is a vector.
- That means $A - \left(\frac{A \cdot B}{B \cdot B}\right) B$ (the difference of two vectors) is a vector.
- Finally, $\left(A - \left(\frac{A \cdot B}{B \cdot B}\right) B\right) \cdot B$ (the dot product of two vectors) is a number.

⟵ Note that since $B \neq O$, $B \cdot B > 0$, so division by $B \cdot B$ is okay.

To see that this number is 0, use Theorem 2.1.

$$\begin{aligned}\left(A - \left(\frac{A \cdot B}{B \cdot B}\right) B\right) \cdot B &= A \cdot B - \left(\left(\frac{A \cdot B}{B \cdot B}\right) B\right) \cdot B \\ &= A \cdot B - \left(\frac{A \cdot B}{B \cdot B}\right) (B \cdot B) \\ &= A \cdot B - A \cdot B = 0\end{aligned}$$

The next two examples show how the basic rules for dot product can be applied to geometry.

Example 8

Problem. Suppose A and B are nonzero orthogonal vectors in $\mathbb{R}^n$ and $c_1 A + c_2 B = O$. Show that $c_1 = c_2 = 0$.

Solution. Take the equation $c_1 A + c_2 B = O$ and dot both sides with A:

$$\begin{aligned}A \cdot (c_1 A + c_2 B) &= A \cdot O \\ A \cdot (c_1 A) + A \cdot (c_2 B) &= 0 \\ c_1 (A \cdot A) + c_2 (A \cdot B) &= 0\end{aligned}$$

Since A is orthogonal to B, $A \cdot B = 0$, so this last equation becomes $c_1 (A \cdot A) = 0$. Since $A \neq O$, $A \cdot A > 0$. Since $c_1 (A \cdot A) = 0$, it follows that $c_1 = 0$. To prove $c_2 = 0$, take the equation $c_1 A + c_2 B = O$ and dot both sides with B.

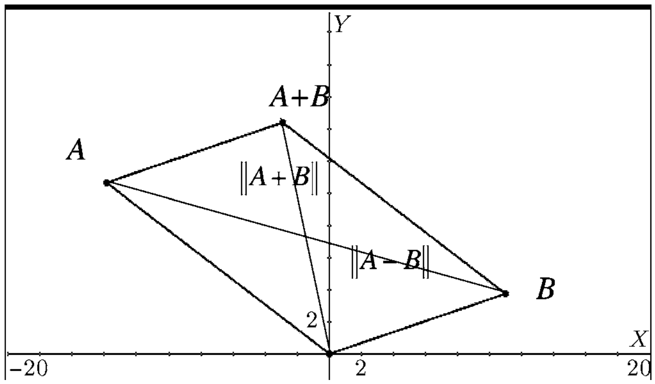
Example 9

Problem. Consider the triangle in $\mathbb{R}^3$ whose vertices are $A = (3, 2, 5)$, $B = (5, 2, 1)$, and $C = (2, 1, 3)$. Show that the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is a right angle.

Solution. You want to show that the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is a right angle. But $\overrightarrow{CA}$ is equivalent to $A - C$ and $\overrightarrow{CB}$ is equivalent to $B - C$. So, you only have to show that $A - C = (1, 1, 2)$ is orthogonal to $B - C = (3, 1, -2)$. And it is: $(A - C) \cdot (B - C) = (1, 1, 2) \cdot (3, 1, -2) = 0$.

One of the most beautiful theorems in mathematics is the Pythagorean Theorem. Does it extend to $\mathbb{R}^n$?

In $\mathbb{R}^2$, if A and B are nonzero vectors that aren't scalar multiples of each other, then $A+B$ and $A-B$ are the two diagonals of the parallelogram whose sides are A and B. So the lengths of the diagonals are $\|A+B\|$ and $\|A-B\|$.



Now, from plane geometry, if the diagonals of a parallelogram have the same length, the parallelogram is a rectangle. So, if the parallelogram determined by A and B is a rectangle, then A is perpendicular to B. Thus, A is perpendicular to B if and only if the diagonals, $A+B$ and $A-B$, have the same length. The same fact is true in $\mathbb{R}^n$, but the proof is via algebra.

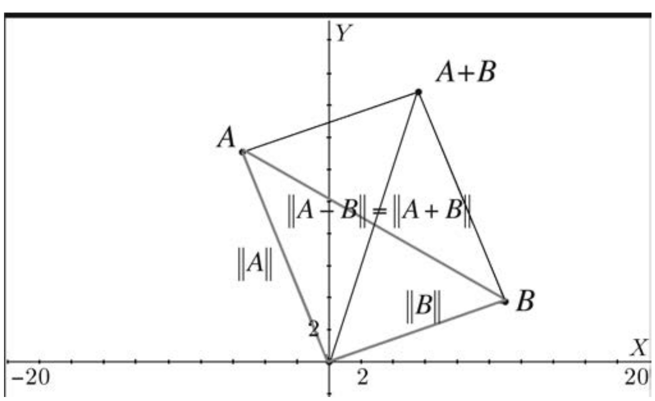
Lemma 2.3

If A and B are vectors in $\mathbb{R}^n$, A is orthogonal to B if and only if $\|A+B\| = \|A-B\|$.

⟵
A **lemma** is a result that's needed to prove another result. In this case, Lemma 2.3 is needed to prove Theorem 2.4. Usually, people discover that they need some fact when they try to prove a theorem, so they call that fact a lemma and prove it separately. The German word for lemma is *hilfsatz*—"helping statement." Note the connection between the words "lemma" and "dilemma."

