Preface

History of the Book. This book started with one purpose and ended with a different purpose. In 2002 a former student, one of the best I had taught, approached me with a book about mathematical finance in his hand. He wanted a reading course about the subject because he was thinking about a career in the area. I flipped through the book briefly and saw that it was too advanced for a reading course even with a very good student. I was aware that the topic combined probability theory and partial differential equations, both interests of mine. Instead of a reading course, I agreed to conduct a seminar on mathematical finance if enough others had an interest. There *were* others, including undergraduates, graduate students in finance and economics, and even some faculty.

I soon found that there were no books or introductions to the subject suitable for mathematics students at the upper undergraduate level. I began to gather my seminar notes and organize them.

After three years of the seminar, it grew into a popular course for senior-level students from mathematics, finance, actuarial science, computer science, and engineering. The variety of students and their backgrounds refined the content of the course. The course focus was on combining finance concepts, especially derivative securities, with probability theory, difference equations, and differential equations to derive consequences, primarily about option prices.

In late 2008 security markets convulsed and the U. S. economy went into a deep recession. The causes were many, and are still debated, but one underlying cause was because mathematical methods had been applied in financial situations where they did not apply [**66**]. At the same time for different reasons, mathematical professional organizations urged a new emphasis on mathematical modeling. The course and the associated notes evolved in response, with an emphasis on uses and abuses of modeling.

Additionally, a new paradigm in mathematical sciences, combining modeling, statistics, visualization, and computing with large data sets, sometimes called "big data" or more formally data analytics, was maturing and becoming common. Data analytics is now a source of employment for many mathematics majors. The topic of finance and economics is a leader in data analytics because of the existing large data sets and the measurable value in exploiting the data.

The result is the current book combining modeling, probability theory, difference and differential equations focused on quantitative reasoning, data analysis, probability, and statistics for economics and finance. The book uses all of these topics to investigate modern financial instruments that have enormous economic influence but are hidden from popular view because many people wrongly believe these topics are esoteric and difficult.

Purpose of the Book. The purpose is to provide a textbook for a capstone course focusing on mathematical modeling in economics and finance. There are already many fine books about mathematical modeling in physical and biological sciences. This text is for an alternative course for students interested in the economic sciences instead of the classical sciences. This book combines mathematical modeling, probability theory, difference and differential equations, numerical solution and simulation, and mathematical analysis in a single course for undergraduates in mathematical sciences. I hope the style is engaging enough that it can also be enjoyably read as an introduction by any individual interested in these topics.

I understand that this introductory modeling approach makes serious concessions to completeness and depth, financial accuracy, and mathematical rigor. Phillip Protter is an expert on mathematical finance, and in a review of an elementary text on mathematical finance [**51**] he makes the following remarks:

> Mathematical finance … is a difficult subject, requiring a broad array of knowledge of subjects that are traditionally considered hard to learn.
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> The mathematics involved in the Black–Scholes paradigm is measure-theoretic probability theory, Brownian motion, stochastic processes including Markov processes and martingale theory, Ito's stochastic calculus, stochastic differential equations, and partial differential equations. Those prerequisites give one entry to the subject, which is why it is best taught to advanced PhD students. One might expect an American undergraduate to know calculus-based probability theory and to have had some exposure to PDEs and perhaps, if one is lucky, an economics course or two, but not much more. Therefore, any attempt to teach such a subject to undergraduates is fraught with compromise …
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> Perhaps it is the same with mathematical finance: it simply is not (yet?) meant to be an undergraduate subject. In a way that is too bad, because the subject is beautiful and powerful, and expertise in it is much needed in industry.

Combining economic and financial modeling with probability, stochastic processes, and differential equations along with quantitative reasoning and data analysis with some simulation and computing provides an inviting entry into deeper aspects of this "beautiful and powerful" subject.

The goals of the book are the following.

(1) To understand the properties of stochastic processes such as sequences of random variables, coin tossing games, Brownian motion, and the solutions of stochastic differential equations as a means for modeling financial instruments for the management of risk.

(2) To use financial instruments for the management of risk as motivations for the detailed study of stochastic processes and solutions of stochastic differential equations.

(3) To introduce standard stochastic processes at the level of the classic references by Karlin and Taylor, and Feller. The book proves some mathematical statements at the level of elementary analysis; some more advanced statements have heuristic motivation without proof, and some advanced results are stated without proof.

(4) To emphasize the mathematical modeling process applied to a modern area that is not based on physical science yet still leads to classical partial differential equations and numerical methods. The field of mathematical finance is only 50 years old, uses leading-edge mathematical and economic ideas, and has some controversial foundational hypotheses. Mathematical finance is also data rich and even advanced results are testable in the market. Using ideas illustrated daily in financial news, the book applies the full cycle of mathematical modeling and analysis in a nontrivial, but still accessible, way that has economic applications.

