2

Arithmetic Growth Models

Recognizing and analyzing patterns is an important part of applying mathematics to real world problems. Accordingly, Chapter 1 focused on number patterns, and particularly, patterns described by simple rules. Now we begin a systematic study of various types of rules. This chapter will look at sequences such as

$$3, 7, 11, 15, 19, 23, 27, \cdots$$

in which each term represents a constant increase (or decrease) from the preceding term. These sequences are said to exhibit *arithmetic growth*. We will work out general properties of their difference equations, functional equations, and graphs, and will see how such sequences can be applied.

Later chapters will look at other types of rules, exploring the properties of corresponding sequences. As we proceed, the rules and properties will become increasingly complicated, and the methods discussed in Chapter 1 will be indispensable. In contrast, most or all of the ideas we develop in this chapter can be understood on the basis of common sense and basic quantitative reasoning. The goal is for students to become familiar with the methods of Chapter 1 in very comfortable surroundings. With this in mind, readers should make it a point to practice using difference equations, functional equations, subscript or parenthesis notation, and the terminology of recursive and direct patterns, even if they do not seem necessary to understand examples or complete homework exercises.

Many applications concern changes that occur over time. These often involve streams of data values that occur at regular intervals, e.g., every hour or every day or every month. In such a context, the concept of arithmetic growth can be stated succinctly as follows:

> **Arithmetic Growth Assumption:** Under the assumption of Arithmetic Growth, equal periods of time result in equal increases of the variable.

For example, in Chapter 1 we considered the average concentration of carbon dioxide (CO_2) for the entire atmosphere on a yearly basis (see page 2). We noticed that each year the concentration increased by approximately the same amount, 2 parts per million. Based on this observation we can formulate a *model* for CO_2 concentration with the assumption that *average atmospheric concentration increases by* 2 *parts per million each year.* That is an arithmetic growth assumption; it leads to an arithmetic growth model. The model only approximates reality, because in our data the annual increases are only approximately equal to 2 parts per million.

2.1 Properties of Arithmetic Growth

Arithmetic growth models all share several common features. For example, they have similar difference equations, graphs, and functional equations. The common features are implications of arithmetic growth, so knowing about them can help us decide whether an arithmetic growth assumption is appropriate. On the other hand, they are also aspects of the model that we can apply once arithmetic growth is assumed. For both reasons, in this section we focus on common features of arithmetic growth models.

Recognizing Arithmetic Growth Sequences. There is an extended version of a data table that makes it clear at a glance whether or not a sequence exhibits arithmetic growth. To illustrate it, consider the sequence $2, 8, 14, 20, 26, 32, \cdots$. Considering the initial term to be a_0, we can list the position numbers and terms in Table 2.1.

Table 2.1. A data table with a column of differences.

n	a_n	Difference
0	2	
		6
1	8	
		6
2	14	
		6
3	20	
		6
4	26	
		6
5	32	

The table also has an extra column for the differences between successive terms. From $a_0 = 2$ to $a_1 = 8$, there is an increase of 6. In other words, the *difference* $a_1 - a_0 = 6$, so we enter that in the difference column between the lines for $n = 0$ and $n = 1$. In the same way, each entry in the difference column is found by subtracting an entry of the a_n column from the entry directly below it. This difference is entered in the table cell that touches the two entries that were subtracted.

In an arithmetic growth sequence each successive term increases by the same amount over the preceding term, so all of the differences are the same. This is immediately apparent in the table. Because the difference column entries are all equal, we can see at a glance that this sequence is an instance of arithmetic growth.

In contrast, let us construct an extended data table for this sequence: 5.1, 7.7, 10.5, 13.0, 15.5, 18.1, defining the starting term as b_0. See Table 2.2.

Because the entries in the difference column are not all the same, this sequence is not an example of arithmetic growth. In this case, because the differences are all pretty close in value, we might consider the sequence to be approximately following an

Table 2.2. The entries in the difference column are not all equal for this sequence. Therefore it is not an arithmetic growth sequence.

n	b_n	Difference
0	5.1	
		2.6
1	7.7	
		2.8
2	10.5	
		2.5
3	13.0	
		2.5
4	15.5	
		2.6
5	18.1	

arithmetic growth pattern. But the rule is: a sequence is an arithmetic growth sequence if and only if all the entries in the difference column are identical.

Difference Equations. For the CO_2 model, if the starting concentration is 381 and we assume an annual increase of 2 parts per million, the resulting sequence is

$$381, 383, 385, 387, \cdots.$$

Introducing c as a label, we can represent the terms using subscript notation. We make the starting position number 0, so that $c_0 = 381, c_1 = 383$, and so on. Then the sequence follows the difference equation

$$c_{n+1} = c_n + 2.$$

This is a typical arithmetic growth difference equation.

This same difference equation is valid whether we call the starting position number 0, 1, or any other whole number. Thus, if we say $c_1 = 381$, $c_2 = 383$, and so on, it will still be true that $c_{n+1} = c_n + 2$. The literal meaning of the equation is *each term is two more than the preceding term*, and that holds for the sequence $381, 383, 385, 387, \cdots$ regardless of the position number attached to 381.

Let us consider some other examples of arithmetic growth. In a study of a flu epidemic, the equation

$$p_{n+1} = p_n + 500$$

might be used to indicate that the number of people who have been infected is going up by 500 per month. Here p_n represents the number of people who have been infected by the start of month n. To use this equation we need an initial value. Taking $p_0 = 1000$ would indicate that at the start time for the model, 1000 people had already been infected.

Similarly, the equation

$$f_{n+1} = f_n + 10$$

could be used to represent the fine for an overdue book at the library where fines increase by 10 cents per day. If the book is returned n days after the due date, the fine will be f_n. In this case, the way we have described the problem implies that $f_0 = 0$, because if the book is returned *zero* days past the due date, there will be no fine.

For a somewhat different example, a sequence g_n could represent the amount (in grams) of available fuel after n months of operation in a model of a satellite propulsion system. The equation

$$g_{n+1} = g_n - 4.35$$

would indicate a consumption of 4.35 grams of fuel each month. Again we need an initial value to use the difference equation. We take $g_0 = 3{,}100$, indicating the amount of fuel available at the start time for the model.[1] In this case the variable is shrinking rather than growing, but because the amount of change is constant we still consider this to be arithmetic *growth*. The successive terms can be thought of as growing smaller. This is also referred to as *negative growth*, or *decay*.

Comparing the difference equations for all of these examples,

$$\begin{aligned} c_{n+1} &= c_n + 2, \\ p_{n+1} &= p_n + 500 \\ f_{n+1} &= f_n + 10 \end{aligned}$$

and

$$g_{n+1} = g_n - 4.35,$$

we see a common form. Each difference equation says that any term of the sequence is obtained by adding a constant to the preceding term. Even the equation for g_{n+1} can be thought of as adding a constant, if we write it in the form

$$g_{n+1} = g_n + (-4.35).$$

All of these examples are instances of a general form that applies in all arithmetic growth models. The general form is given as follows.

Arithmetic Growth Difference Equation: Every arithmetic growth sequence follows a difference equation of the form

$$a_{n+1} = a_n + d, \tag{2.1}$$

where d represents a constant added amount, also called the *common difference.*

This serves as a template for difference equations in all arithmetic growth models. We use a as a generic label, understanding that in any specific application of arithmetic growth, some other label might take a's place. Similarly, we use d as a place holder for whatever added amount appears in any specific case.

Looked at in this way, (2.1) defines a whole family of closely related difference equations, specifically, the arithmetic growth difference equations. Notice that d is a special kind of variable. In any specific arithmetic growth model, d will be replaced by an actual number, and will remain constant throughout our analysis of the model. The value of d only changes if we modify the model. Thus, d is useful for understanding how different models are related to each other, rather than for analyzing a specific model. Such a variable is referred to as a *parameter*. Parameters will arise in later chapters in connection with other families of difference equations. They are also familiar in algebra where they are used in families of equations. For example, in the generic linear equation $y = mx + b$, m and b are parameters. Like the d in (2.1), they must be replaced with numbers to obtain the equation of a particular line.

The letter d should be thought of as standing for the word *difference*. In any arithmetic growth sequence, the numerical difference between one term and the next is a constant. The parameter d is equal to this constant *difference*.

[1]The difference equation and initial value for this model are based on data from the Algerian satellite ALSAT-2A. See [**34**].

The assumption of arithmetic growth, as described verbally in the box on page 39, leads to a difference equation of the form of (2.1). Actually, the arithmetic growth assumption implies more than the difference equation. In the CO_2 example, the difference equation says that the concentration of CO_2 increases by the same amount every year. But the verbal statement would also indicate constant increases every month, or week, or day. This broader idea of arithmetic growth will be taken up in greater detail later. The point of emphasis here is that if you agree to use an arithmetic growth model, as described verbally, then you are led to the difference equation (2.1).

This illustrates an important idea—using a simple assumption about how terms of a number sequence vary to devise a difference equation. Here it is the arithmetic growth assumption (equal growth occurs in equal periods of time) that leads to (2.1). In later chapters we will consider other sorts of growth assumptions and study the difference equations they inspire.

Numerical and Graphical Properties. One way to study a sequence in a model is to systematically compute the values of terms. Note though that these computations are part of the *model*, as opposed to the observed data on which the model is based. Usually, additional data values are *collected*, not computed.

For the CO_2 example, we have already found the terms $381, 383, 385, 387, \cdots$. These can be incorporated in the data table

n	0	1	2	3	4	5
c_n	381	383	385	387	389	391

,

which reflects our decision to label the starting term c_0. Often questions about a model can be answered by reference to such a table. For the CO_2 model, we might like to know how long it will take the concentration to reach 400, or what the concentration will be in a future year, say 2025.

Both questions can be answered by extending the table systematically term by term. For the first question, we continue until we find a term c_n that is at or above 400. For the second question, we have to remember that c_0 is the value for 2006. Then 2025 is 19 years later, and we have to continue the table until we reach c_{19}.

Systematic computation of this sort is referred to as a *numerical* approach or method. This can involve computing only a few terms, or hundreds or thousands. As long as we are working directly with numerical values, we are using a numerical approach.

What can we observe from numerical methods in arithmetic growth models? For one thing, the terms of the sequence increase (or decrease) with perfect regularity. This is, after all, the underlying assumption of arithmetic growth. As soon as we see the first two terms, 381 and 383, and observe an increase of 2, we know that the increases for all successive terms will be the same.

In fact, the terms of an arithmetic growth sequence are completely determined by two quantities, the starting term a_0 and the constant difference d. If you are told that $a_0 = 100$ and $d = 7$, you can immediately compute sequence terms of $100, 107, 114, 121$, and so on.

Computing a large number of terms by hand can be tedious. Fortunately, calculators and computers make the computation almost effortless. This is true whenever we are working with a difference equation, but is especially true for arithmetic growth sequences. See the supplementary technology guide [**23**] for further discussion of numerical computation.

Numerical Method Example. In the model for satellite fuel, how long will it take to use up all the fuel? To analyze this, we produce a data table as shown below.

Table 2.3. Using a numerical method to determine how long the fuel will last in the satellite model.

n	g_n
0	3,100.00
1	3,095.65
2	3,091.30
3	3,086.95
4	3,082.60
⋮	⋮
710	11.50
711	7.15
712	2.80
713	−1.55

We have left out all the lines between $n = 4$ and $n = 710$ to save space, but they were all computed as part of our numerical method. The last few lines show that after 712 months only 2.80 grams of fuel would remain, according to the model. The last of the fuel would then be used up in the following month. Thus, we predict that the fuel would be used up after 713 months, or about 59.4 years.

Graphs. Another aspect of arithmetic growth sequences is that their graphs are straight lines. Specifically, when we graph the terms of the sequence, we see the individual points are all in a line. Often, to emphasize this fact, we display the line as part of the graph, with or without the individual data points.

Figures 2.1–2.3 show graphs for three of the sequences we have been discussing.

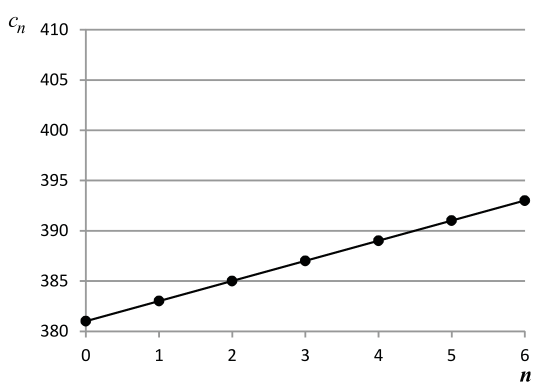


Figure 2.1. Graph for the CO_2 sequence.

The first graph depicts c_n (CO_2 concentration). It shows individual points as well as the line on which they fall. For f_n (library fine amounts) the second graph shows only the individual points. In contrast, for g_n (amount of fuel on a satellite) the graph

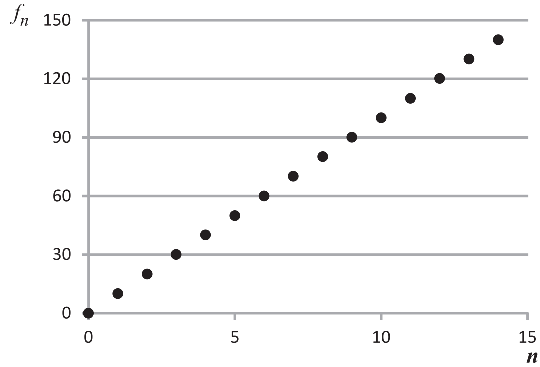


Figure 2.2. Graph for the library fine sequence.

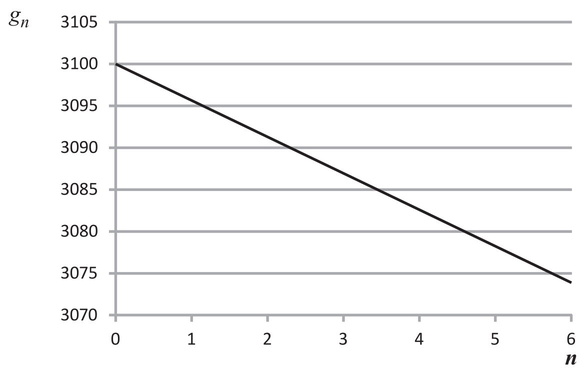


Figure 2.3. Graph for the satellite fuel sequence.

does not display any individual points. The points were used to find the line, but only the line appears on the graph. These are all valid formats. Which one is best in a particular application depends on what you wish to emphasize.

The fact that the individual points line up is a direct consequence of the arithmetic growth assumption. Each successive term increases or decreases by the same amount, d, so each successive point of the graph rises or falls by the same amount, again d. Moreover, the value of the parameter d is reflected in the steepness of the line. If d is positive, the line slopes up to the right, whereas for negative d the line slopes down to the right. The bigger d is, the more steeply the line slopes. Later we will discuss this in greater detail. Because the graphs always involve straight lines, arithmetic growth models are sometimes referred to as *linear* models.