Proof. Suppose $\|A+B\| = \|A-B\|$. By Theorem 2.2 and squaring both sides, you have $(A+B)\cdot(A+B) = (A-B)\cdot(A-B)$. This simplifies to $4(A\cdot B) = 0$, so $A\cdot B = 0$, and A is orthogonal to B. The proof of the converse is just as simple. ■

In the next figure, A, B, and $A-B$ are three sides of a triangle. Since A is orthogonal to B, it follows from Lemma 2.3 that $\|A-B\| = \|A+B\|$, so the hypotenuse of the right triangle whose legs are A and B has length $\|A+B\|$.



Theorem 2.4 (The Pythagorean Theorem)
If A and B are vectors in $\mathbb{R}^n$ and A is orthogonal to B, then $\|A+B\|^2 = \|A\|^2 + \|B\|^2$.

⟵ The converse of the Pythagorean Theorem also holds in $\mathbb{R}^n$ (see Exercise 9).

Proof. Since A is orthogonal to B, $A \cdot B = 0$. So,

$$\begin{aligned} \|A+B\|^2 &= (A+B)\cdot(A+B) \\ &= A\cdot A + 2(A\cdot B) + B\cdot B \\ &= A\cdot A + B\cdot B \\ &= \|A\|^2 + \|B\|^2 \end{aligned}$$

■

Developing Habits of Mind

Use algebra to extend geometric ideas. The extension program is now fully underway. Look at the proof of the Pythagorean Theorem. It looks just like an algebraic calculation with numbers—the difference is that the letters stand for vectors and the operation is dot product, so it is carried out using a different set of basic rules.

There are two advantages to a proof like this:

1. It is extremely simple and compact.
2. It establishes the result for *any* dimension.

A disadvantage is that it doesn't seem very geometric—gone are the lovely "squares upon the hypotenuse" from plane geometry. With time and practice, you'll be able to look at a calculation like this and *see* the geometry.

Exercises

1. For each A and B, find

(i) $A \cdot B$
(ii) $(A+B)\cdot(A+B)$
(iii) $(A+B)\cdot(A-B)$
(iv) $(2A+3B)\cdot(A-B)$
(v) $(A+B)\cdot(3A-3B)$

a. $A = (1,4,2,1)$, $B = (-2,1,3,2)$
b. $A = (-2,3)$, $B = (5,1)$
c. $A = (-2,3,0)$, $B = (5,1,0)$
d. $A = (1,4,2)$, $B = (2,1,-3)$
e. $A = (1,5,2,3,1)$, $B = (1,4,-2,0,-3)$

2. If A and B are vectors in $\mathbb{R}^n$ and c is a number, characterize each of the following by one of the words "vector" or "number."

a. $A\cdot(cB)$
b. $(A\cdot B)A$
c. $(A\cdot A)B + (B\cdot B)A$
d. $(cA+cB)\cdot A$
e. $((cA\cdot B)B)\cdot A$
f. $\dfrac{A\cdot B}{B\cdot B}B \quad (B \neq O)$

3. Find a nonzero vector X in $\mathbb{R}^3$ orthogonal to $(1,3,2)$.

4. Characterize all vectors X in $\mathbb{R}^3$ orthogonal to $A = (1,3,2)$ and $B = (-1,-2,1)$.

5. Characterize all vectors X in $\mathbb{R}^3$ orthogonal to $A = (1,3,2)$, $B = (-1,-2,1)$, and $C = (0,1,3)$.

6. Characterize all vectors X in $\mathbb{R}^3$ orthogonal to $A = (1,3,2)$, $B = (-1,-2,1)$, and $C = (0,1,4)$.

7. Let $A = (5,3,3)$, $B = (1,3,1)$, and $C = (2,6,-1)$. One angle of $\triangle ABC$ is a right angle. Which one is it?

8. In $\mathbb{R}^4$, let $A = (4,2,5,3)$, $B = (1,1,1,1)$, and $C = (0,4,2,-1)$. Show that $\triangle ABC$ is a right triangle.

9. Prove the converse of the Pythagorean Theorem: if A and B are vectors in $\mathbb{R}^n$ so that $\|A+B\|^2 = \|A\|^2 + \|B\|^2$, then A is orthogonal to B.

10. Suppose A and B are vectors in $\mathbb{R}^n$ and X is a vector orthogonal to both A and B. Show that X is orthogonal to every vector of the form $c_1A + c_2B$.

11.
- **a.** If $A_1, A_2, \ldots, A_r$ are vectors in $\mathbb{R}^n$, a **linear combination** of $A_1, A_2, \ldots, A_r$ is a vector B which can be written as $c_1A_1 + c_2A_2 + \cdots + c_rA_r$ for some numbers $c_1, c_2, \ldots, c_r$. Show that $(3,9,4,7)$ is a linear combination of $(1,3,0,1)$, $(2,1,4,2)$, and $(-1,2,0,3)$, while $(3,9,4,8)$ is not.
- **b.** If $A_1, A_2, \ldots, A_r$ are vectors in $\mathbb{R}^n$, and if X is orthogonal to A_i for each i, show that X is orthogonal to every linear combination of $A_1, A_2, \ldots, A_r$.
- **c.** If A_1, A_2, A_3 and B_1, B_2 are two sets of vectors in $\mathbb{R}^n$ so that each B_j is orthogonal to all the A_i's, show that any linear combination of A_i's is orthogonal to any linear combination of the B_j's.

12. Let A and B be nonzero vectors in $\mathbb{R}^n$ and suppose C is a linear combination of A and B. If C is orthogonal to both A and B, show that $C = O$.

13. Suppose $A = (2,11,10)$. Find
- **a.** $\|A\|$
- **b.** another vector B that has the same length as A
- **c.** a vector B that has the same length as A and is orthogonal to A
- **d.** a vector B that has the same length as A, that is orthogonal to A, and that has integer coordinates

⟵ A point whose coordinates are all integers is called a **lattice point**.

14. Two adjacent vertices of a square are at O and $A = (-14, -2, 5)$.

a. How many such squares are there?
b. Find two vertices that will complete the square.
c. Find two vertices that complete the square and that are lattice points.