(5) To reach a point where the students thoroughly understand the derivation and modeling of financial instruments, advanced financial models, advanced stochastic processes, partial differential equations, and numerical methods at a level sufficient for beginning graduate study in mathematics, finance, economics, actuarial science, and for entry-level positions in the sophisticated financial services industry.

The general area of stochastic processes and mathematical finance has many textbooks and monographs already. This book differs from them in the following ways:

- Most books on stochastic processes have a variety of applications, while this book concentrates on financial instruments for the management of risk as motivations for the detailed study of mathematical modeling with stochastic processes. The emphasis is on the modeling process, not the financial instruments.

- Most books on mathematical finance assume either prerequisite knowledge about financial instruments or sophisticated mathematical methods, especially measure-based probability theory and martingale theory. This book serves as a introductory preparation for those texts.

- This book emphasizes the practice of mathematical modeling, including post-modeling analysis and criticism, making it suitable for a wider audience.

Overall, this book is an extended essay in mathematical modeling applied to financial instruments from the simplest binomial option models to the Black–Scholes–Merton model, with some excursions along the way.

Intended Audience and Background. This book is primarily for undergraduate students in mathematics, economics, finance, and actuarial science. Students in physical sciences, computer science, and engineering will also benefit from the book with its emphasis on modeling and the uses and limits of modeling. Graduate students in economics, finance, and business will benefit from the nonmeasure theoretic based introduction to mathematical finance and mathematical modeling.

This book is for students after a course on calculus-based probability theory. To understand the explanations and complete the exercises:

(1) the reader should be able to calculate joint probabilities of independent events.

(2) the reader should be able to calculate binomial probabilities and normal probabilities using direct calculation, tables, and computer or calculator applications.

(3) the reader should be able to recognize common probability distributions such as binomial probabilities and calculate probabilities from them.

(4) the reader should be able to calculate means and variances for common probability distributions.

(5) the reader should be familiar with common statistical concepts of parameter point evaluations and of confidence intervals and hypothesis testing.

(6) the reader should have a familiarity with compound interest calculations, both continuous compounding and periodic compounding.

(7) the reader should be able to be able to perform interest calculations to find present values, future values, and simple annuities.

The text also assumes general knowledge of linear algebra, especially about solutions of linear nonhomogeneous equations in linear spaces. A familiarity with solving difference equations, also called recurrence equations and recursions, is helpful but not essential. Where needed, the solution of the specific difference equations uses elementary methods without reference to the general theory. Likewise, a familiarity with differential equations is helpful but not essential since the text derives specific solutions when necessary, again without reference to the general theory. Naturally, a course in differential equations will deepen understanding and provide another means for discussing mathematical modeling, since that is often the course where many students first encounter significant mathematical modeling of physical and biological phenomena. Concepts from linear algebra also enter into discussions about Markov processes, but this text does not make the deeper connections. Ideas from linear algebra pervade the operators and data structures in the program scripts.

Program Scripts. An important feature of this book is the simulation scripts in the R language that accompany most sections. Scripts are in the R language because it is a popular open source language widely used for data analysis. The simulation scripts illustrate the concepts and theorems of the section with numerical and graphical evidence. The scripts are part of the “Rule of 3” teaching philosophy of presenting mathematical ideas symbolically, numerically, and graphically.

The programs are springboards for further experimentation, not finished apps for casual everyday use. The scripts are minimal in size, in scope of implementation and with minimal output. The scripts are not complete, stand-alone, polished applications, rather they are proof-of-concept starting points. The reader should run the scripts to illustrate the ideas and provide numerical examples of the results in the section. The scripts provide a seed for new scripts to increase the size and scope of the simulations. Increasing the size can often demonstrate convergence or increase confidence in the results of the section. Increasing the size can also demonstrate that although convergence is mathematically guaranteed, sometimes the rate of convergence is slow. The reader is also encouraged to change the output, to provide more information, to add different graphical representations, and to investigate rates of convergence.

The scripts are not specifically designed to be efficient, either in program language implementation or in mathematical algorithm. Efficiency is not ignored, but it is not the primary consideration in the construction of the scripts. Similarity of the program algorithm to the mathematical idea takes precedence over efficiency. One noteworthy aspect of both the similarity and efficiency is the use of vectorization along with other notational simplifications such as recycling. Vectorized scripts look more like the mathematical expressions found in the text, making the code easier to understand. Vectorized code often runs much faster than the corresponding code containing loops.