On the other hand, many phenomena have graphs that are not straight lines. A famous example from economics is the Laffer curve (Figure 2.4). It shows one conception of how government revenue (i.e., income) varies with the tax rate. We emphasize that the shape of the curve is meant to be figurative, rather than a quantitatively accurate plot of revenue versus tax rate. The point is the general appearance of an arch, and definitely not a straight line, inferred from the fact that a tax rate of either zero or 100% will result in zero revenue. In more general terms, the Laffer curve warns not to use a linear model for the growth of revenue as the tax rate is increased. Indeed, looking

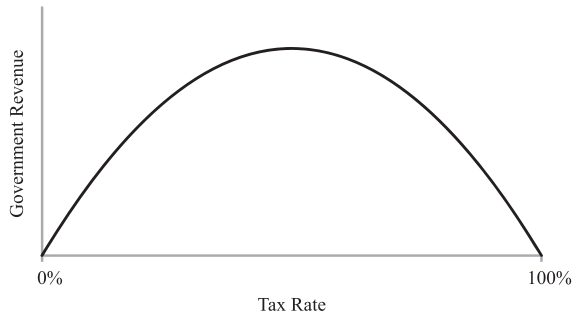


Figure 2.4. The Laffer curve. A tax rate of 0% means no taxes are due, so government revenue is 0. At the other extreme, a 100% tax rate means no one keeps any earnings. Assuming a rational person would not work under those conditions, government revenue would be 0 in this case as well. For intermediate tax rates, we know that government revenue is positive. These considerations suggest a curve of the sort shown, though this graph should not be taken as an accurate depiction of the actual revenue curve. See [**19**]. There is much to be said about this topic, including arguments refuting the entire concept of a Laffer curve. See [**27**, pp. 130-136].

only at the right side of the curve, some economists argued that decreasing the tax rate could increase revenue.

Specifics of tax policy aside, our goal here is to recognize that the straight line graph of arithmetic growth models is a special characteristic, and indicates that these models only apply in certain contexts. Phenomena and models with graphs other than straight lines are called *nonlinear*. They will be considered in later chapters. Now we continue our discussion of arithmetic growth models, where the graphs *are* linear.

Graphical Methods. Earlier we used a numerical method to predict how long the fuel in the satellite model would last. This question (and similar ones) can also be answered using a graphical method. It requires that we reformulate the question in terms of the graph, and then try to find the answer visually. For the question at hand, we want to know when g_n will equal zero. The points on the graph all have the form (n, g_n), and we particularly want to find one of the form $(n, 0)$. Such a point would lie on a horizontal line at a height of 0. In a standard xy graph, that would be the x-axis. But in our graph (Figure 2.3), the lowest horizontal line is at a level of 3070, so the x-axis doesn't even appear in the figure.

A more complete picture is provided by Figure 2.5, where the horizontal axis extends to $n = 750$, and corresponds to a g_n value of zero. In particular, the value of g_n equals 0 at the point where the sequence's graph crosses the axis. That appears to occur where $n = 720$, approximately.

As this example shows, solving a problem graphically requires that the point or points of interest appear in the graph. Moreover, the accuracy of answers obtained graphically is limited by the graph's resolution. In Figure 2.5, for example, we can estimate values on the n-axis to the nearest multiple of 25, but it is not possible to accurately read the values to the nearest whole number.

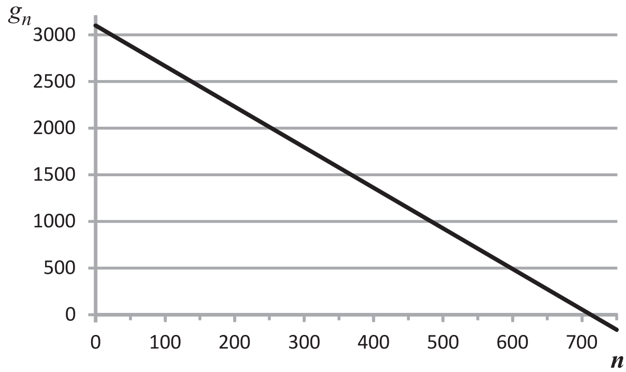


Figure 2.5. Extended graph for the satellite fuel sequence.

These are limitations of graphical methods when applied to static images in print and on line.[2] Fortunately, with graphical computer and calculator applications, the resolution and extent of graphs can be adjusted interactively. For example, by entering the satellite model difference equation and initial value in a graphing calculator, the user can dynamically extend or shrink the range shown on each axis, zooming out for the big picture to reveal where features of interest occur, and zooming in to locate those features with great accuracy. These ideas are explored more fully in the supplementary technology guide [**23**].

In general, a graphical method always involves visually inspecting a graph. For static graphs, the accuracy with which the graph can be read must limit the accuracy of the answers we obtain. But even when that is true, graphical methods are often helpful to get an approximate answer. In addition, expressing a question in terms of some feature of a graph provides a different view point, and this often contributes to the overall understanding of a model. We will see this later in the chapter when we consider linear functions and equations. But it will take on even greater significance in later chapters when we study non-linear phenomena.

Functional Equations. Do you recall the distinction between functional equations and difference equations discussed in Chapter 1? The difference equation is used *recursively*; the initial term is used to find the second term, which in turn is used to find the third term, and so on. To find the 100th term in the sequence, we have to compute all of the 99 terms that precede it. A functional equation is direct. We can use it to find the 100th term in a single computation. For this reason, a functional equation is very useful.

Every arithmetic growth model has a functional equation. For the CO_2 model, the equation is

$$c_n = 381 + 2n.$$

How was this equation found? Here is one approach: use the difference equation to compute the first several terms of the sequence, but without carrying out any of the

[2]In this context, *static* means fixed or unalterable. Computer graphics that can be modified by the user are called *dynamic*.

operations, as in the following equations.

$$\begin{aligned} c_0 &= 381 \\ c_1 &= 381 + 2 \\ c_2 &= 381 + 2 + 2 \\ c_3 &= 381 + 2 + 2 + 2. \end{aligned}$$

For each succeeding term of the sequence, we simply added 2 to the right side of the preceding equation. But notice that we didn't actually perform those additions. We just wrote them down with + signs. This reveals a simple pattern. Following that pattern, we could write the next line as

$$c_4 = 381 + 2 + 2 + 2 + 2.$$

But more importantly, we can jump ahead and predict an equation for any term, for example c_7 without working out equations for c_5 and c_6. The number of added 2's on each line is the same as the subscript on the left side of the equation, so for c_7 we write seven 2's,

$$c_7 = 381 + 2 + 2 + 2 + 2 + 2 + 2 + 2.$$

This is going to get unwieldy quickly, so let us combine the added 2's to obtain

$$c_7 = 381 + 2 \cdot 7.$$

With the same approach the earlier equations can be rewritten as

$$\begin{aligned} c_0 &= 381 \\ c_1 &= 381 + 2 \cdot 1 \\ c_2 &= 381 + 2 \cdot 2 \\ c_3 &= 381 + 2 \cdot 3 \\ c_4 &= 381 + 2 \cdot 4. \end{aligned}$$

Now the pattern is so simple, we can immediately jump to

$$c_{19} = 381 + 2 \cdot 19$$

or

$$c_{42} = 381 + 2 \cdot 42$$

or even

$$c_{1000} = 381 + 2 \cdot 1000.$$

Such an equation could be written for any term of the sequence. With algebra we express this idea economically with the single equation

$$c_n = 381 + 2n,$$

which we understand to be valid when n is replaced by any whole number.

At this point we have a strong guess about the functional equation for the sequence. But all we have to justify it is an observed pattern, and sometimes patterns are misleading, as we saw in Chapter 1. Fortunately, there is an alternative way to think about the functional equation. Suppose we wish to predict the value of c_{14}. With $c_0 = 381$ as the starting value, we can interpret c_{14} as the value after 14 years. Since the CO_2 concentration increases by 2 every year, after 14 years there will be an increase of $2 \cdot 14$. This shows that $c_{14} = 381 + 2 \cdot 14$. And similar reasoning applies for any number of years.

This leads us to the same functional equation we found before: $c_n = 381 + 2n$. But this time we reached the equation by reasoning about the computations, rather than by merely extending an observed pattern.

Throughout this book we will use patterns to understand functional equations for models. Proper mathematical analysis requires finding logical reasons to validate an observed pattern. For the case of arithmetic growth, because the pattern is so simple, the logical validation is reasonably simple. As we progress to more complex models in later chapters, logical validation of functional equations will become correspondingly complicated. In some instances we will merely hint at a necessary validation, or even omit it altogether. But the reader should understand the difference between observing a pattern and validating it. Furthermore, be assured that the authors have formulated proper validations for all the patterns we use, even though those validations are not all presented.

Using either patterns or logical reasoning, we can find functional equations for the library fine and satellite fuel examples. The table below shows the respective initial terms and the difference and functional equations for each example, for comparison purposes.

Example:	CO_2	Library Fine	Satellite Fuel
Initial Term	$c_0 = 381$	$f_0 = 0$	$g_0 = 3{,}100$
Difference Equation	$c_{n+1} = c_n + 2$	$f_{n+1} = f_n + 10$	$g_{n+1} = g_n - 4.35$
Functional Equation	$c_n = 381 + 2n$	$f_n = 0 + 10n$	$g_n = 3{,}100 - 4.35n$

The functional equations for all of these examples can again be united in a single generic equation, just as we found for difference equations on page 42.

Arithmetic Growth Functional Equation: If an arithmetic growth sequence follows the difference equation $a_{n+1} = a_n + d$, then its functional equation is given by

$$a_n = a_0 + dn. \tag{2.2}$$

As before, a is a generic label for the sequence.[3]

For the functional equation, we have an additional parameter, a_0. In any application of arithmetic growth, such as the three examples represented in the table above, we may use a different letter in place of a, and specific numbers will take the place of d and a_0 in the functional equation.

Like the functional equation in the CO_2 model, the generic functional equation can be found in two ways. The examples in the table reveal a pattern, and that alone is enough to suggest the generic functional equation. Going further, the generic functional equation can also be derived logically by similar reasoning as used earlier, but with symbolic parameters d and a_0 rather than specific numbers. Although this is more abstract than the earlier argument, it demonstrates that the generic functional equation is valid for *all* possible parameter values, and for all whole numbers n.

Functional Equations and Numerical Methods. Earlier, we used a numerical method to predict when the fuel would run out in the satellite model. There we used the difference equation, and systematically computed all the values of g_n until we found

[3] In (2.2) note that dn means $d \times n$, not d_n; this n is not a subscript.

the point where negative terms first appear. Now we have the functional equation, $g_n = 3{,}100 - 4.35n$, providing another possible approach: systematic trial and error. Will the fuel last one year? Compute $g(12) = 3{,}100 - 4.35 \cdot 12 = 3047.8$ to see that yes, there will still be fuel after a year. Will it last for ten years? That would be 120 months, and $g(120) = 3{,}100 - 4.35 \cdot 120 = 2578$. So there will still be fuel. Will it last for 100 years? This time we find $g(1{,}200) = 3{,}100 - 4.35 \cdot 1{,}200$, which is negative. We cannot actually have a negative amount of fuel, but this shows that the fuel will not last for 100 years. In this way, we can systematically refine our questions, getting closer and closer to the point where the fuel runs out. The table below shows how the entire process might play out. In the n column are our successive guesses about how long the fuel might last, while the entries in the g_n column are computed from the functional equation.

Table 2.4. Systematic trial and error to discover when the satellite fuel will run out.

n	g_n
12	3,047.8
120	2,578.0
1,200	−2,120.0
600	490.0
800	−380.0
700	55.0
750	−162.0
712	2.8
713	−1.5

For this example systematic trial and error requires much less calculation than the earlier approach using the difference equation. But functional equations may be difficult or impossible to derive in some models, so that trial and error is not possible. Both approaches are instances of numerical methods because they work directly with the numerical values of the terms of the sequence.

Functional Equations and Theoretical Methods. We have seen both numerical and graphical methods for answering questions about a model. There is another important approach, referred to in this book as a *theoretical method.*[4] To illustrate, we consider again the question: when will the fuel run out in the satellite model? The answer is the point in the sequence at which $g_n = 0$. This equation, combined with the functional equation for g_n, leads to

$$3{,}100 - 4.35n = 0.$$

Now we can use algebra to solve the equation, and thus determine the unknown n, as follows. First, we add $4.35n$ to both sides of the equation, obtaining

$$3{,}100 = 4.35n.$$

Next, exchange the two sides of the equation,

$$4.35n = 3{,}100.$$

[4]Some authors refer to this as a symbolic method.

Finally, dividing both sides of the equation by 4.35 shows that

$$n = 3{,}100/4.35,$$

which is approximately 712.6. Thus, according to the model the satellite fuel will last for between 712 and 713 months. Dividing by 12 we see that is 59 years and between 4 and 5 months.

Although we used numbers in this analysis, it differs from a *numerical* approach because we also used algebra. But the concept of a theoretical method encompasses more than just using algebra to solve an equation, as indicated in the following points.

First, theoretical methods can involve many types of analysis, including algebraic manipulation, solution of equations and inequalities, geometry, statistics, and more advanced types of mathematics. The most common use of theoretical methods in this book involves solving functional equations algebraically, but you will also see instances where these methods are used to derive properties of models. We have already seen one example, the derivation of a functional equation for the CO_2 model. Second, theoretical methods often help us understand models on a deeper level than the answers we obtain using numerical and graphical methods, because they reveal relationships among variables in a model. Finally, numerical, graphical, and theoretical methods can and should be used in combination, each contributing part of our understanding of a model. This idea will be developed further in Section 2.2.

Profile for Arithmetic Growth. We have considered quite a few aspects of arithmetic growth models. In later chapters we will consider other kinds of models. One of our goals in doing so is to see how different kinds of models compare, what sorts of properties they can have, and how to recognize when they are appropriate for applications. To facilitate comparisons we will develop a profile of each type of model. Here is part of the profile for arithmetic growth models.

Table 2.5. Profile for arithmetic growth sequences.

Verbal Description:	Each term increases (or decreases) from the preceding term by a constant amount
Parameters:	Initial term a_0, constant difference d
Difference Equation:	$a_{n+1} = a_n + d$
Functional Equation:	$a_n = a_0 + dn$
Graph:	Straight line

2.1 Exercises

Several problems require the creation of graphs. It is recommended that these graphs be produced using computer software (such as Excel) or an online graphing application, and printed for inclusion with your solutions. Alternatively, graphs may be produced by hand on graph paper, but if so, it is important to plot points with the greatest possible accuracy.

Reading Comprehension.

(1) Write a short essay on the topic of arithmetic growth. Tell what features are shared by all arithmetic growth models, and give details about graphical properties, difference equations, and functional equations of these models. How can you tell whether a particular application might (or might not) be an appropriate place to use an arithmetic growth model?

(2) Why are arithmetic growth models often referred to as linear models? Think of your answer as an explanation to help a fellow student who is struggling with this topic. Your answer should be as specific as possible and include at least one example.

(3) Find a recent newspaper or magazine article that uses either a numerical, graphical, or theoretical approach to draw a conclusion about a model. Explain the example you have chosen briefly, using your own words. It does not necessarily need to be an arithmetic growth model.