15. Show that the triangle whose vertices are $A = (4, 3, 0, 1)$, $B = (5, 4, 1, 2)$, and $C = (5, 2, 1, 0)$ is an isosceles right triangle.

16. If A and B are vectors in $\mathbb{R}^n$, show that $(A + B) \cdot (A - B) = A \cdot A - B \cdot B$.

17. If A and B are vectors in $\mathbb{R}^n$, show that $(A+B)\cdot(A+B) = A\cdot A+B\cdot B$ if and only if A is orthogonal to B.

18. Show that if A is orthogonal to B, A is orthogonal to every scalar multiple of B.

19. Let $A_1, A_2, \ldots, A_r$ be mutually orthogonal nonzero vectors in $\mathbb{R}^n$. If $c_1A_1 + c_2A_2 + \cdots + c_rA_r = O$, show that each $c_i = 0$.

20. Let A and B be vectors so that $(A+B)\cdot(A+B) = (A-B)\cdot(A-B)$. Show that A is orthogonal to B.

21. Let $A = (2, 1, 3, 2)$ and $B = (2, 1, 4, 1)$. Show that $A - \left(\frac{A\cdot B}{B\cdot B}\right) B$ is orthogonal to B.

22. True or false? If $A \cdot B = A \cdot C$ and if $A \neq O$, then $B = C$.

23. Find all vectors X that have length 3 and that are orthogonal to both $(-1, 0, 1)$ and $(3, 2, -4)$.

24. Prove part (2) of Theorem 2.1.

25. Derman wrote down the incorrect definition of dot product. His notes say that

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2$$

TONY: Derman, it's supposed to be $a_1b_1 + a_2b_2 + a_3b_3$.

DERMAN: OK, but my dot product obeys the same basic rules as the ones in Theorem 2.1.

Is Derman right? Explain.

26. If A and B are vectors in $\mathbb{R}^n$, show that

$$A \cdot A + B \cdot B \geq 2(A \cdot B)$$

$\longleftarrow$ When will both sides be equal?

27. Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and

$$C = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

Show that C is orthogonal to both A and B.

28. Use Theorem 2.2 and the Basic Rules for Dot Product (Theorem 2.1) to prove parts (1) and (2) of Theorem 1.5 from Lesson 1.4.

> **Theorem 1.5.** *Let A and B be vectors in $\mathbb{R}^n$ and let c be a real number. Then*
> (1) $\|A\| \geq 0$, *and* $\|A\| = 0$ *if and only if* $A = O$
> (2) $\|cA\| = |c| \, \|A\|$

29. If A and B are vectors, show that

a. $\|A+B\|^2 = \|A\|^2 + \|B\|^2 + 2(A \cdot B)$
b. $\|A+B\|^2 - \|A-B\|^2 = 4(A \cdot B)$
c. $\|A+B\|^2 + \|A-B\|^2 = 2\|A\|^2 + 2\|B\|^2$

⟵ What does this say in $\mathbb{R}^2$? Draw a picture.

30. If A is a scalar multiple of B, show that

$$(A \cdot A)(B \cdot B) - (A \cdot B)^2 = 0$$

31. If A, B, and C are vectors in $\mathbb{R}^n$ so that $d(A, B) = d(C, B)$, show that

$$\frac{A \cdot A}{2} - A \cdot B = \frac{C \cdot C}{2} - C \cdot B$$

32. If A and B are orthogonal vectors in $\mathbb{R}^n$ so that $\|A\| = \|B\| = 1$, show that $d(A, B) = \sqrt{2}$.

33. Let A and B be vectors in $\mathbb{R}^n$, and let c and d be numbers. Prove the following identities.

⟵ The last two of these identities will be useful in later sections.

a. $(A+2B) \cdot (A-B) = A \cdot A + A \cdot B - 2B \cdot B$
b. $(cA+B) \cdot (cA+B) = c^2(A \cdot A) + 2c(A \cdot B) + B \cdot B$
c. $(cA+dB) \cdot (cA+dB) = c^2 A \cdot A + 2cdA \cdot B + d^2 B \cdot B$
d. $(\|B\| A + \|A\| B) \cdot (\|B\| A + \|A\| B)$
$= 2\|A\| \|B\| (\|A\| \|B\| + A \cdot B)$
e. $\left(\frac{A \cdot B}{B \cdot B} B\right) \cdot \left(\frac{A \cdot B}{B \cdot B} B\right) = \frac{(A \cdot B)^2}{B \cdot B} \qquad (B \neq O)$

34. Show that if $A \cdot B = A \cdot (B + C)$, then A is orthogonal to C.

35. If A and B are vectors in $\mathbb{R}^n$, $B \neq O$, and c is a number so that $A - cB$ is orthogonal to B, show that $c = \frac{A \cdot B}{B \cdot B}$.

36. If A and B are vectors in $\mathbb{R}^n$, $B \neq O$, and $P = \frac{A \cdot B}{B \cdot B} B$, show that

a. $P \cdot P = A \cdot P$
b. $A \cdot A = P \cdot P + (A - P) \cdot (A - P)$

37. If A and B are vectors in $\mathbb{R}^n$, $B \neq O$, and c is a number so that $A - cB$ is orthogonal to $A + cB$, show that

$$c = \pm \frac{\|A\|}{\|B\|}$$

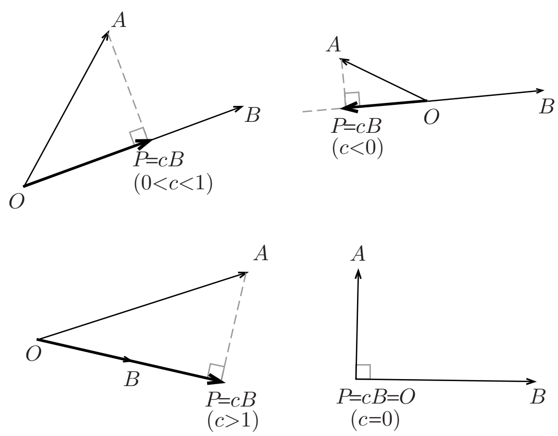
2.3 Projection

An important use of dot product is to determine the projection of a vector onto another vector. Projection has a number of applications throughout the study of linear algebra.