The scripts are not intended to be a tutorial on how to do mathematical programming in R. A description of the algorithm used in the scripts is in each section. The description is usually in full sentences rather than the more formal symbolic representation found in computer science pseudo-code. Given the description and some basic awareness of programming ideas, the scripts provide multiple examples for study. The scripts provide a starting point for investigating, testing, and comparing language features from the documentation or from other sources. The scripts use good programming style whenever possible, but clarity, simplicity, and similarity to the mathematics are primary considerations.

Connections to MAA CUPM guidelines. The nature of the text as an interdisciplinary capstone text intentionally addresses each of the cognitive and content recommendations from the Mathematical Association of America (MAA) Committee on the Undergraduate Program in Mathematics (CUPM) curriculum for courses and programs in mathematical sciences.[1]

Cognitive Recommendations.

(1) Students should develop effective thinking and communication skills.

An emphasis in the text is on development, solution, and subsequent critical analysis of mathematical models in economics and finance. Exercises in most sections ask students to write comparisons and critical analyses of the ideas, theories, and concepts.

(2) Students should learn to link applications and theory.

The entire text is committed to linking methods and theories of probability and stochastic processes and difference and differential equations to modern applications in economics and finance. Each chapter has a specific application of the methods to a model in economics and finance.

(3) Students should learn to use technological tools.

Computing examples in the modern programming language R appear throughout and many exercises encourage further adaptation and experimentation with the scripts.

(4) Students should develop mathematical independence and experience open-ended inquiry.

Many exercises encourage further experimentation, data exploration, and up-to-date comparisons with new or extended data.

Content Recommendations.

(1) Mathematical sciences major programs should include concepts and methods from calculus and linear algebra.

The text makes extensive use of calculus for continuous probability and through differential equations. The text extends some of the ideas of calculus to the domain of stochastic calculus. Linear algebra appears throughout, from the theory of solutions of linear nonhomogeneous equations in linear spaces to operators and data structures in the program scripts.

(2) Students majoring in the mathematical sciences should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major.

    Mathematical proofs are not emphasized, because rigorous methods for stochastic processes need measure-theoretic tools from probability. However, when elementary tools from analysis are familiar to students, then the text provides proofs or proof sketches. Derivation of the solution of specific difference equations and differential equations appears in detail but without reference to general solution methods.

(3) Mathematical sciences major programs should include concepts and methods from data analysis, computing, and mathematical modeling.

    Mathematical modeling in economics and finance is the reason for this book. Collection and analysis of economic and financial data from public sources is emphasized throughout, and the exercises extend and renew the data. Providing extensive simulation of concepts through computing in modern scripting languages is provided throughout. Exercises encourage the extension and adaptation of the scripts for more simulation and data analysis.

(4) Mathematical sciences major programs should present key ideas and concepts from a variety of perspectives to demonstrate the breadth of mathematics.

    As a text for a capstone course in mathematics, the text uses the multiple perspectives of mathematical modeling, ideas from calculus, probability and statistics, difference equations, and differential equations, all for the purposes of a deeper understanding of economics and finance. The book emphasizes mathematical modeling as a motivation for new mathematical ideas and the application of known mathematical ideas as a framework for mathematical models. The book emphasizes difference and differential equations to analyze stochastic processes. The analogies between fundamental ideas of calculus and ways to analyze stochastic processes is also emphasized.

(5) All students majoring in the mathematical sciences should experience mathematics from the perspective of another discipline.

    The goal of the text is to focus on mathematics modeling as a tool for understanding economics and finance. Collection and analysis of economic and financial data from public sources using mathematical tools is emphasized throughout.

(6) Mathematical sciences major programs should present key ideas from complementary points of view: continuous and discrete, algebraic and geometric, deterministic and stochastic, exact and approximate.

    The text consistently moves from discrete models in finance to continuous models in finance by developing discrete methods in probability into continuous time stochastic process ideas. The text emphasizes the differences between exact mathematics and approximate models.

(7) Mathematical sciences major programs should require the study of at least one mathematical area in depth, with a sequence of upper-level courses.

    The text is for a capstone course combining significant mathematical modeling using probability theory, stochastic processes, difference equations, and differential equations to understand economic and finance at a level beyond the usual undergraduate approach to economic and finance using only calculus ideas.

(8) Students majoring in the mathematical sciences should work, independently or in a small group, on a substantial mathematical project that involves techniques and concepts beyond the typical content of a single course.

    Many of the exercises, especially those that extend the scripts or that call for more data and data analysis, are suitable for projects done either independently or in small groups.

(9) Mathematical sciences major programs should offer their students an orientation to careers in mathematics.

    Financial services, banking, insurance and risk, financial regulation, and data analysis combined with some knowledge of computing are all growth areas for careers for students from the mathematical sciences. This text is an introduction to all of those areas.

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