(4) @What are parameters and how do they differ from variables? Give a description and examples.

Mathematical Skills.

(5) @For each of the following items a sequence is given either numerically or graphically. Determine whether or not the sequence is arithmetic. If not, explain why not. If so, give the values for the constant difference, d, and the initial term, a_0.

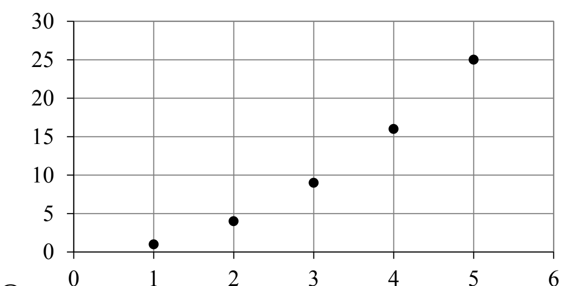
a. @3, 5, 7, 9, 11, 13, $\cdots$

b. 7.1, 8.3, 9.5, 10.7, 11.9, 13.1, $\cdots$

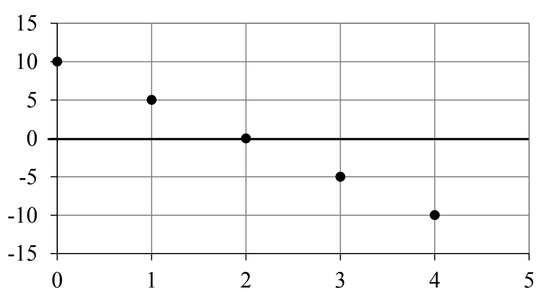
c. @1, 4, 16, 64, 256, $\cdots$

d. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \cdots$

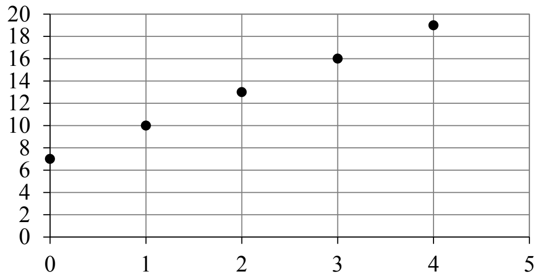
e. @150, 145, 140, 135, 130, 125, $\cdots$



f. @



g.



h.

(6) @For each functional equation below, find the corresponding difference equation and the initial term.

a. @$a_n = 15 - 3n$
b. $b_n = 7 + \frac{1}{3}n$
c. $c_n = 20 - 5n$
d. $d_n = 20 + 5n$
e. @$e_n = 10n - 3$
f. $f_n = 10n + 3$

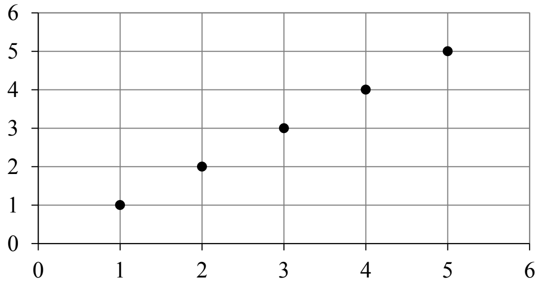
(7) @Each part below gives a difference equation and the value of an initial term for a sequence. Find the functional equation for each sequence.

a. @$a_{n+1} = a_n + 2;\ \ a_0 = 1$
b. $b_{n+1} = b_n + 6.2;\ \ b_0 = 5$
c. $c_{n+1} = c_n + \frac{1}{5};\ \ c_0 = -312$
d. $d_{n+1} = d_n - 1.3;\ \ d_0 = 100$
e. @$e_{n+1} = e_n - \frac{1}{2};\ \ e_0 = 70$
f. $f_n = f_{n-1} - 10;\ \ f_1 = 23$
g. $g_n = g_{n-1} + 0.8;\ \ g_1 = 11.3$

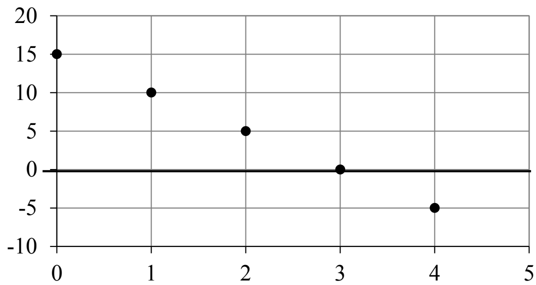
(8) Graph each of the sequences, $a_n, \cdots, g_n$, defined in the previous problem. Your graphs should include at least five clearly labeled points. Graphs may be produced carefully by hand or printed after being created with computer software. See the technology guide [**23**] for more information.

(9) @For each of the following items a sequence is given either numerically or graphically. For each sequence find functional and difference equations.

a. $20,\ 24,\ 28,\ 32,\ 36 \cdots$
b. @$\frac{1}{4},\ \frac{1}{3},\ \frac{5}{12},\ \frac{1}{2},\ \frac{7}{12}, \cdots$
c. $-18,\ -28,\ -38,\ -48,\ -50, \cdots$
d. $1.1,\ 2.4,\ 3.7,\ 5,\ 6.3, \cdots$



e.



f. @

(10) @In each part below find specified terms for an arithmetic growth sequence.

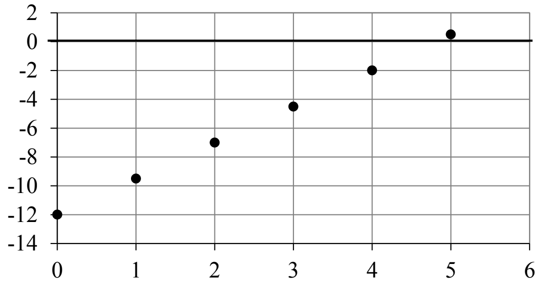
a. @The initial value is $a_0 = 5$ and the sequence begins 5, 8, 11, 14, 17, $\cdots$. Find the terms a_6 and a_{400} in the sequence.

b. The initial value is $b_0 = 10$ and the sequence begins 10, 15.6, 21.2, 26.8, 32.4, $\cdots$. Find the terms b_9 and b_{150} in the sequence.

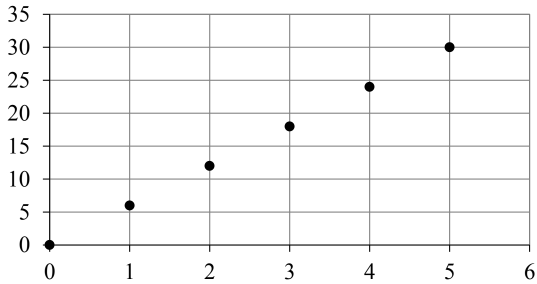
c. @Each term decreases from the preceding term by 12. The initial value is $c_0 = 325$. Find c_7 and c_{100}.

d. Each term increases from the preceding term by 3.4. The initial value is $d_0 = 9$. Find d_{10} and d_{200}.

e. @Find e_9 and e_{250} for sequence e_n whose graph is shown here.



f. Find f_7 and f_{300} for sequence f_n whose graph is shown here



(11) Each equation below defines a sequence. For each,

i. compute the value of the sequence for $n = 0, 1, 2$, and $n = 10$,
ii. find a difference equation, and
iii. graph the sequence. Your graph should include at least five clearly marked points.

a. $a_n = 10n - 2$
b. $b_n = 2 - 3n$
c. $c_n = 5 - n$
d. $d_n = 1 + 2.7n$

Problems In Context.

(12) @Flu Epidemic. In this section one of the examples of arithmetic growth is

$$p_{n+1} = p_n + 500; \quad p_0 = 1{,}000,$$

where p_n represents the number of people who have been infected with flu by the start of month n.

a. @Graph at least the first 5 terms of the sequence and leave room to extend the graph several more terms.
b. @Use a graphical approach to determine how long it will be before at least 9300 people are infected with flu.
c. @Use a numerical approach to determine how long it will be before at least 9300 people are infected with flu and compare the answer to the one you found in the previous part of this problem.

(13) Ebola Outbreak. In a 2014 outbreak of the Ebola virus in West Africa, one report included the statement "At current rates of transmission, each infected person is passing the disease to about two more people"[**9**]. In trying to understand this situation, a student reasons as follows. Suppose that there are 100 people infected initially. Each of those infects two more people, so then there will be 200 new patients (they are in generation 2), and a total of 300 patients. Now each of the 200 new patients infects two more, for a new set of 400 patients (generation 3) and a total of 700. Continuing in this way, we can track the number of newly infected people in each generation finding $100, 200, 400, 800, \cdots$, or we can track the total number of infected people after each new generation, finding $100, 300, 700, 1500, \cdots$. Would

an arithmetic growth model be appropriate for either of these sequences? If so, find appropriate difference and functional equations, identifying all your variables. If not, explain why not.

(14) @Diving Pressure. As a diver descends into the ocean ambient pressure builds at a rate of about 1 atmosphere (atm) for every 10 meters (m) of depth. A diver will experience 1 atm at the surface, 2 atm at a depth of 10 m, 3 atm at a depth of 20 m, and so on.

a. @Let d_n be the depth in meters at which a diver experiences n atms of pressure. Compute d_n for $n = 1,\ 2,\ 3$, and 4.
b. @Graph the sequence from part a.
c. @Explain why this situation can be accurately modeled by an arithmetic growth sequence. Are there any limitations we should consider? For example, does it make sense to allow $n = 0$?
d. @Find a difference and a functional equation for d_n.
e. @A diver has a camera that can withstand pressures up to 25 atm. How deep can the camera be safely taken?
f. @A team of divers descends to a depth of 27 meters. What pressure will they experience?

(15) Simple Interest. A student borrows \$5,000 from his aunt. He promises to repay the loan as soon as possible. It is agreed that he will pay the original \$5,000, plus simple interest, in one lump sum payment. The interest will be calculated at one quarter percent per month. Let p_n be the amount of money the student will have to pay the aunt if he makes the payment after n months. For example, if he makes the payment after a year, that is 12 months and the amount to be repaid would be p_{12}. At a quarter percent per month, the interest will be 3 percent in 12 months. Now 3 percent of \$5, 000 is $0.03 \times 5{,}000 = 150$. So if the loan is paid back after 12 months, the amount to be repaid is \$5, 150, the original loan amount plus the \$150 in interest. That is p_{12}. In contrast, p_6 is what the student has to pay if he makes the payment after 6 months.

a. What is p_0? Does that make sense?
b. What is the difference equation for p_n?
c. What is the functional equation for p_n?
d. Use the functional equation to determine how much must be repaid if the payment is made after 18 months.
e. Eventually, the student repaid his Aunt at a cost of \$6,000. How long was that after he borrowed the original \$5,000? That is, find n so that $p_n = \$6{,}000$.

Digging Deeper.

(16) @When we know the initial term a_0 and the common difference d for an arithmetic growth sequence, the functional equation $a_n = a_0 + dn$ can be immediately formulated. In this problem you will be asked to find functional equations using other information. Find a functional equation for each part below. Show your work or explain your reasoning.

a. @$a_1 = 12$ and $d = 4$
b. $b_3 = 8$ and $b_{n+1} = b_n + 1.4$
c. $c_{n+1} = c_n - \frac{1}{2}$ and $c_3 = 12$
d. @$e_2 = 8$ and $e_6 = 52$
e. The first term of the sequence is $f_1 = 6$. There is no f_0 term. The second term is $f_2 = 0$.

(17) @Two equations involving a single variable are called equivalent if they are both true for exactly the same values of the variable. For example $x+1 = 6$ and $10/x = 2$ are equivalent because they are both true when x is 5, and for no other values of x. However $x(x + 3) = 5x$ and $x + 3 = 5$ are not equivalent because the first is true for both $x = 0$ and $x = 2$, whereas the second is only true for $x = 2$.
For each part below an equation is given, and an operation to transform the equation. In each case, tell whether the original equation and the result of the operation are equivalent.

a. @$x - 2 = 7$; add 2 to both sides
b. $3x - 4 = 8$; divide both sides by 3
c. @$(x + 1)(2x) = (x + 1)(x - 3)$; divide both sides by $(x + 1)$
d. $5 + 6x = -(x + 1)$; multiply by -1 on the right side
e. $4x = 11$; take the absolute value of both sides.
f. $3x + 6 = 4x + 8$; factor the right side

(18) For each of the following, write an equation for n as a function of a_n. For example, for part a, since $a_n = 15 - 3n$, we have $3n = 15 - a_n$ so $n = 5 - a_n/3$. Thus we have expressed n as a function of a_n.

a. $a_n = 15 - 3n$ b. $a_n = 15 + 3n$
c. $a_n = -3n + 20$ d. $a_n = 3n + 20$
e. $a_n = 20 - 5n$ f. $a_n = 20 + 5n$
g. $a_n = 20n - 3$ h. $a_n = 20n + 3$

2.2 Applications of Arithmetic Growth

In Section 2.1 we considered mathematical properties of arithmetic growth models, such as difference equations, functional equations, and graphs. Now we turn to applications.

The applications we have in mind begin with a problem situation or context, about which we wish to answer one or more questions. Our overall strategy is to introduce a mathematical framework, called a mathematical model, that resembles the real context. Questions about our problem situation are translated into questions about the model, which we answer (if possible) using mathematical operations and analysis. The answers are mathematically correct conclusions about the model.

We recognize that the model is not an exact portrayal of the problem situation, and that the conclusions of the model may only provide approximate answers to our original questions. Indeed, validating and/or improving models are important facets of the mathematical modeling approach. But we will not delve into them here. Our focus will be on how to formulate models and use them to answer questions about a problem context.

A Modeling Outline. Many mathematical operations are expressed in terms of an *algorithm*, which is a set of prescribed steps to be performed in a specified order. Examples include procedures for addition, subtraction, multiplication, and division of numbers, and algebraic manipulations such as expanding $(x+1)^2$ to x^2+2x+1. Conceptually, an algorithm is supposed to be clear-cut and mechanical—at each step one knows exactly what to do, with no uncertainty and no need to exercise judgment.

The process for adding two numbers illustrates the idea of an algorithm. To add 1,768 and 421, we first write them like so.

$$\begin{array}{r} 1768 \\ +\quad 421 \\ \hline \end{array}$$

Then we work from right to left, adding digits in each vertical column, recording the result below the line. The actions we take at each step may depend on the results of preceding steps—sometimes we have to carry, sometimes we don't, but at every step we know exactly what to do. And we know that when all of the steps have been carried out in the prescribed manner, we will have the desired result. That is how an algorithm works.

There is no algorithm for developing and using mathematical models. Even in the restricted context of this section, where arithmetic growth sequences are our sole concern, applications will require judgment and insight. The information given and questions posed will vary from problem to problem. It will be up to the student to choose appropriate notation, and identify or adopt assumptions, questions, and methods of analysis.

Our primary goal for this section is to explain the modeling process, preparing students to develop their own models for exercises at the end of the section. Listed below are common steps that usually occur in developing and using models. We emphasize that this list should be viewed as a rough outline, and not as a step-by-step algorithm. Most problems will involve all the steps in some form, but in a way that varies from one problem to another. Nor should the order of the steps be rigidly followed. For example, graphs and tables are mentioned in the sixth step of the outline, but they may be useful in the preceding steps. Also, sometimes working on a later step of the outline prompts you to revise an earlier step. The point of the outline is to help you organize your work and to highlight important components of the modeling process.