In this lesson, you will learn how to

- find the component of a vector along another vector
- find the projection of a vector along another vector

In $\mathbb{R}^2$, if A and B are vectors and $B \neq 0$, the projection of A along B is the vector obtained by intersecting the line through A perpendicular to the line along B with that line. This is illustrated for several situations in the following figure.



In each case, P is the projection of A along B. To extend this notion to $\mathbb{R}^n$, you need to describe P with vector algebra.

(1) $P = cB$ for some scalar c.

(2) $\overrightarrow{PA}$ meets the line along B at right angles.

Condition ((2)) can be reformulated.

$2'$. $A - P$ is orthogonal to B.

Since $P = cB$, you will have a formula for P if you can determine c. To this end, use condition ((2)′).

$$\begin{aligned} (A-P)\cdot B &= 0 \\ A\cdot B - P\cdot B &= 0 \\ P\cdot B &= A\cdot B \\ cB\cdot B &= A\cdot B \\ c(B\cdot B) &= A\cdot B \\ c &= \frac{A\cdot B}{B\cdot B} \qquad \text{(since } B \neq O, B\cdot B \neq 0) \end{aligned}$$

⟵
Fill in a reason for each step.

Hence $P = \dfrac{A\cdot B}{B\cdot B}B$. This formula makes sense in $\mathbb{R}^n$.

Definition

Let A and B be vectors in $\mathbb{R}^n$, with $B \neq 0$.

- The **component** of A along B, written $\text{comp}_B\, A$, is the *number*

$$\text{comp}_B\, A = \frac{A\cdot B}{B\cdot B}$$

- The **projection** of A along B, written $\text{Proj}_B\, A$, is the *vector* defined by the formula

$$\text{Proj}_B\, A = (\text{comp}_B\, A)B = \frac{A\cdot B}{B\cdot B}\, B$$

Habits of Mind

Find general purpose tools. The projection ties together quite a bit of geometry into one little package. You'll see in the exercises and in the next sections that it's a very useful tool.

Example 1

Problem. In $\mathbb{R}^2$, let $A = (5, 1)$ and $B = (-3, 0)$. Find $\text{Proj}_B\, A$.

Solution. You might expect that $\text{Proj}_B\, A = (5, 0)$. (Why?) Use the definition to find that

$$\text{comp}_B\, A = \frac{-15}{9} = \frac{-5}{3}$$

so

$$\text{Proj}_B\, A = \frac{-5}{3}(-3, 0) = (5, 0)$$

Example 2

Problem. In $\mathbb{R}^4$, let $A = (-3, 1, -2, 4)$ and $B = (1, 1, 2, 0)$. Find $\text{Proj}_A\, B$ and $\text{Proj}_B\, A$.

Solution.

$$\text{comp}_B\, A = \frac{-6}{6} = -1 \qquad \text{so} \qquad \text{Proj}_B\, A = (-1, -1, -2, 0)$$

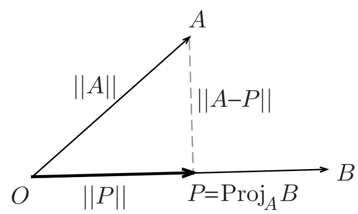
Similarly, $\text{Proj}_A\, B = -\frac{1}{5}A = \left(\frac{3}{5}, \frac{-1}{5}, \frac{2}{5}, -\frac{4}{5}\right)$.

Note that if A and B are vectors in $\mathbb{R}^n$ and $P = \operatorname{Proj}_B A$, then P satisfies conditions ((1)) and ((2)′), which were used to characterize projections in $\mathbb{R}^2$. Clearly, P is a multiple of B, and Example 7 from Lesson 2.2 shows that $A - P$ is orthogonal to B.

Example 3

Problem. If A and B are vectors in $\mathbb{R}^n$ and $B \neq 0$, let $P = \operatorname{Proj}_B A$ and show that

$$\|A\|^2 = \|P\|^2 + \|A - P\|^2$$



Solution 1. Let $c = \operatorname{comp}_B A$ so that $P = cB$. Since $A - P$ is orthogonal to B, $A - P$ is also orthogonal to P (Lesson 2.2, Exercise 18), so $P \cdot (A - P) = 0$. But then

$$\begin{aligned}
\|P\|^2 + \|A - P\|^2 &= P \cdot P + (A - P) \cdot (A - P) \\
&= P \cdot P + A \cdot A - 2A \cdot P + P \cdot P \\
&= 2P \cdot P - 2A \cdot P + A \cdot A \\
&= -2P \cdot (A - P) + A \cdot A \\
&= -2\,(P \cdot (A - P)) + A \cdot A \\
&= A \cdot A = \|A\|^2
\end{aligned}$$

Solution 2. Since $A - P$ is orthogonal to P, you can apply the Pythagorean Theorem to $A - P$ and P.

$$\|A - P\|^2 + \|P\|^2 = \|A - P + P\|^2 = \|A\|^2$$

Exercises

1. For each of the given points A and B, find $d(A, B)$, $\operatorname{Proj}_B A$, and $\operatorname{Proj}_A B$.

a. $A = (3, 1), B = (4, 2)$ **b.** $A = (1, 0, 1), B = (0, 1, 0)$
c. $A = (1, 3, 2), B = (4, 1, 3)$ **d.** $A = (1, 3, -1, 4), B = (2, 1, 3, 8)$

2. Suppose A, B, and C are vectors. Characterize each expression with one of the words "vector," "number," or "meaningless."

a. $\|A + B\|$ **b.** $A \cdot (B + C)$ **c.** $\|A \cdot B\|$
d. $\operatorname{Proj}_B A$ **e.** $\|\operatorname{Proj}_B A\|$ **f.** $\operatorname{Proj}_A(\operatorname{comp}_B A)$
g. $(A \cdot B) \cdot A$ **h.** $A - A \cdot B$ **i.** $\|(A \cdot B)C\|$
j. $d(A, \operatorname{Proj}_B A)$ **k.** $A \cdot \operatorname{Proj}_A B$ **l.** $(\operatorname{comp}_B A\ \operatorname{comp}_A B)C$

3. If $A = (1, 0, 3)$ and $B = (-1, 2, 0)$, find

a. $d(A, \text{Proj}_B A)$ **b.** $\|\text{Proj}_A B\|$
c. $\text{Proj}_A(\text{Proj}_B A)$ **d.** $\text{comp}_B A \ \text{comp}_A B$
e. $(\text{Proj}_B A - A) \cdot B$ **f.** $A \cdot \text{Proj}_B A$

4. If A and B are nonzero vectors, show that A is orthogonal to B if and only if $\text{Proj}_B A = O$.

5. If A and B are nonzero vectors, show that $\text{comp}_B A$ and $\text{comp}_A B$ cannot have opposite signs.

⟵ Is there a geometric interpretation of this common sign?

6. If A and B are nonzero vectors, show that

$$\frac{\text{comp}_B A}{\text{comp}_A B} = \left(\frac{\|A\|}{\|B\|}\right)^2$$

7. If A and B are vectors in $\mathbb{R}^n$ $(B \neq O)$, show that $\|\text{Proj}_B A\| = \frac{|A \cdot B|}{\|B\|}$.

8. Show that if A and B are nonzero vectors,

$$\frac{\|A\|}{\|\text{Proj}_B A\|} = \frac{\|B\|}{\|\text{Proj}_A B\|}$$

What is the value of this common ratio?

9. If A and B are vectors in $\mathbb{R}^n$ $(B \neq O)$, and A is a scalar multiple of B, show that $\text{Proj}_B A = A$.

10. Suppose $A = (1, 4, -1)$. Find

a. the projection of A on the x–y plane
b. the projection of A on the x–z plane
c. the projection of A on the y–z plane

⟵ What does it mean to project a vector on a plane? Part of this problem is for you to figure out a reasonable answer.

11. Suppose A and B are nonzero points in $\mathbb{R}^2$. Show that the area of the triangle whose vertices are A, B, and O is $\frac{1}{2}\sqrt{(A \cdot A)(B \cdot B) - (A \cdot B)^2}$.

Hint: Show that the area is $\frac{1}{2}\|B\|\sqrt{\|A\|^2 - \|\text{Proj}_B A\|^2}$ and then simplify.

12. Use Exercise 11 to show that if A and B are vectors in $\mathbb{R}^2$, $(A \cdot A)(B \cdot B) - (A \cdot B)^2 \geq 0$.

13. Suppose A and B are nonzero points in $\mathbb{R}^n$ and let $P = \text{Proj}_B A$.

a. Show that

$$\|A\|^2 \geq \|P\|^2$$

with equality if and only if $A = P$.

b. Use this to show that

$$(A \cdot A) \geq \frac{(A \cdot B)^2}{(B \cdot B)}$$

⟵ See Example 3 in this lesson.

14. Use Exercise 11 to find the area of the triangle whose vertices are

a. $(0, 0), (3, 1), (7, 0)$
b. $(0, 0), (4, -2), (5, 3)$
c. $(0, 0), (5, 2), (-1, -3)$
d. $(1, 3), (2, 1), (7, -2)$

⟵ For part **d**, translate to $(0, 0)$.

15. Suppose $A = (1, 4, -1)$ and $B = (-4, 0, 2)$. Let $\mathfrak{P}$ be the parallelogram whose vertices are O, A, B, and $A + B$.

a. Find the vertices of $\mathfrak{P}'$, the projection of $\mathfrak{P}$ on the x-y plane.
b. Find the vertices of $\mathfrak{P}''$, the projection of $\mathfrak{P}$ on the x-z plane.
c. Find the vertices of $\mathfrak{P}'''$, the projection of $\mathfrak{P}$ on the y-z plane.
d. Find the areas of $\mathfrak{P}'$, $\mathfrak{P}''$, and $\mathfrak{P}'''$.

$\longleftarrow$ $\mathfrak{P}'$, $\mathfrak{P}''$, and $\mathfrak{P}'''$ are also parallelograms. Can you prove it?

2.4 Angle

The geometric image of vectors in $\mathbb{R}^2$ allows you to think about the angle between two vectors. Working with such an image even lets you measure that angle in a familiar way.

In this lesson, you will learn how to

- find the angle between two vectors in any dimension
- understand and use the triangle inequality in $\mathbb{R}^n$

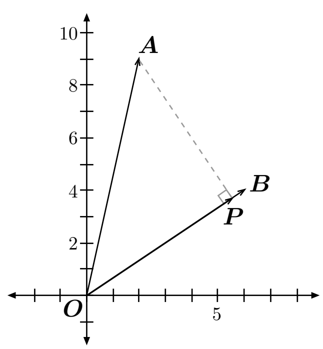
Minds in Action Episode 6

Sasha and Tony are thinking about Exercise 5 from Getting Started (Lesson 2.1).

SASHA: Say, Tony, I was thinking about projection. I bet we can use it to find the angle between two vectors. Remember in the Getting Started, we tried to find the angle between $A = (2, 9)$ and $B = (6, 4)$?