Modeling Outline:

(1) Understand the general characteristics of the problem situation, including given information and questions to be answered.

(2) Define variables.

(3) Identify assumptions.

(4) Find values of parameters.

(5) Formulate difference and functional equations.

(6) Where appropriate, create diagrams, tables, and graphs representing important aspects of the model.

(7) Formulate specific questions about the model.

(8) Find answers to the questions.

(9) Translate conclusions about the model into conclusions about the problem situation.

(10) Reflection and reconsideration. Are the results reasonable? Are the assumptions? Were there any errors? Are there ways to improve the model?

We proceed to consider several examples.

Example 1: Time of Death. According to the Writer's Medical and Forensics Lab [**36**], body temperature can be used to estimate time of death. Quoting the source,

> Normal body temperature during life is 98.6 degrees F. After death, the body loses heat progressively until it equilibrates with that of the surrounding medium. The rate of this heat loss is approximately 1.5 degrees per hour until the environmental temperature is attained, then it remains stable.

The expression *degrees F* means *degrees Fahrenheit*. This is also represented symbolically, as in 98.6°F.

Suppose that a dead body is discovered in a hotel room at 9 PM. The temperature in the room is 68°F. Upon discovery, the temperature of the corpse is found to be 84°F. Find the time of death.

Solution.

General Characteristics. The problem poses a single question: when did death occur? This asks us to find a particular time, so time will be one variable. We are also given information about the temperature of the corpse, so temperature will also be a variable. We can formulate a model by understanding the given context in terms of a sequence of data values. One way to do this is to imagine a sequence of hourly temperature readings, starting with the temperature at the time of death. We can conceptualize the situation in this way, even though the problem does not say that such data have been collected. We note that the problem statement includes two additional facts: normal body temperature during life, and rate of heat loss after death.

Define Variables. Choosing the letter T (for temperature), we denote our sequence of temperatures $T_0, T_1, T_2, \cdots$, where T_0 is the temperature at the time of death, T_1 is the temperature one hour later, and so on. In this context, the position number n represents elapsed time, in hours, from the time of death. For reference purposes, we incorporate this information into a formal definition of variables:

> Let T_n be the temperature of the body, in degrees F, n hours after death.

Identify Assumptions. By the end of the course you will have learned about several different kinds of models, and part of each modeling problem will be to decide which kind of model to use. But at this point of the course we have only seen arithmetic growth models, so that is the kind of model we will use here. With that in mind, we note three assumptions:

(1) T_n is an arithmetic growth sequence,

(2) body temperature is initially 98.6°, and

(3) body temperature decreases by 1.5° each hour.

We are also given the temperature of the room, but we shall see that this fact does not enter into the model. As you work on a problem, it may not be obvious what facts are relevant, and there is no harm in including something that later is not needed. So a fourth assumption about room temperature could be included in the list above. But your solution will be easier to follow if you create a final draft that excludes references to unnecessary information.

Parameters. We know that two parameters are included in an arithmetic growth model, the initial value T_0 and the difference d. These are both given directly in our list of assumptions. The initial temperature is 98.6, the temperature of the body when death occurred. And a decrease of 1.5° every hour means that $d = -1.5$. If these facts are not at first apparent, a good strategy is to work out the first several terms of the sequence, as follows. By definition, T_0 is the temperature at the time of death. What is that? Right up to the moment of death the victim is alive, and so has a temperature of 98.6. Proceeding, T_1 is the temperature an hour later. But we know the temperature decreases by 1.5° each hour. So we must have $T_1 = 98.6 - 1.5 = 97.1$. By similar reasoning, succeeding terms are $T_2 = 97.1 - 1.5 = 95.6$, $T_3 = 95.6 - 1.5 = 94.1$, and $T_4 = 94.1 - 1.5 = 92.6$. Computing these values should help you recognize arithmetic growth at work, and identify the parameters.

Difference and Functional Equations. The difference and functional equations for this model are

$$T_{n+1} = T_n - 1.5$$

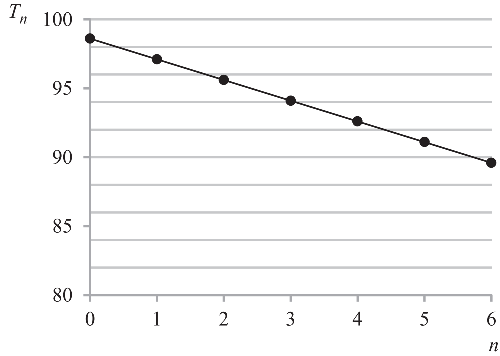
and

$$T_n = 98.6 - 1.5n,$$

respectively. The difference equation formalizes the recursive pattern of repeated 1.5 degree reductions in temperature. The functional equation provides a means for direct (as opposed to recursive) determination of T_n for any n. Because they offer alternative ways to find the same thing, we can use the equations together to check for errors.

Diagrams, Tables, Graphs. We display below a table of values and a graph for the sequence T_n. These were created using the methods discussed in Section 2.1. Tables and graphs offer additional perspectives on a model, sometimes revealing aspects that are not easily recognized working with equations alone. They provide yet another alternative way to answer specific questions about the model. Also, they should generally be included as part of the documentation of a model, because they help communicate what you have done to a broader audience. For all of these reasons, you should create tables and graphs for your models, even if you do not find them strictly necessary to find specific answers.

n	T_n
0	98.6
1	97.1
2	95.6
3	94.1
4	92.6
5	91.1
6	89.6

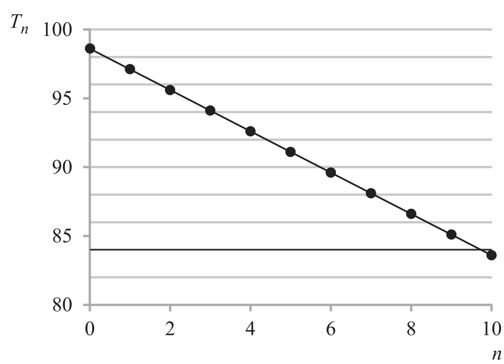


Tables and graphs may be created at any convenient point in the development of the model. For example, as soon as the first several terms of the sequence were computed, the table and graph could have been produced. On the other hand, in the process of answering specific questions about the model, it might prove worthwhile to extend the table and graph to additional data points.

Questions About the Model. The original problem statement asks us to find the time of death. How do we restate this as a question about the model? Observe that in the model, the start time is the time of death, and that is not known. However, we do know the temperature of the body was 84° when it was discovered. Is that the value of T_n for some n? Suppose, for example, we find that $T_8 = 84$. That would tell us that the discovery of the body occurred 8 hours after death. Notice how the definitions of variables makes this clear: T_8 is the temperature of the body 8 hours after death. And since we are given the time at which the body was discovered, we can also determine the time of death. In essence, we can change the original question to *How long had the victim been dead when the corpse was discovered?* That translates directly to a question about the model: *For what n does* $T_n = 84$?

Find Answers. One approach to finding the answer is to look at the table and graph already produced. Is there an entry in the table or a point on the graph where $T_n = 84$? No. But we see immediately that extending the table and graph for a few additional terms should reveal the desired information. Accordingly, we create the following revised table and graph.

n	T_n
0	98.6
1	97.1
2	95.6
3	94.1
4	92.6
5	91.1
6	89.6
7	88.1
8	86.6
9	85.1
10	83.6



Referring to the table, we see that $T_9 = 85.1$ and $T_{10} = 83.6$, so $T_n = 84$ should occur somewhere between $n = 9$ and $n = 10$. The graph portrays the same information visually. Note that the line corresponding to a T value of 84 has been highlighted.

For a more exact answer, we combine the functional equation with $T_n = 84$ to obtain

$$98.6 - 1.5n = 84.$$

Solving this equation for n we find

$$n = (98.6 - 84)/1.5 = 9.73$$

to two decimal places. An exact answer is $n = 146/15 = 9\frac{11}{15}$.

Translate Conclusions. We have found an exact answer to the question *In the model, for what n does* $T_n = 84$? This indicates a time of death 9.73 hours before 9 PM, or at approximately 11:15 AM. Or, using the exact answer, we can convert the fractional part 11/15 from hours to minutes, observing that 1/15 of an hour is 4 minutes. Thus, death occurred 9 hours and 44 minutes before 9 PM, and hence at 11:16 AM.

Reflection and Reconsideration. We have found an answer, but is it correct? Is it reasonable? Perhaps we made a mathematical error. Or, even if the mathematical steps were performed perfectly, it may be that the assumptions in the model are questionable. Turning a critical eye on your own work is an important aspect of mathematical modeling.

We can be fairly confident that the mathematical steps were correctly done, because we actually found answers three different ways. Both the table and graph indicated that n should be between 9 and 10, and by visual inspection, the graph suggests a value of n more than 9.5. These observations are consistent with the exact answer we found using algebra. At least we can see that our different views of the model are consistent.

But we can also check the conclusion by a different approach. Supposing that the death did occur at 11:16 AM, what temperature would we expect to find at 9 PM? We know that the temperature should decrease by 1.5° every hour, or by 3° every two hours. So, with a temperature of 98.6 at 11:16, we expect a temperature of 95.6 at 1:16, 92.6 at 3:16, 89.6 at 5:16, and so on. Continuing in this way, we predict a temperature of 83.6 at 9:16. This is consistent with the given information, and indicates that our answer is in the right ballpark.

Because we know that our model is only an approximate representation of the true situation, we should not expect an answer that is exactly correct. Just on the basis of common sense, we would expect there to be some uncertainty about the time the body was discovered. So even if the model is reasonably accurate, the exact time of death will probably be somewhat different than what the model indicates. Also, in establishing the time of death, the investigators probably do not require accuracy down to the second. Both of these factors influence how we interpret the results. In particular, we found an approximate solution of $n = 9.73$ as well as an exact figure of $n = 9\frac{11}{15}$. The exact value may be preferred for aesthetic reasons, but in practical terms, 9.73 is just as believable. Similarly, it is as reasonable to round the time of death to 11:15 as it is to report the exact value of 11:16. The one minute difference is almost certainly insignificant.

Another point worth considering is the use of a fractional value for n. In our original formulation of the model, n is a position number, and hence represents a whole number. On the other hand, n is also interpreted as a number of hours, and so fractional values make sense. But the derivation of the arithmetic growth functional equation is based on n representing a whole number. Does the equation remain valid for values of n that are not whole numbers?

In the context of the current problem, this is essentially asking whether the temperature decrease each hour is evenly distributed during the hour. In other words, if the temperature decreases by 1.5° in an hour, can we further assume a decrease of 0.75° every half hour? Or of 0.5° every 20 minutes? Adopting this plausible assumption, using the functional equation with fractional values of n is justified. We will discuss this issue at greater length later, in connection with continuous and discrete variables. As we will see, although permitting fractional values of n is valid in this problem, it will not always be so.

Finally, we might also question the assumptions of the model. In the context of a textbook exercise, we can legitimately accept the given information as correct. But in an actual application in the real world, assumptions are always subject to discussion and criticism. For the problem under discussion, two obvious questions are whether the body will really cool by the same amount every hour, and if so, whether the 1.5 degree figure is correct. For the first question, notice that the model does not agree with the verbal problem description in one important respect: in the model the body continues to cool even after it reaches the environmental temperature. Everyday experience (in other contexts) tells us that the body will cool down to the temperature of the surrounding room, but not lower. Indeed, the quoted passage from the Medical and Forensics Lab states as much. Yet in the model, with $n = 40$ we find $T_{40} = 98.6 - 40 \cdot 1.5 = 38.6$, much colder than the room termperature of 68. The model may still be accurate for the time period of the problem context, but we should be aware that it cannot be valid in the long term.

Regarding the second question, we again rely on our everyday experience. We know that a warm object will cool more quickly in a refrigerator than in a heated room. Analogously, in the context of the model, we expect a body's cooling rate to depend on the temperature of its surroundings. Since the model does not take the surrounding temperature into account, we should question the accuracy of the 1.5 degree figure. One response would be to investigate how our conclusions might change under different assumptions. For example, if the body cools by 2° per hour rather than 1.5°, how much earlier or later would we place the time of death?

Deeply analyzing questions of this sort is beyond the scope of our discussion here. They are mentioned to illustrate the meaning of the last step of the outline, and to enrich the reader's understanding of the methods of mathematical modeling.

Many students will wonder how much reflection is reasonable in homework exercises or on examinations. Certainly it is worthwhile to check for mathematical errors and consider whether your answers are reasonable. For more conceptual considerations, such as whether the assumptions of the model are valid, students should not expect to find definitive conclusions. However, they should be prepared to indicate what assumptions might be questioned, and why. In the context of this specific example, there is another assumption that might reasonably be questioned, and the reader is challenged to identify it. A question about this assumption will appear in the exercises at the end of the section.

As the preceding discussion shows, working through an applied modeling problem involves many considerations, requires some judgment and insight, and probably does not proceed in a linear fashion. As you proceed you may wish to modify earlier steps, for example. At the end you may discover inconsistencies that reveal errors. For all of these reasons, it is often helpful to work out a problem in a rough form, making changes as necessary, and then write up a final draft of the solution at the end. This not only will result in clear and coherent solutions for evaluation on assigned work, but will also reinforce your understanding of how mathematical modeling is actually used in the real world.

Homework Solution. In the preceding discussion our goal has been to expose the rationale for our approach. The reader is encouraged to think in similar terms in solving problems in this section. However, it is not always necessary to include as much explanation in your solutions as has been provided above. In Figure 2.6 we display a sample solution for the time of death problem, as a student might include in a homework assignment. The table and graph were produced using computer software, and attached to the page. If you are studying this text as part of a course, consult the instructor for guidelines on writing up solutions to applied problems.

Example 2: College Tuition. Table 2.6 shows the annual in-state tuition at the College of William and Mary, for each year from 2001 to 2012. Develop an arithmetic growth model based on this information, and estimate the expected tuition in 2015 and in 2020. If a child is 5 years old in 2015, how much will four years tuition at William and Mary be when the child reaches the age of 18, according to the model?