TONY: Vaguely. How did we solve it then?

SASHA: Using the Law of Cosines. But look:



If I drop a perpendicular from A to B, I get a right triangle. And P is the projection of A onto B. Now finding the cosine is pretty basic.

TONY: Wow, Sasha, how do you come up with these crazy things! So that new triangle has sides $\|A\|$, $\|\text{Proj}_B\, A\|$, and—

SASHA: We don't need that third side. I can use cosine with just those two:

$$\cos\theta = \frac{\|\text{Proj}_B\, A\|}{\|A\|}$$

TONY: But wait, how is this easier?

SASHA: Well, we've also seen that numerator before. Here it is, Exercise 7 from Lesson 2.3: *If A and B are vectors in $\mathbb{R}^n$ ($B \neq O$), show that* $\|\text{Proj}_B\, A\| = \frac{|A \cdot B|}{\|B\|}$. So, in our case,

$$\frac{|A \cdot B|}{\|B\|} = \frac{|(2,9)\cdot(6,4)|}{\sqrt{(6,4)\cdot(6,4)}} = \frac{|12+36|}{\sqrt{36+16}} = \frac{48}{\sqrt{52}}$$

TONY: And $\|A\| = \sqrt{(2,9)\cdot(2,9)} = \sqrt{85}$. So $\cos\theta = \frac{48}{\sqrt{52}} \cdot \frac{1}{\sqrt{85}}$. That makes $\theta = \cos^{-1}\left(\frac{48}{\sqrt{52}\sqrt{85}}\right)$.

SASHA: Yeah, that's what I got before. It works out to be about 43.8°.

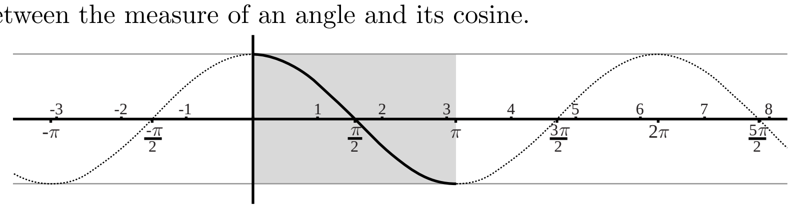
TONY: Hmm . . . I bet we're on to something here.

Sasha and Tony have started using the new facts about vectors to algebraically find the measure of an angle in $\mathbb{R}^2$. The next step is to see if their discovery helps to think about angles in higher dimensions.

Developing Habits of Mind

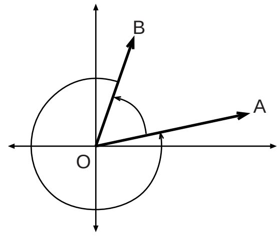
Use algebra to extend geometric ideas. You are looking for a way to measure the angle between two vectors in $\mathbb{R}^n$, but you don't want to be restricted to the geometry of $\mathbb{R}^2$. So you want to use the geometry of $\mathbb{R}^2$ to measure an angle algebraically in a way that lets you extend it to any dimension. The best tool for that job? As Sasha and Tony discovered, it's trigonometry.

As Sasha and Tony discovered, you can calculate the measure of an angle between vectors using trigonometry, basing your calculation on the lengths of vectors. Since you already know how to find the length of the projection of one vector onto another, cosine is the best choice. Cosine also has another benefit: on the interval $0 \le \theta \le \pi$, there is a one-to-one correspondence between the measure of an angle and its cosine.



So, for angles within that interval, knowing the cosine of an angle is enough to tell you what the angle is. And, if you can find a way to compute the cosine of the angle between two vectors from what you know about the vectors, that will provide a way of defining the angle between two vectors in $\mathbb{R}^n$.

When you look at two vectors in $\mathbb{R}^2$, you might see two different angles that could be considered "between A and B": one could be considered "the angle from A to B" and one "the angle from B to A." For any two vectors in $\mathbb{R}^2$, these two measurements will always add to 2π. One will measure between 0 and π, and the other will measure between π and 2π. It doesn't



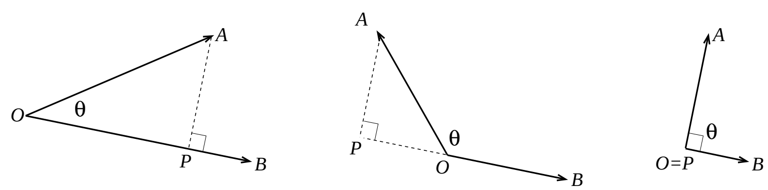
⟵ When measuring angles, you could also "wrap around" the axis any number of times, adding (or subtracting) multiples of 2π (or, using degrees, 360°) to the angle. But your goal is to find a *unique* number, so you can ignore these equivalent measures here.

necessarily matter which you choose as long as you're consistent. So you can consider the unique angle between two nonzero vectors A and B in $\mathbb{R}^2$ to be the unique angle θ determined by A and B that satisfies the restriction $0 \leq \theta \leq \pi$.

⟵ In fact, for any angle θ, $\cos(2\pi - \theta) = \cos\theta$, so the cosine would be the same whether you pick the larger or smaller angle.

So now, to invoke the extension program, you want to finalize a formula for the cosine of the angle between two vectors in $\mathbb{R}^2$ and use that formula as the *definition* for the cosine of the angle between two vectors in $\mathbb{R}^n$.

Suppose A and B are nonzero vectors, and θ is the angle between A and B.



In each figure, $P = \operatorname{Proj}_B A$. In the first case, where θ is acute, you can use right triangle trigonometry, like Sasha and Tony did, to say that $\cos\theta = \frac{\|P\|}{\|A\|}$. Since $P = cB$, where $c = \operatorname{comp}_B A$, you can say that

$$\cos\theta = \frac{\|cB\|}{\|A\|} = \frac{|c|\,\|B\|}{\|A\|}$$

Now, since θ is acute, $c > 0$, so $|c| = c$. Also, from the definition of component, you know that $c = \frac{A\cdot B}{B\cdot B}$, so

$$\cos\theta = \left(\frac{A\cdot B}{B\cdot B}\right)\frac{\|B\|}{\|A\|}$$

Finally, since $\|B\| = \sqrt{B\cdot B}$, then $\|B\|^2 = B\cdot B$, and so

$$\cos\theta = \left(\frac{A\cdot B}{\|B\|^2}\right)\frac{\|B\|}{\|A\|} = \frac{A\cdot B}{\|A\|\,\|B\|}$$

For You to Do

1. You just saw that in $\mathbb{R}^2$,

$$\cos\theta = \frac{A\cdot B}{\|A\|\,\|B\|}$$

when $0 < \theta < \frac{\pi}{2}$. Show that this formula works for any angle θ where $0 \leq \theta \leq \pi$.

⟵ Split your work into these four additional cases:

- $\frac{\pi}{2} < \theta < \pi$
- $\theta = \frac{\pi}{2}$
- $\theta = 0$
- $\theta = \pi$

For any angle θ where $0 \leq \theta \leq \pi$, you can calculate its cosine using the formula

$$\cos\theta = \frac{A\cdot B}{\|A\|\,\|B\|}$$

This formula uses only the length of a vector and the dot product, both of which can be calculated in $\mathbb{R}^n$ for any n, not just $\mathbb{R}^2$. So it appears to be a great candidate for the extension program: you can define the angle between any two vectors A and B in $\mathbb{R}^n$ to be the unique angle between 0 and π that satisfies this equation. But there's one more issue to check: in $\mathbb{R}^2$, you know that the cosine of an angle ranges between -1 and 1. How can you be sure the formula will always produce numbers in that range in any $\mathbb{R}^n$?

In other words, to extend this formula to $\mathbb{R}^n$, you first need to show that $-1 \leq \frac{A \cdot B}{\|A\|\|B\|} \leq 1$.

Theorem 2.5 (Cauchy-Schwarz Inequality)
If A and B are vectors in $\mathbb{R}^n$, then $\|A\| \, \|B\| \geq |A \cdot B|$.

⟵
The inequality is named after Augustin-Louis Cauchy (1789–1857) and Herman Schwarz (1843–1921). It has other names as well, and it is used all over mathematics.

Proof. Let A and B be vectors in $\mathbb{R}^n$ ($B \neq 0$) and let $P = \text{Proj}_B \, A$. From Example 3 from Lesson 2.3, you know that

$$\|A\|^2 = \|P\|^2 + \|A - P\|^2$$

Since $\|A - P\| \geq 0$, you can thus say that $\|A\|^2 \geq \|P\|^2$. Now, $P = \frac{A \cdot B}{B \cdot B} B$, so

⟵
Make sure you understand the reason for each step in this proof.

$$\begin{aligned}
\|A\|^2 &\geq \left\| \frac{A \cdot B}{B \cdot B} B \right\|^2 \\
\|A\|^2 &\geq \left(\frac{A \cdot B}{B \cdot B} \right)^2 \|B\|^2 \\
\|A\|^2 &\geq \left(\frac{A \cdot B}{\|B\|^2} \right)^2 \|B\|^2 \\
\|A\|^2 &\geq \frac{(A \cdot B)^2}{\|B\|^2} \\
\|A\|^2 \|B\|^2 &\geq (A \cdot B)^2 \\
(\|A\| \, \|B\|)^2 &\geq (A \cdot B)^2 \\
\|A\| \, \|B\| &\geq |A \cdot B|
\end{aligned}$$

■

For You to Do

2. Use the Cauchy-Schwarz Inequality to show that

$$-1 \leq \frac{A \cdot B}{\|A\| \, \|B\|} \leq 1$$

Because $\frac{A \cdot B}{\|A\|\|B\|}$ is the cosine of a unique angle θ in $\mathbb{R}^2$ so that $0 \leq \theta \leq \pi$, and because you have shown that, even in $\mathbb{R}^n$, it never exceeds the range of the cosine function, you can feel confident using it as a definition of cosine in $\mathbb{R}^n$.

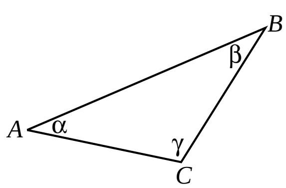
Definition

The **angle between two nonzero vectors** in $\mathbb{R}^n$, A and B, is the unique angle θ that satisfies $0 \le \theta \le \pi$ and $\cos\theta = \frac{A \cdot B}{\|A\|\|B\|}$.

For You to Do

3. Suppose $A = (5, 5) and B = (-3, 0)$. Find the angle θ between A and B.

Example



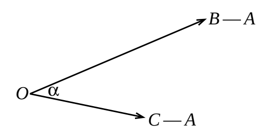
Problem. Find angle α of $\triangle ABC$, as shown in this figure, where $A = (-13, -66, 76)$, $B = (5, 60, 31)$, and $C = (27, 46, 60)$.

Solution. To find $\cos\alpha$, find the cosine of the angle between $B - A$ and $C - A$. So,

$$\cos\alpha = \frac{(B - A) \cdot (C - A)}{\|B - A\| \, \|C - A\|} = \frac{15552}{135 \cdot 120} = \frac{24}{25}$$

And thus, $\alpha = \cos^{-1} \frac{24}{25} \approx 16.26°$.

⟵ First move A to the origin.



For You to Do

4. Find angles β and γ of $\triangle ABC$ from the example above.

5. Find the angle between $A = (1, 3, -1, 2)$ and $B = (4, 1, -3, 0)$.

Here's another application of the Cauchy-Schwarz Inequality. You can use it to prove part (3) of Theorem 1.5 from Lesson 1.4, which is typically called the Triangle Inequality.