Table 2.6. In-state tuition charges for the College of William and Mary, in thousands of dollars.[5]

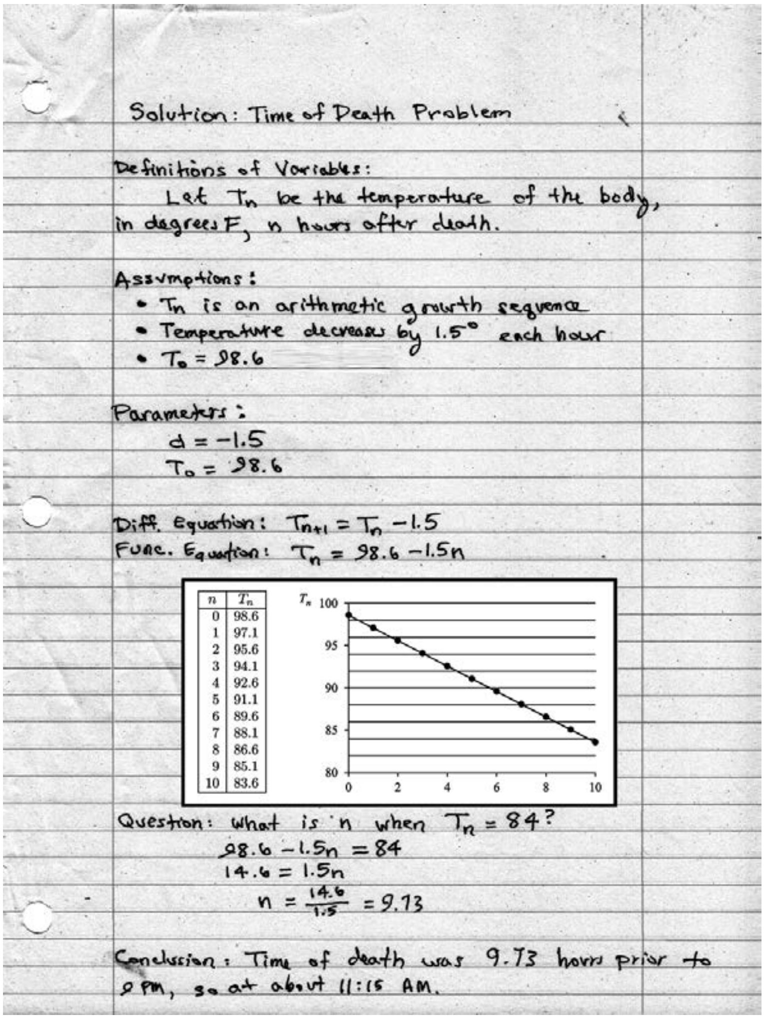
Year	Tuition	Year	Tuition
2001	4.780	2007	9.164
2002	5.528	2008	10.426
2003	6.430	2009	10.800
2004	7.096	2010	12.188
2005	7.778	2011	13.132
2006	8.490	2012	13.570

Solution.

General Characteristics. In an arithmetic growth model there will be a sequence of values. We can think of the annual tuition figures as terms of such a sequence: $4.780, 5.528, 6.430, \cdots$. The problem statement directs us to develop an arithmetic growth model. For the model to fit the data exactly, each term should increase by the same amount over the preceding term. To investigate whether this is true, we add

[5]Data source is reference [**55**].

a column to the table (see Table 2.7) showing how much the tuition increases for each year.



Solution: Time of Death Problem

Definitions of Variables:
Let T_n be the temperature of the body, in degrees F, n hours after death.

Assumptions:
- T_n is an arithmetic growth sequence
- Temperature decreases by 1.5° each hour
- $T_0 = 98.6$

Parameters:
$d = -1.5$
$T_0 = 98.6$

Diff. Equation: $T_{n+1} = T_n - 1.5$
Func. Equation: $T_n = 98.6 - 1.5n$

n	T_n
0	98.6
1	97.1
2	95.6
3	94.1
4	92.6
5	91.1
6	89.6
7	88.1
8	86.6
9	85.1
10	83.6

Question: What is n when $T_n = 84$?
$98.6 - 1.5n = 84$
$14.6 = 1.5n$
$n = \frac{14.6}{1.5} = 9.73$

Conclusion: Time of death was 9.73 hours prior to 9 PM, so at about 11:15 AM.

Figure 2.6. Sample homework solution for the Time of Death problem.

We see that the yearly increases are not all the same, and that there is a fair amount of variation. This tells us that the model will not fit the data exactly. But it may still be the case that an arithmetic growth model can approximate the given data.

Here, it will be worthwhile to examine a graph of the data, as shown in Figure 2.7. We have included a straight line joining the first and last data points, merely as a visual reference, and it does appear that the data points are all pretty close to this line. Thus, we will proceed to develop an arithmetic growth model. The points of the model will fall exactly on a straight line, and will be close to, but not exactly equal to, the original data points.

Table 2.7. Tuition data with yearly increases, in thousands of dollars.

Year	Tuition	Increase
2001	4.780	
		0.748
2002	5.528	
		0.902
2003	6.430	
		0.666
2004	7.096	
		0.682
2005	7.778	
		0.712
2006	8.490	
		0.674
2007	9.164	
		1.262
2008	10.426	
		0.374
2009	10.800	
		1.388
2010	12.188	
		0.944
2011	13.132	
		0.438
2012	13.570	

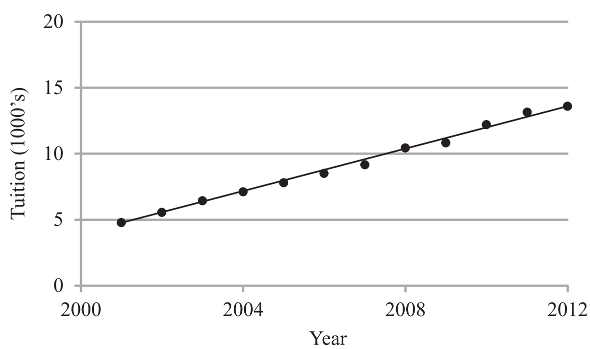


Figure 2.7. Graph of the tuition data.

Define Variables. We again choose the letter T (for tuition in this example), so that T_n is the tuition in year n, in units of thousands of dollars. We emphasize here that T_n is a term in the sequence in our *model*, not the actual data value given in the table.

It will be convenient to take the starting term to be T_1, so that T_1 is the tuition in 2001, T_2 the tuition in 2002, and so on. Then it will be particularly simple to translate between a year and the corresponding position number, e.g., T_{17} corresponds to 2017.

As in the earlier example, position number n represents elapsed time, but now in units of years. Mimicking the approach of the preceding example, we state the variable definitions formally, for future reference.

Let T_n represent in-state tuition, in thousands of dollars, at the College of William and Mary for year n, where year 1 is 2001.

Identify Assumptions. As specified in the directions, we assume that the model sequence $T_1, T_2, T_3, \cdots$ is an arithmetic growth sequence.

Parameters. As always for an arithmetic growth model, we wish to identify the initial value T_0 and the common difference d. We would like the terms T_1 through T_{12} to approximate the corresponding data values very closely. Since it is not obvious how to find the parameters that are the best possible, we start with a simple but plausible approach and hope that it produces reasonably accurate results. First, we compute the parameter d as the average of all of the differences in Table 2.7. Rounded to three decimal places, that gives us $d = 0.799$. Second, we choose T_0 so that T_1 equals the first data value, 4.780. Using our arithmetic growth assumption, T_1 must be 0.799 units higher than T_0. That means $T_0 = T_1 - 0.799 = 4.780 - 0.799 = 3.981$.

Difference and Functional Equations. Now that we have found the parameters, we can substitute them into the standard arithmetic growth difference and functional equations, to find

$$T_{n+1} = T_n + 0.799$$

and

$$T_n = 3.981 + 0.799n.$$

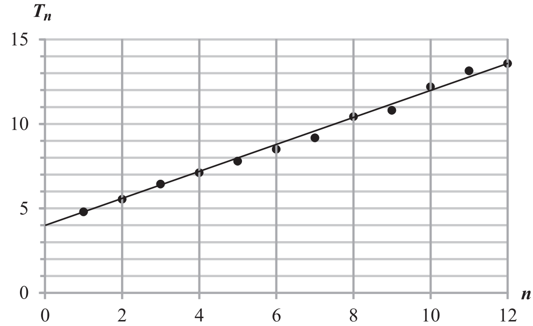
These are the difference and functional equations for our model.

Diagrams, Tables, Graphs. A table of values and a graph are shown below for our sequence T_n. For later reference we also include in the table the original data values and the errors, that is, the differences between each term of T_n and the corresponding data value. To be more precise, each error is found by subtracting a data value from the corresponding term T_n of the model. Thus, nonzero errors can be either positive or negative. A positive error indicates that the model value is too high; negative indicates that the model value is too low.

In the graph the original data values appear as dots, while the model is represented by a straight line. The individual terms T_n are not shown on this line, because they would be too close to many of the original data points. Comparing the dots with the line, we obtain a visual portrayal of the errors. Where a data point is below the line, the corresponding error is positive; a data point above the line indicates the error is negative. And the vertical distance between each data point and the line is the absolute value of the error.

Notice that the worst error is 0.411. For this point the model over-estimates the tuition by \$411, out of \$9, 575. For about half the points, the model is off by less than \$100.

n	Data	T_n	Error
1	4.780	4.780	0.000
2	5.528	5.579	0.051
3	6.430	6.378	−0.052
4	7.096	7.177	0.081
5	7.778	7.976	0.198
6	8.490	8.775	0.285
7	9.164	9.574	0.410
8	10.426	10.373	−0.053
9	10.800	11.172	0.372
10	12.188	11.971	−0.217
11	13.132	12.770	−0.362
12	13.570	13.569	−0.001



Questions About the Model. The problem statement asks for tuition estimates for 2015 and 2020. As we have defined our variables, this corresponds to finding T_{15} and T_{20}. We are also asked for the four-year tuition total at age 18 for a child age 5 in 2015. Since age 18 is 13 years older than age 5, we want the tuition total for 2028 through 2031. Thus, we would like to compute $T_{28} + T_{29} + T_{30} + T_{31}$.

Find Answers. Using the functional equation we compute the following values:

n	15	20	28	29	30	31
T_n	15.966	19.961	26.353	27.152	27.951	28.750

This gives us T_{15} and T_{20} directly. We can also use the values in the table to compute

$$T_{28} + T_{29} + T_{30} + T_{31} = 110.206.$$

Translate Conclusions. The interpretation of these results is as follows. T_{15} represents the tuition for 2015. Thus, the model predicts tuition will be \$15, 966 for 2015, and similarly, \$19, 961 for 2020. The sum of T_{28} through T_{31} indicates a four year tuition total of \$110, 206 at age 18 for the child who is 5 in 2015.

Reflection and Reconsideration. This application is very similar to the prior example, in terms of potential sources of error. We can again review our findings using graphical and numerical methods, checking that all of the methods produce consistent results. We can also question whether future tuition increases are likely to follow the same trends as the past twelve years. Many possible factors can influence tuition at a public university, including the general health of the economy and political forces in state government. Accordingly, we should not be surprised to see sudden changes that render our arithmetic growth model incorrect. In contrast, for the prior example, the goal was to model a physical process. We expect the rate of cooling of an inert body to depend on surrounding conditions, but otherwise to follow a predictable and consistent course of events. Thus, two bodies that start at the same temperature and are subjected to the same conditions should cool off at the same rate. Also, the validity of an arithmetic growth model is asserted by a forensic expert, presumably based on observations in many cases. These factors should give us greater confidence in the model for that example.

As observed earlier, the data presented in this problem closely approximate an arithmetic growth sequence. Our model produces annual tuition projections that are very close to the actual data for the first 12 terms. So we might be fairly confident that the model will continue to be accurate for the first few years after 2012. However, using the model to project as much as 13 years into the future should not be counted on to produce accurate results.

One aspect of the reflection phase is gaining new insights about a particular problem or the methods employed. For this example it is striking that the model reproduces the first and last data points almost exactly. And if we had defined d as the *exact* average of the first differences, without rounding, the model would have reproduced the first and last data values *exactly*. Is this just a coincidence?

No. The same thing will happen in any arithmetic growth model defined in the same way. That is, with any given set of data, if we define a_0 to equal the first data point exactly, and define d to be the average of all the differences between consecutive data values, then the model will agree exactly with the last data point as well. This assertion can be justified with a little thought, taking into account how an average is computed, and the interested reader is invited to think it out. Our main purpose in mentioning it here, though, is to illustrate the discovery of new ideas during the reflection phase of modeling.

Example 3: Making T-shirts. An established business manufactures various items of clothing. For this problem we focus on a single factory set up to produce t-shirts, and on the cost of production for a single typical day. With the existing equipment and facilities, it is possible to make at most $125,000$ shirts per day. But depending on other variables (such as sales demand) the company might decide to make fewer shirts, or even suspend operations for a while and make no shirts at all. Our goal is to develop a model that shows how the costs vary as we increase the number of shirts produced. The board of directors has tasked us to answer two specific questions:

(1) What will it cost to produce $90,000$ shirts?

(2) If the budget is limited to $\$\,60,000$, how many shirts can be produced?

We distinguish between two different types of cost. First, there are costs that do not depend on how many shirts are made. They include costs of the building that are incurred whether or not any shirts are produced, such as the cost of insurance, taxes, security, and maintenance. These are called *fixed* costs. Company analysts have determined that fixed costs for a typical day amount to about $\$\,17,500$.

Second, there are costs that do depend on the number of shirts made, such as the cost of fabric, wages for the workers who make the shirts, and energy costs to operate equipment in the factory. These are called *variable* costs, and are calculated on a per-shirt basis. The analysts have calculated that current variable costs total about $\$\,0.40$ per shirt.[6]

[6]Variable cost and factory capacity are based on a 2010 report about a factory in Bangladesh. See [**43**].

Solution.

General Characteristics. Does it make sense to use a sequence model in this problem? To get an idea of the situation, let us look at some examples. If the company is idle, and produces no shirts, then there are only fixed costs, totaling \$ 17, 500. If one shirt is produced, the variable cost will be \$ 0.40, so the total cost will be \$ 17, 500.40. For two shirts, the variable costs will be \$ 0.80, so the total cost in this case is \$ 17, 500.80. Continuing in this fashion, we can find the total cost to produce 3, 4, or 5 shirts. However, this approach is unrealistic. Can we expect a factory capable of producing 125, 000 shirts to produce just one or two? In addition, mathematically and conceptually, it will be inconvenient to work with quantities as small as a single shirt.

Accordingly, let us represent the quantity of shirts in units or lots of 10, 000. Then the fixed costs will be \$ 17, 500 and variable costs will be \$ 4,000 per lot. Arguing as before, we find the cost to produce nothing will be \$ 17, 500, to produce one lot will be \$ 21, 500, two lots will be \$ 25, 500, and so on. Continuing in this fashion we obtain Table 2.8.

Table 2.8. Cost (in dollars) to produce shirts in lots of 10, 000.

Number of Lots	Total Cost (\$)
0	17, 500
1	21, 500
2	25, 500
3	29, 500
4	33, 500
5	37, 500

We make two observations. First, arranged in this way our results form a data table for a sequence. The entries in the first column are position numbers, and those in the second column are the terms. Second, this sequence exhibits arithmetic growth. Each time we increase the number of lots by 1, the total cost increases by a constant amount, \$ 4,000. We conclude that an arithmetic growth sequence should be a good model for this context.

Define Variables. Just as we elected to represent shirts in lots of 10, 000, so too it will be convenient to represent monetary amounts in units of \$1,000. We use the letter c for *cost* for the terms of the sequence. Accordingly, we define our variables as follows:

> Let c_n represent the cost to produce n lots of 10, 000 shirts each, and express c_n in units of thousands of dollars.

Identify Assumptions. We assume that c_n is an arithmetic growth sequence. This is consistent with the table of values, reproduced below in units of thousands of dollars.

In graphing those values, we note that the individual points line up, further justifying the use of an arithmetic growth model (see Figure 2.8). Going beyond what is in the table or the graph, the problem context tells us that with each additional lot, c_n will increase by 4. That implies that the sequence terms do in fact grow arithmetically.

Table 2.9. c_n is the cost in thousands of dollars to produce n lots of 10,000 shirts each.

n	c_n
0	17.5
1	21.5
2	25.5
3	29.5
4	33.5
5	37.5

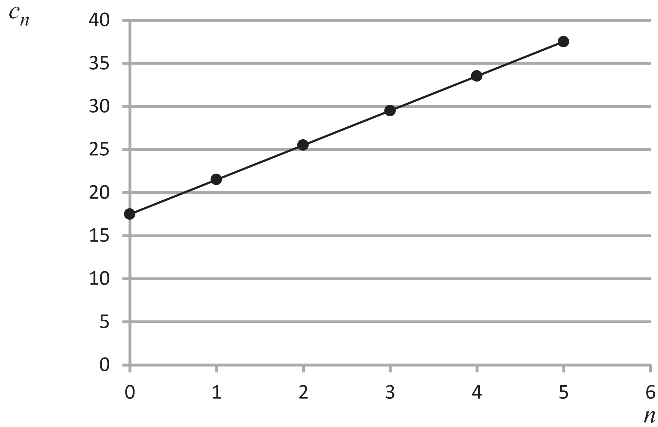


Figure 2.8. Individual points from the data table lie on a straight line.

We also assume that $n \leq 12.5$ because the factory can only produce 125,000 shirts per day.

Parameters. The arithmetic growth parameters are $c_0 = 17.5$ and $d = 4$.

Difference and Functional Equations. The difference and functional equations are

$$c_{n+1} = c_n + 4$$

and

$$c_n = 17.5 + 4n.$$

Diagrams, Tables, Graphs. We have already produced a table and a graph for this sequence. They show how the model behaves up to $n = 5$. This corresponds to 50,000 shirts, far less than the factory's 125,000 shirt capacity. To answer the questions posed in the problem statement, it might be informative to extend both the table and the graph to $n = 13$.

Questions About the Model. In the problem statement, question 1 asks for the cost of producing 90,000 shirts. That corresponds to 9 lots of 10,000, so the question becomes *In the model, what is* c_9? The second question asks how many shirts can be produced with a budget of \$ 60,000 . In the context of the model this becomes *For what value of n does* $c_n = 60$?

Find Answers. For question 1, applying the functional equation, with $n = 9$, we find

$$c_9 = 17.5 + 9 \cdot 4 = 53.5.$$

For question 2 we again use the functional equation. This time we set c_n equal to 60 and leave n as an unknown. That gives us

$$60 = 17.5 + 4n.$$

Using algebra to solve for n results in

$$n = (60 - 17.5)/4 = 10.625.$$

Translate Conclusions. For the first question, $c_9 = 53.5$ means a cost of \$ 53, 500 to produce 90, 000 shirts. For the second question, $n = 10.625$. But n is the number of 10, 000 shirt lots. Therefore, the total number of shirts is $10.625 \cdot 10,000 = 106,250$ shirts.

Reflection and Reconsideration. For this problem we do not have specific data to work with. We have no way to assess the validity of the fixed and variable costs described in the problem statement. Within the context of the given information, an arithmetic growth model is appropriate.

Are our answers to the questions in the problem statement reasonable? Returning to the original data, to make 125, 000 shirts at 0.40 dollars per shirt would cost \$ 50, 000, and with fixed costs of \$ 17, 500, the total would be \$ 67, 500. That is what it costs to operate the factory at maximum capacity. To produce only 90, 000 would cost less, so our figure of \$ 53, 500 is at least in a reasonable range.

We can also verify our conclusions using alternate methods. For example, in Figure 2.9 we show an extended graph for the model. We can answer the stated questions approximately by reading this graph. It shows that c_9 is about 53, and that c_n reaches 60 for n around 10.5. These agree well with our earlier results of 53.5 and 10.625. Extending the data table provides another alternate method, but we leave that to the reader.

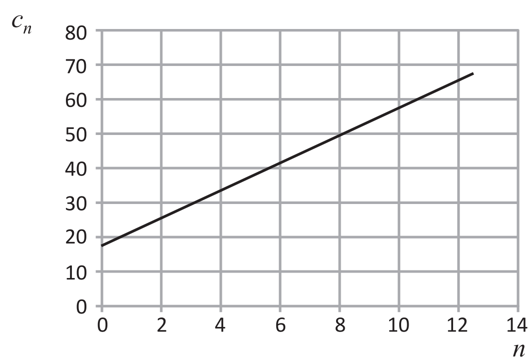


Figure 2.9. Extended graph for the t-shirt cost model.

Because n is in units of 10, 000, it makes sense to consider fractional values of n. For example, a single shirt is one ten-thousandth of a lot, so increasing the number of shirts by one corresponds to increasing n by 0.0001. This means that a four decimal place figure, such as 5.6871, corresponds to a whole number of shirts. On the other hand, extending n to additional decimal places would be questionable, since it would require quantifying fractional parts of a single shirt.

Does the functional equation still apply when we allow non-whole number values of n? The given information specifies the variable cost per single shirt, which we used in the model to establish the cost of each lot of 10,000. This implies that the functional equation will remain valid for non-whole numbers n having 4 or fewer decimal digits.

The experience of working through this example shows that an arithmetic growth model can be applied in any analogous problem. As long as we agree to analyze costs in terms of a fixed component and a variable (per item) component, an arithmetic growth model will be feasible.

Additional Considerations. The three preceding examples illustrate how arithmetic growth models are developed and used. We have seen common features of the models, such as the use of difference and functional equations. We have also seen that each problem is a little different from the others, and that judgment and understanding are needed.

We conclude this section with some observations about functions and variables. Our goal is to recognize considerations from the examples that arise frequently in arithmetic growth models.

Functions. In any arithmetic growth model, an important role is played by the functional equation $a_n = a_0 + dn$. When the parameters a_0 and d are replaced with specific numbers, two variables remain, n and a_n. As we have seen, given a value for either variable, the functional equation allows us to determine the other.

In some cases the goal is to find a particular term a_n of the sequence. Thus, a value for n is given and we want to know the value of the corresponding a_n. These are called *find-a_n questions*. For example,

What will the tuition be in 2020?

and

What will it cost to produce 90,000 t-shirts?

are both find-a_n questions. In the context of the model, each question reduces to finding a_n for a specific n.

In contrast, when we are given the value of a_n and asked to find n, that is referred to as a *find-n question*. Both

How long after death would it take for body temperature to reach 85°?

and

How many t-shirts can be manufactured for $ 60,000?

are find-n questions.

In many models position numbers represent times, and a find-n question asks when something occurs. Thus, the first of the two preceding questions might be restated as *When does body temperature reach* 85°? In a similar way, whenever n represents time, a find-n question is also a find-*when* question. But keep in mind that n won't always indicate a time. In the t-shirt model, n indicates how many shirts are to be produced.

The distinction between find-n and find-a_n questions reflects an important aspect of working with functions. To illustrate, we momentarily put aside the idea of number sequences, and look at a familiar equation of the sort that is studied in algebra and precalculus classes:

$$y = x^2 + \frac{5}{x}. \tag{2.3}$$

Notice that the two variables do not play comparable roles here. If we replace x with a specific value, such as $x = 8$, we can immediately calculate y. The reverse is not true. If we replace y by 20 for example, a corresponding value of x cannot be immediately calculated. Indeed, it is not immediately obvious whether such a value even exists, or if one does exist, how to find it.

We use special terminology to emphasize the distinct roles played by x and y in (2.3). We say the equation defines y as a function of x, meaning that we can immediately compute y as soon as a value of x is given. Another way to say the same thing is this: y is determined by x.

The simplest way to use the equation is first to *choose* a value of x then *compute* the value of y. In choosing x we don't need to know y, so the choice of x is independent of y, and we call x the *independent* variable. When we compute the value of y, the result depends on the value of x, so we call y the *dependent* variable.

Notice that the dependent variable appears by itself on one side of the equation, whereas the independent variable appears in a more complicated algebraic combination on the other side of the equation. That might seem backward if you consider being *by itself* to be an aspect of *independence*. But that is incorrect. The terms *dependent* and *independent* refer to how we *use* each variable, not to the visual appearance of the variable in the equation. And while we can identify which variable is which based on appearance, we have to do so carefully. It is incorrect to say, *y is all by itself and therefore is independent*. The correct analysis is *y is all by itself, and therefore dependent, because the equation tells us how the value of y depends on the other variables.*

Starting with a numerical value for x and computing the corresponding y is referred to as *function evaluation*. We can often reverse the process, first choosing y and then trying to find corresponding values of x. This is referred to as *function inversion*.

This terminology can be used to distinguish between find-a_n questions and find-n questions using a functional equation. The functional equation expresses a_n as a function of n, because a_n appears by itself on one side of the equation while n appears in a more complicated algebraic combination on the other side. In other words, n is the independent variable and a_n is the dependent variable. To answer a find-a_n question, we are given the value of the independent variable (n) and want to find the dependent variable. This is a direct computation, and an instance of function evaluation. On the other hand, to answer a find-n question, we have to reverse the roles of the variables. This is an instance of function inversion, and generally involves more than direct calculation.

Continuous and Discrete Variables. When we work with a sequence in a model, we understand the position number n to be a whole number. It makes sense to refer to the third term a_3 or the fourth term a_4, but there is no three-and-a-half-th term. On the other hand, we have also seen that the position number can have a second interpretation that remains valid when n is not a whole number. For example, in the time of death model, we identify n with elapsed time, so that T_3 is the temperature at a time 3 hours after death. In that context it makes sense to interpret $T_{3.5}$ as the temperature 3.5 hours after death. And as we argued in discussing this model, it is reasonable to assume that the hourly decrease in temperature, 1.5°, occurs uniformly during each hour. Thus, in half an hour the body should cool by 0.75°, so $T_{3.5}$ is 0.75° cooler than T_3. With this assumption, the functional equation gives correct values for T_n even when n is not a whole number.

But that sort of logic is not always valid. Consider the tuition model, where T_3 is the tuition for 2003 and T_4 is the tuition for 2004. To be more specific, T_3 is the tuition for the academic year starting in the fall of 2003. But what might $T_{3.2}$ mean? Since there is no academic year corresponding to 3.2, it is not clear what meaning, if any, $T_{3.2}$ could have. In this case, the model only makes sense for whole number values of n.

This leads us to the distinction between continuous and discrete variables. In the first example, where n can assume any numerical value, we say that n is a *continuous variable*. In the second example the value of n is restricted to be a whole number, and we say n is a *discrete variable*. Discrete variables need not be restricted to whole number values. For the t-shirt model n represents the number of $10,000$ shirt lots produced by a factory. As discussed earlier, that means n can be a non-whole number with up to four decimal places. In other words, the only permitted values of n are $0, 0.0001, 0.0002, 0.0003, \cdots, 0.9999, 1.0000, 1.0001, \cdots$. This is another instance of a discrete variable n. Significantly, n has a specific first value, second value, third value, and so on, with no values in between. This is the key idea of a discrete variable: the permitted values are separate from one another. In contrast, for any value of a continuous variable, there is no *next* value. In the time of death model, we understand that T_2 is the temperature for a time of 2 hours. We can also conceive of the temperature after $2.1, 2.01$ or 2.001 hours. Thus we can understand the meaning of n equal to $2.1, 2.01, 2.001$, or any other decimal of the form $2.00\cdots01$. In this instance it is not possible to list the permitted values of n. Visually, they make up a continuous set of points on a number line, with no intervening gaps.

Interpreting n as a discrete variable has significant implications for find-a_n and find-n questions. To illustrate, let us consider the tuition model, recalling the functional equation

$$T_n = 3.981 + 0.799n.$$

Although the right-hand side of this equation can be computed for any value of n, in the model n only makes sense as a whole number. Accordingly, we should only ask find-T_n questions for whole number values of n. Although we can ask *What is T_n after* 3.2 *years*?, and compute $T_{3.2} = 3.981 + 0.799 \cdot 3.2 = 6.5378$, neither the question nor the answer is meaningful: there is no academic year corresponding to $n = 3.2$, so how can there be a tuition figure for $n = 3.2$? In general, for the tuition model, find-T_n questions are meaningful only when n is a whole number.

The situation for find-n questions is similar. To see why that is so, consider first $T_7 = 9.574$ and $T_8 = 10.373$. Because n can only be a whole number, it cannot have any value between 7 and 8. Likewise, the tuition cannot have any value between 9.574 and 10.373. We might ask *For what n is $T_n = 10$?* But there is no meaningful solution. Although the equation

$$3.981 + 0.799n = 10$$

is algebraically solvable, the solution $n = 7.533\cdots$ is not a whole number, and so is not an acceptable value for n in this model. Consequently, there is no solution to the question *In what year is the tuition equal to* $\$\,10,000$*?*. On the other hand, suppose we ask *What is the first year that tuition is* $\$\,10,000$ *or higher?* This time there is a perfectly valid answer: $n = 8$. Before $n = 8$ the tuitions are all less than $\$\,10,000$, and for $n \geq 8$ the tuitions are all more than $\$\,10,000$. This illustrates that questions about a model may need to be carefully formulated when n is a discrete variable.

The distinction between defining n to be continuous or discrete can be understood in relation to the graph of a_n. When n is discrete, the graph consists of individual points, separated by spaces, because the possible values of n are separated on the horizontal axis. This is shown in the graph for the library fine sequence (Figure 2.2 on page 45). In the graph, n is assumed to be a whole number of days, and each point of the graph shows the fine for a book returned some specific number of days late.

In contrast, we have seen that n has a meaningful interpretation as a continuous variable in the time of death problem. Conceptually, graphing points for all possible T_n will produce a continuous line. It will have no gaps because the possible n values cover the horizontal axis with no gaps. This is illustrated in the graph on page 61. The dots on the graph highlight the values of T_n for whole numbers n. But the other points on the line are also meaningful. They show how the temperatures T_n vary for values of n between consecutive whole numbers.

This graphical distinction between continuous and discrete variables is useful conceptually, but it is not always observed in practice. Sometimes, for convenience or emphasis we might draw a discrete model's graph as a continuous line, as we did in the graph of the tuition model on page 68. There, as discussed earlier, n is only meaningful as a whole number. Accordingly, on the graph, only the points of the line with whole number n-coordinates are meaningful. Nevertheless, we represented the model with a solid line to facilitate a visual comparison of the model and the data.

As a final variation on these ideas, let us look again at the library fine model, where f_n is the fine when a book is returned n days past the due date. Earlier we considered this as an example where n should be interpreted as a discrete variable. But now we ask, is there a way to interpret this when n is not a whole number? Here is one approach. Let us assume that the library has an after-hours book return facility, so that books can be returned at any time. And for simplicity, also assume that the library closes at the same time every day. If a book is returned before closing time on the date it is due, there is no fine. After that if the book is returned any time in the next 24 hours, that is, before closing time on the next day, the fine is 10 cents. If the book is returned any time in the next 24 hours, the fine is 20 cents. Thinking in this way, we can define n to represent the elapsed time, in days, between when the book was due and when it was returned. And because the book can be returned at any time, 24 hours per day, n need not be restricted to whole numbers. Defining f_n as the fine for a book returned n days past its due date makes as much sense when $n = 3.12$ as it does when $n = 3$.