Theorem (The Triangle Inequality)

If A and B are vectors in $\mathbb{R}^n$, then $\|A + B\| \le \|A\| + \|B\|$.

Proof.

$$\begin{aligned}
\|A + B\|^2 &= (A + B) \cdot (A + B) \\
&= A \cdot A + 2(A \cdot B) + B \cdot B \\
&= \|A\|^2 + 2(A \cdot B) + \|B\|^2 \\
&\le \|A\|^2 + 2\,|A \cdot B| + \|B\|^2 \\
&\le \|A\|^2 + 2\,\|A\|\,\|B\| + \|B\|^2 \\
&= (\|A\| + \|B\|)^2
\end{aligned}$$

⟵ Give a reason for every step.

So, $\|A+B\|^2 \le (\|A\|+\|B\|)^2$; since both $\|A+B\|$ and $\|A\|+\|B\|$ are positive, you can take the square root of both sides, giving the desired result. ■

Developing Habits of Mind

Use vectors to prove ideas about numbers. The Cauchy-Schwarz Inequality makes a statement about vectors. You can restate it using coordinates: if $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, then

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \le \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

This remarkable statement is about ordinary numbers—in this form, it has nothing to do with vectors. While proving it can be quite difficult using only the algebra of numbers, as you've seen, it is pretty straightforward to do so using linear algebra.

Exercises

1. Find $\cos\theta$ if θ is the angle between A and B.

a. $A = (3,4)$, $B = (0,7)$
b. $A = (1,1,1)$, $B = (1,1,0)$
c. $A = (2,1,0)$, $B = (5,-3,4)$
d. $A = (-3,1,2,5)$, $B = (4,1,3,-4)$
e. $A = (-2,1,3,2), B = (1,1,1,-1)$
f. $A = (-5,0), B = (1,\sqrt{3})$

2. Find the cosine of each angle of $\triangle ABC$ where

a. $A = (1,5,2)$, $B = (2,6,3)$, $C = (2,5,1)$
b. $A = (1,-1)$, $B = (\sqrt{3},\sqrt{3})$, $C = (\sqrt{3}+1,0)$
c. $A = (10,68,56), B = (-22,-156,136), C = (-150,100,-120)$

3. **a.** Use trigonometry to show that the angle between $A = (1,1)$ and $B = (1,\sqrt{3})$ is $\frac{\pi}{12}$.
b. Show that $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

4. If A and B are nonzero vectors in $\mathbb{R}^n$, and c and d are positive scalars, show that the angle between cA and dB is the same as the angle between A and B.

5. If A and B are nonzero vectors in $\mathbb{R}^n$, show that

$$\text{comp}_B\, A \;\; \text{comp}_A\, B = \cos^2\theta$$

where θ is the angle between A and B.

6. If A and B are nonzero vectors in $\mathbb{R}^n$ and θ is the angle between A and B, show that

$$\|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2\,\|A\|\,\|B\|\cos\theta$$

(In $\mathbb{R}^2$, this is the Law of Cosines.)

7. In $\mathbb{R}^3$, let $A = (\sqrt{3}, \sqrt{3}, 1)$, $B = (-1+\sqrt{3}, 1+\sqrt{3}, 1)$, and $C = (-1, 1, 1)$. Show that the angles of the triangle whose vertices are A, B, and C are $90°$, $60°$, and $30°$. Verify that the length of the shorter leg is one half the length of the hypotenuse.

8. Let A and B be nonzero vectors in $\mathbb{R}^n$, and suppose $\|\text{Proj}_B A\| = \frac{\sqrt{3}}{2}\|A\|$. Show that the angle between A and B is $30°$.

9. If A and B are nonzero vectors in $\mathbb{R}^n$ so that $\|A\| = \|B\|$, show that $A + B$ bisects the angle between A and B. Draw a picture in $\mathbb{R}^2$.

10. Suppose A and B are nonzero vectors in $\mathbb{R}^n$, and θ is the angle between A and B.

a. If $\cos\theta = 1$, show that $A = cB$ where $c > 0$.
b. If $\cos\theta = -1$, show that $A = cB$ where $c < 0$.

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Hint: Let $c = \text{comp}_B A$. Show that $(A - cB) \cdot (A - cB) = 0$.

11. If A and B are points in $\mathbb{R}^n$ and $A = cB$ where $c > 0$, show that $\|A + B\| = \|A\| + \|B\|$.

12. If A, B, and C are points in $\mathbb{R}^n$, show that $\|A + B + C\| \leq \|A\| + \|B\| + \|C\|$.

13. If A and B are points in $\mathbb{R}^n$, show that $\|A - B\| \geq \|A\| - \|B\|$.

14. Find X in $\mathbb{R}^3$ so that X is orthogonal to $(2, 0, -1)$, $\|X\| = 9$, and X makes a $45°$ angle with $(0, 1, 1)$.

15. Find a vector A in $\mathbb{R}^3$ so that $\|A\| = 9$, A is orthogonal to $(4, 0, -1)$, and $\cos\theta = \frac{28}{45}$ where θ is the angle between A and $(4, 3, 0)$.

16. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$$

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Hint: Use the Cauchy-Schwarz Inequality.

17. **Take It Further.** In vector language, the triangle inequality says that for any vectors A and B in $\mathbb{R}^n$, it is true that $\|A + B\| \leq \|A\| + \|B\|$. In geometric language, the Triangle Inequality says that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

a. Working only in $\mathbb{R}^2$, show the *geometric* interpretation of $\|A + B\| \leq \|A\| + \|B\|$. In other words, show what $\|A + B\| \leq \|A\| + \|B\|$ has to do with triangles.
b. Use $\|A + B\| \leq \|A\| + \|B\|$ to prove the Cauchy-Schwartz Inequality.