Although our definition of f_n remains valid for fractional values of n, the functional equation, $f_n = 10n$, does not. For $n = 1.5$, this becomes $f_{1.5} = 15$. But that is not the fine the library charges for a book that is returned one and a half days late. On the first day after the due date, the fine would be ten cents any time up to closing. Then the fine jumps to 20 cents, where it remains for the next 24 hours. But a fine of 15 cents would never be incurred. The library rules do not assess a partial fine for a half day, or any other fraction. So in this model, we can permit non-whole number values of n conceptually, but we cannot use the functional equation unless n is a whole number.

We can recognize the same thing visually, by creating a graph based on the analysis in the preceding paragraph (Figure 2.10). The result is a series of horizontal lines, each

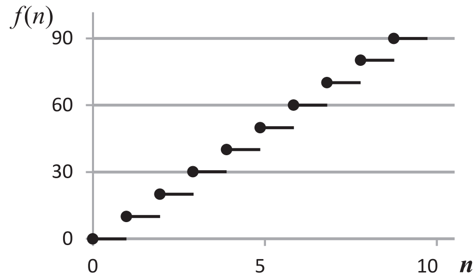


Figure 2.10. Graph for library fines when n is a continuous variable.

of which represents one 24-hour period. During the first 24 hours the fine is zero; for the next 24 hours the fine is 10 cents, and so on. The points corresponding to whole number values of n are depicted as dots. These are the same points that we originally saw in Figure 2.2.

For values of n between whole numbers, the correct values of $f(n)$ do not fill in the line connecting the dots. This shows that the correct graph of $f(n)$ is not a straight line, and therefore cannot be the graph of the functional equation $f(n) = 10n$. It also illustrates a general graphical fact. In an arithmetic growth model, when we use the functional equation with values of n that are not whole numbers, that is equivalent to assuming we can connect the individual points of the graph with a straight line. The library model shows that this is not always correct.

In most of the models you will encounter in this book it will be possible to interpret n as a continuous variable, and the functional equations will be equally valid for whole number and fractional values of n. However, as the examples above show, there are exceptions. In some cases, like the tuition model, it just will not make sense to consider n as a continuous variable. In others, like the library fine model, even though we can interpret n meaningfully as a continuous variable, the functional equation will not be valid unless n is a whole number. Unfortunately, there is no simple rule for deciding which situation holds in a particular model. Therefore, when you are developing and analyzing models, you should give some thought to whether it makes sense to treat n as a continuous variable.

Which is better, discrete or continuous? Throughout the book, we will study situations where the models are most easily formulated in terms of sequences and difference equations. That means we will begin with the idea of n as a whole number. But in most of these cases we will also derive a functional equation within which we will want to interpret n as a continuous variable. If that is feasible in the model context, formulating and answering find-a_n and find-n questions is simplified. If not, we have to be more careful analyzing the model.

2.2 Exercises

Reading Comprehension.

(1) There are various methods for analyzing data to create a model and using the model to make predictions. The methods presented in this section were categorized as numerical, graphical, and theoretical. Compare and contrast these types of methods. Your answer may be organized in table or paragraph form. For each method your answer should include at least a description, an example, some advantages, and some disadvantages.

(2) This section includes a list of common steps used to create a model. Which of these steps seems the most obvious? Which seems the most challenging? Briefly explain your choices.

(3) @In the time of death example the reader was challenged to identify a questionable assumption included in the model (see page 64). What is the questionable assumption, why might it be incorrect, and how might that affect the conclusions from the model? [Hint: Suppose death resulted from a severe bacterial infection.]

(4) @Suppose $a_n = 2n+3$ and we want to find a_3. Is that a case of function evaluation or function inversion? Briefly justify your answer.

(5) @Questions about continuous and discrete variables.

a. Describe continuous and discrete variables in your own words.

b. @In a fish population model, p_n is the number of fish after n years, in units of thousands. Thus, the equation $p_2 = 3$ means that there are 3,000 fish after 2 years. In this model, would it make sense for n to be a continuous variable? A discrete variable? What about p_n?

c. Suppose d represents an amount of money, in units of dollars, taking values such as 0.35, 199.99, and 1,000,000. Is d a continuous or discrete variable? Explain.

d. Re-visit your descriptions for part a. Did they capture everything you thought about in parts b and c? If not, revise your descriptions.

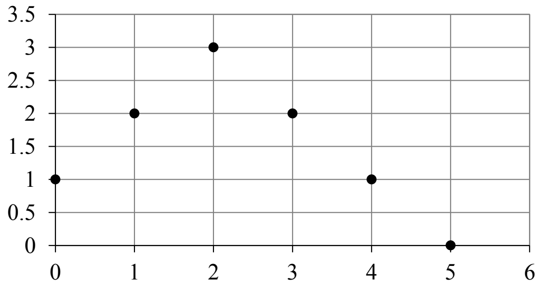
Math Skills.

(6) @In each part a sequence is given, either numerically or graphically. For each one, tell whether the sequence is exactly, approximately, or not at all an example of arithmetic growth. Briefly explain your answer.

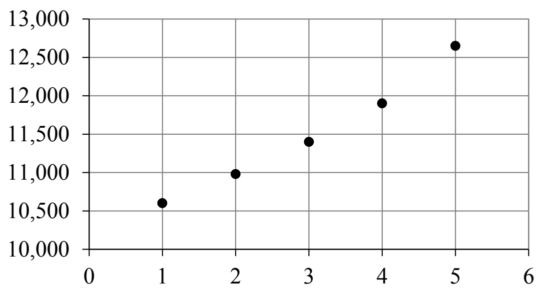
a. @$1,\ 4,\ 9,\ 16,\ 25, \cdots$

b. $7.7,\ 17,\ 26.3,\ 35.6,\ 44.9, \cdots$

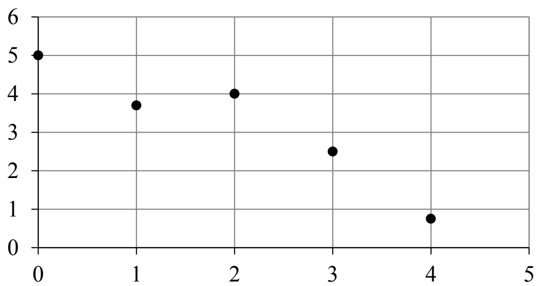
c. @$5,\ 8,\ 10,\ 12,\ 15,\ 17, \cdots$



d.



e. @



f.

(7) @For each equation below find n such that the statement is satisfied. Use either a numerical or theoretical approach. State which method you used.

a. @$a_{n+1} = a_n + 6$ and $a_0 = 10$, find n such that $a_n = 34$
b. @$b_n = 52 - 3n$, find n such that $b_n = 7$
c. $c_n = 2(5 + 3n)$, find n such that $c_n = 40$
d. $d_{n+1} = d_n - 8$ and $d_0 = 100$, find n such that $d_n = 20$
e. $e_n = 1 + 7n$, find n such that $e_n = 57$

Problems in Context.

(8) @Battery Power. A weather balloon carries a battery-powered radio transmitter which sends weather data back to the ground. When the balloon is sent up, the battery carries a charge of 30 units. It uses up 2.4 units of charge per hour. Let q_n represent the charge on the battery n hours after the balloon is sent up.

a. @Using a numerical method, find q_1, q_2, and q_3.
b. @What is the difference equation for q_n?
c. @What is the functional equation for q_n?
d. @What will the charge be 4 hours after launch?

e. @The radio transmitter cannot continue to work once the charge on the battery falls below 4 units. How many hours will that take?

(9) Wetlands. In 2011 the US Department of Interior released a five-year survey on wetlands and found that from 2004 to 2009 America's wetlands declined slightly. The report says, "The net wetland loss was estimated to be 62,300 acres between 2004 and 2009, bringing the nation's total wetlands acreage to just over 110 million acres in the continental United States, excluding Alaska and Hawaii" [**52**]. Let's suppose that in one county the number of acres of wetlands has been decreasing by about 200 acres per year for the past several years and there were 36,000 acres of wetlands in 2014. An interested ecologist uses the notation w_n for the number of acres of wetlands n years after 2014. Develop an arithmetic growth model for this problem as follows:

a. Formulate the arithmetic growth difference equation for w_n.
b. Formulate the initial value and any restrictions on n.
c. Create a graph for the model. Your graph may be produced with technology and printed or created by hand using graph paper and plotting the points as accurately as possible.
d. Formulate the functional equation for w_n.
e. Use the model to predict how many acres of wetlands will remain in the county in the year 2050.
f. Use the model to determine when there will be only $25,000$ acres of wetlands in the county.
g. Reflect on your conclusions. Do they seem plausible? What other considerations should be taken into account? What other questions could this model be used to answer? How might we want to extend the model?

(10) @Housing Cost. An economist is studying the cost of housing. In one county, the data show that despite some ups and downs the average price of a new home increased by about $2,000 per year from 2004 to 2010, starting at $135,000 in 2004. The economist uses the notation h_n for the average price of a new home n years after 2004. Develop an arithmetic growth model for this problem by completing the steps below.

a. @Formulate a difference equation for h_n. State the corresponding initial value and any restrictions on n.
b. @Formulate a functional equation for h_n.
c. @Create a graph for the model.
d. @Use the model to predict the average home price in the year 2025.
e. @Use the model to predict when the average home price will reach $250,000.
f. @Reflect on your conclusions. Do they seem plausible? What other considerations should be taken into account? Does it seem plausible that the price of a new home would increase consistently every year? What other questions could this model be used to answer? How might we want to extend the model?

(11) Auto Theft. According to the San Antonio, Texas, Uniform Crime Report there were 5,893 cases of auto theft in 2011 and 6,577 cases of theft in 2013 [**46**]. Using c_n to represent the number of car thefts n years after 2011, develop an arithmetic growth model for this situation.

a. What is the difference equation and initial value for the model?

b. What is the functional equation for this model?

c. According to the model, how many car thefts would have occurred in 2014?

d. According to the model, when will the number of car thefts in a year first exceed 10,000?

e. Reflect on your conclusions. What other considerations should be taken into account? What other questions could this model be used to answer? How might we want to extend the model?

(12) @Cooling System Malfunction. On board a UFO, far out in space, the cooling system for the hyper-drive develops a malfunction. The engineers studying the problem estimate that temperature is rising at 2.8 degrees per hour. The temperature was 148 degrees at 9 AM Standard Galactic Time on Sunday. If the temperature reaches 200 degrees, the reactor will have to be shut down. Applying appropriate steps from the modeling outline on page 58, develop an arithmetic growth model for this situation and use your model to answer the following questions. You may use any combination of graphical, numerical, and theoretical methods.

a. @What will the temperature be by 4 PM Standard Galactic Time on Sunday?

b. @According to the model, when will the reactor have to be shut down?

c. Reflect on your conclusions. Do they seem plausible in the UFO context? What other considerations should be taken into account? What other questions could this model be used to answer? How might we want to extend the model?

(13) Spread of Disease. A scientist studying the spread of a new disease in a small town decides to use an arithmetic growth model. She estimates that 3,700 people have the disease at the start of her study, and that there are 45 new cases each day.

a. Applying appropriate steps from the modeling outline on page 58, develop an arithmetic growth model for this situation, and use your model to predict when the number who are or have been infected will reach 10,000.

b. The scientist has also found that about 3 percent of the people who get the disease require treatment with a special medicine. Before the outbreak, the local hospital had a limited supply of the medicine, amounting to 500 doses. This was provided to patients as necessary both before, and during, the study. According to the model, how long after the start of the study will the special medicine last?

c. Suppose that the small town in the study is isolated—very few people arrive or leave. Given what you know about the way diseases spread, do you think an arithmetic growth model is reasonable? Consider both predictions made over a short period of time, and those over much longer periods of time.

d. How would your answer to the previous question change if the town were in a popular tourist area? Would it matter how long the tourists stay in the town?

Digging Deeper.

(14) @At the end of Example 2: College Tuition, on page 69, the assertion is made that, "with any given set of data, if we define a_0 to equal the first data point exactly, and define d to be the average of all the differences between consecutive data values, then the model will agree exactly with the last data point as well." Think about this assertion, try a few examples of your own, and convince yourself it is a true statement. Write an explanation of your work that would help elucidate the ideas for a classmate struggling with them. If possible, use algebra to show that the statement is true.

(15) Consider again the "Spread of Disease" exercise about the epidemic in a small town. Modify the previous discussion as follows. Suppose that the spread of the disease is increasing, and that the researcher observes 45 new cases on the first day of the study, 50 new cases on the next day of the study, and 55 new cases the day after that. She models the number of *new* cases per day using an arithmetic growth model. Here, let c_n be the number of new cases of the disease in day n of the study. So $c_1 = 45$, $c_2 = 50$, and so on. Create a model for the number of new cases each day, then use numerical methods to figure out how many days the hospital's supply of medicine will last.

(16) @A modified version of the library fine model was discussed on page 76. In that discussion, it was determined that the amount of the fine could be calculated using $f(n) = 10n$ when n is a whole number. The value of the expression $10n$ is defined for fractional values of n, but as shown in Figure 2.10, the graph for the amount of the library fine is a collection of line segments, not a single line. Therefore the equation $f(n) = 10n$ is not valid when n is not a whole number. If we decide to use the equation anyway, how large an impact might that have on the results? That is, what is the largest possible error that can arise in predicting a library fine if the equation $f(n) = 10n$ is used for a value of n that is not a whole number?

2.3 Linear Functions and Equations

In Section 2.1 we found that the graphs of arithmetic growth models always appear as straight lines. We also found arithmetic growth models always have functional equations that can be expressed in a form similar to

$$g_n = 3{,}100 - 4.35n. \tag{2.4}$$

This is an example of a *linear equation*, so called because the graph is a straight line. It is a variant of the familiar form $y = mx + b$ studied throughout the mathematics curriculum. In this section we will study linear equations in depth, covering ideas connected with functions, graphs, and solving equations.

Recall that (2.4) is the functional equation we found in the satellite fuel example. This equation expresses g_n as a function of n because it permits us to compute a value of g_n as soon as we know n. To be more specific, we refer to this as a *linear* function because the functional equation is a linear equation.

A word is in order here about terminology, and in particular about the terms *equation*, *expression*, and *function*. It is a common error to use the word *equation* for just

The next example illustrates how such models are used, without going into the details of how their parameters are determined.

North Polar Ice Cap. In the preceding examples, nearly constant second differences prompted us to adopt quadratic growth models. But data can suggest using a quadratic growth model even when second differences do not appear to be nearly constant. To illustrate this point, we consider a model for the shrinking north polar ice cap, as presented by Witt [**56**].

Our data consist of yearly figures for the extent (in millions of square kilometers) of the north polar ice cap in the month of September each year from 1979 to 2012 [**41**]. See Table 3.8 and Figure 3.7.

Table 3.8. North polar sea ice extent in September, each year from 1979 through 2012. Extent is given in units of millions of square kilometers.

Year	Extent	Year	Extent	Year	Extent	Year	Extent	Year	Extent
1979	7.20	1986	7.54	1993	6.50	2000	6.32	2007	4.30
1980	7.85	1987	7.48	1994	7.18	2001	6.75	2008	4.73
1981	7.25	1988	7.49	1995	6.13	2002	5.96	2009	5.39
1982	7.45	1989	7.04	1996	7.88	2003	6.15	2010	4.93
1983	7.52	1990	6.24	1997	6.74	2004	6.05	2011	4.63
1984	7.17	1991	6.55	1998	6.56	2005	5.57	2012	3.63
1985	6.93	1992	7.55	1999	6.24	2006	5.92		

In Figure 3.7, we have included a straight line representing a possible arithmetic growth model. The data points are scattered fairly widely around the line, in contrast to Figure 3.5 where the data points appear to lie almost exactly on the model curve. Here we recognize that there is a good deal of variability in the extent of sea ice from one year to the next, and our goal is not to predict the exact amount of ice in any particular year. Rather, we would like a description of the underlying trend. These data points do not appear to be scattered at random. There is a visible downward aspect as we proceed from older to more recent values. The line is meant to represent that downward trend, and so is referred to as a *trend line*. In adopting a linear model, we are implicitly assuming that the spread of data points about the trend line in the future will be similar to what we see in the figure. If so, even though we cannot predict the exact amount of ice in any specific year, we *can* predict that the amount of ice will fall within a certain range.

This is illustrated in Figure 3.7 by a shaded area centered on the trend line. For the extant data, nearly all of the points lie in the shaded area, and we expect that to remain true in the future. Applying this in a specific example, we would expect the data point for 2020 to fall within the shaded region. Therefore, based on the figure, we predict that the sea ice extent for 2020 will be between 3.5 and 5.1 million square kilometers.

For this type of analysis, where we look for a trend in a scattered set of points, there are a great many possible lines that might be drawn. On the basis of a casual visual inspection, it would be difficult to single out one specific model. This is where the ability to find the best possible model is useful. The mathematics behind choosing

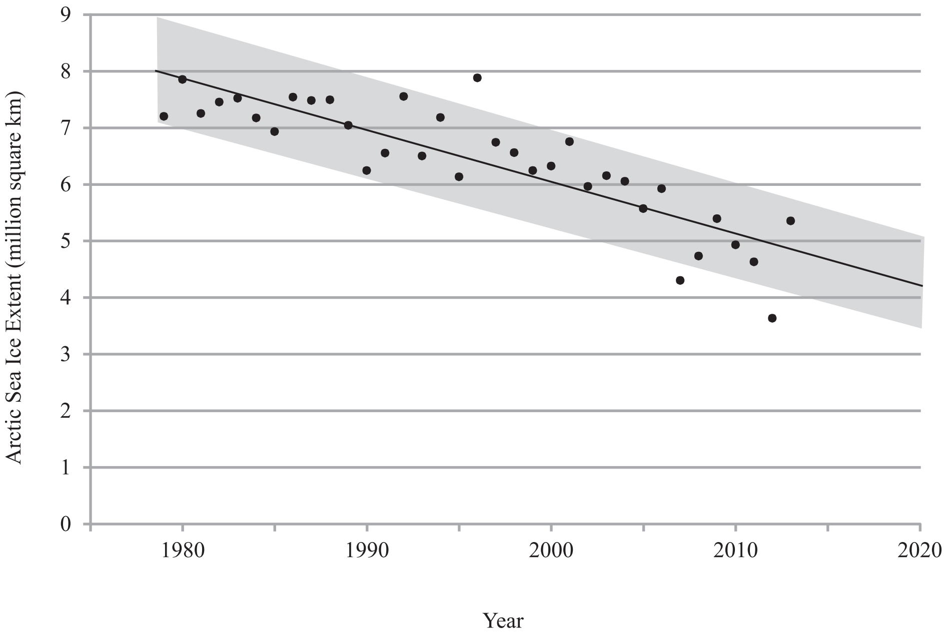


Figure 3.7. September north polar sea ice extent, 1979–2012, shown as dots. The straight line represents the best possible arithmetic growth model. Nearly all of the data points lie in the shaded area, and we suppose that future observations of sea ice extent will continue this pattern.

the best parameters is a bit beyond the scope of the present discussion, but the main idea is roughly to choose a model so that the average vertical distance from the dots to the line is as small as possible. That is the model represented by the line in Figure 3.7.

But why should we choose the best *straight line*? Or equivalently, why should we choose the best arithmetic growth model? Perhaps some other type of model would be better. And indeed, it is not difficult to visualize an arching trend *curve* (instead of a trend line) as in Figure 3.8. This suggests considering a quadratic growth model, even though an analysis of the second differences shows they are nowhere near to being constant.

Just as in the case of a linear model, there are known procedures for defining a quadratic model with the best possible parameters. As before, that means roughly that the average vertical distance from the data points to the graph of the model is as small as possible. However, with a quadratic model, the graph is a parabolic trend curve, not a straight trend line.

The procedures for finding the best quadratic model are pre-programmed in many popular graphing calculators and computer spreadsheet applications. Carrying out this analysis with the sea ice data produces the quadratic model shown in Figure 3.8. As before, the data points are shown as dots, and the quadratic model is shown as a curve. There is also a shaded area indicating how the data points are spread about the curve.

To the eye, the quadratic model does seem to track the cluster of data points more accurately than the linear model. That is not conclusive proof that the quadratic model

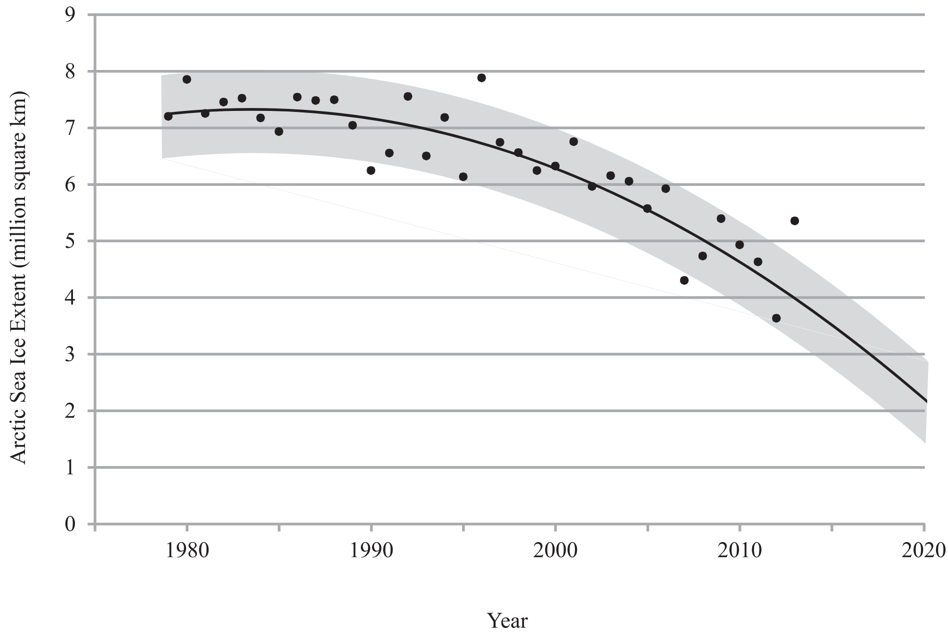


Figure 3.8. The sea ice data with a possible quadratic growth model.

is correct, but without further information, it suggests that the quadratic model is probably more credible than the linear model.

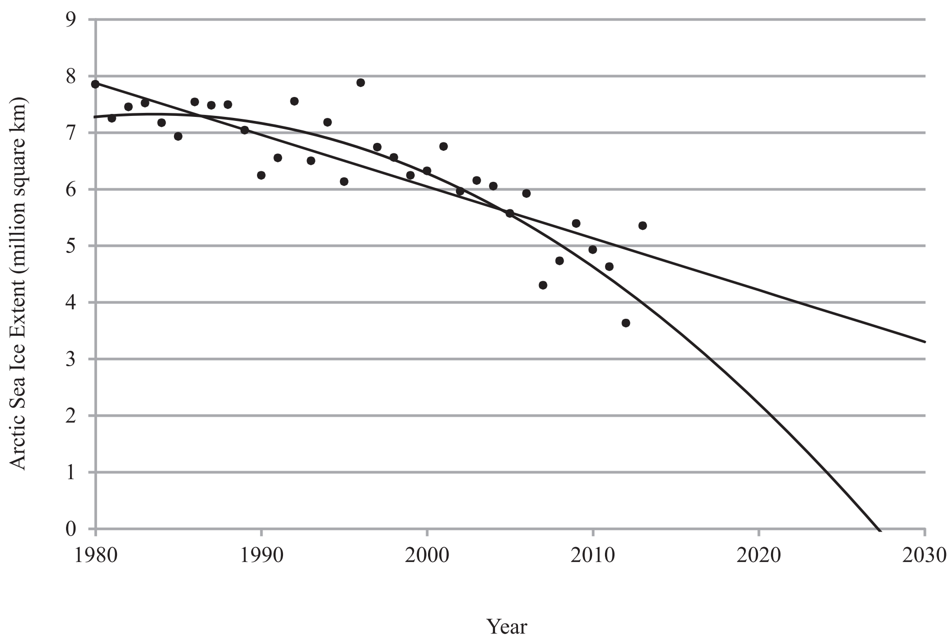


Figure 3.9. Sea ice extent data with linear and quadratic models.

The two models differ greatly in predicting future levels of September sea ice extent. To highlight the difference, we have included both models in a single graph, with the time line extended to 2030. See Figure 3.9. Because the parabolic curve bends away from the line, the further we look into the future, the wider is the separation between the two models. The quadratic model predicts that sea ice could completely disappear in the summer by 2025, whereas the linear model predicts that sea ice extent should still be in a range between 2 and 5 million square kilometers as late as 2030. Remember that we do not predict future levels of sea ice to follow either model curve exactly, but rather to range within an area centered on the curve. Even so, it is interesting to compare where each curve touches the horizontal axis. For the quadratic model that appears to be around 2027; for the linear model the line will not reach the axis until 2066.

As you can see, the two models lead to widely different predictions on the future course of summer sea ice disappearance. Here we are looking at extremely simple models and using relatively unsophisticated methods. But what we have seen hints at the importance and difficulty of validating accurate models for climate change phenomena. At the very least, we can see how a quadratic model might be considered as one possibility, even when the data are far from following the constant second difference rule. And we can see how dramatic an impact the nature of the model can have on what it predicts for the future.

In all of the examples discussed so far, a quadratic model was selected based on a set of data. But quadratic models can be suggested by other considerations. In some applications, the structure of the problem context leads naturally to the selection of a quadratic model. In particular, it may be possible to work out a difference equation based on a logical analysis of the context. If the difference equation has the right algebraic form, we can recognize it as an instance of quadratic growth. We will see several examples of that next.

Recursive Analysis and Network Problems. Quadratic growth difference equations arise naturally in a variety of problems featuring networks. A network can be thought of as a collection of objects connected together in some way. They may be cities connected by roads or airline routes, telephones connected by phone lines, computer processors connected by data pathways, or satellites connected by radio links. As an illustration, we will consider here a network of computers connected together to exchange email. A very naive approach would be to connect each pair of computers directly with a dedicated physical link, say a telephone line.[9] That might require a lot of phone lines for large numbers of computers. But how many? Suppose there are 100 computers, how many phone lines are needed? What if there are 1,000 computers? This problem can be analyzed using difference equations, and we will see that the type of equation that is developed is a quadratic growth difference equation.

[9]This is not a realistic way to connect large networks of computers, but it is an easily visualized example that illustrates the idea of quadratic growth in network problems. A more realistic problem is simply to enumerate the number of different pairs of computers. This might be of interest in analyzing the email traffic within the network. As a first step in monitoring the flow of data it would be natural to keep track of how much mail passed from any one computer to each of the others. This can be recorded for each pair of computers. How many pairs are there? Analyzing that question is mathematically identical to counting how many wires are needed to connect each pair of computers.

b. @Using the parameters found in the preceding step, formulate a difference equation for p_n.

c. @Investigate the behavior of this model using several different values of p_0, considering both numerical and graphical results. How do these results compare with the original logistic growth model developed for the mold example, as found in problem 22 of the Problems in Context section?

d. @Determine where the new model levels off by finding a fixed point of the difference equation.

(32) Repeat the preceding problem for the second fish population model (page 372). Recall that we assumed $r = 2.8$ when $p = 10{,}000$ and $r = 1$ when $p = 50{,}000$ for that model. Using these same assumptions, and the methods of the preceding problem, express r as a function of p in the form of equation (6.8). Then derive the corresponding difference equation and investigate the behavior of the model. How do your results compare with the results in Figure 6.6?

6.2 Chaos in Logistic Growth Sequences

Now we come to the topic of chaos, a subject that is the focus of research today in important areas of mathematics, engineering, and the physical and social sciences. Chaos in mathematics is closely related to the subject of fractals, widely publicized in the form of stunningly intricate and fascinating color images. James Gleick's popular book on the subject [**29**] was a best-seller. Chaos has even been referenced in popular culture, including in the film *Jurassic Park,* and an episode of The Simpsons.

But what exactly is chaos? And why is it of so much interest? Answering these questions in a limited way is one of the goals for this section. Following a brief general account of the main ideas of chaos, we will look in detail at chaos as it can arise in logistic models. Although this is only the tip of the chaos iceberg, it should give you an idea of several important aspects of chaos. Gleick's book discusses chaos in many other contexts, and describes the historical development of the subject, without going into much mathematical detail.

The Modeling Methodology. A key theme of this text is the application of mathematics to real world problems through the methods of mathematical modeling. We hope that the reader will gain a realistic understanding of how and why mathematics is useful, and that this understanding will extend beyond the specific details of the various types of models we have considered. In particular, we hope readers will witness the power and applicability of the modeling methodology.

It is appropriate to review this methodology here. We will highlight common aspects of the models we have studied up to this point. Note that these are all nonchaotic models. As we will see, the modeling methodology breaks down for chaotic models.

One key idea we have seen is the distinction between an actual phenomenon or problem context and a mathematical model. The model is a mathematical framework with functions, equations, graphs, and so on. We can apply mathematical procedures (such as algebra) to derive definite conclusions about the model. But we should always remember that these are not directly conclusions about the original problem context.

In almost all cases, models are based on simplifying assumptions. We have seen how such assumptions lead to models of several different types, for example arithmetic growth, geometric growth, etc. We have also seen the use of parameters, and how parameter values can be adjusted to improve the agreement between a model and observed data values. But even when our model agrees very closely with the data, errors remain possible. If our simplifying assumptions are only approximately true, if the observed data can only be measured approximately, then our model can only be expected to approximate the actual context or situation.

Given the approximate nature of a model, we are sometimes more interested in qualitative descriptions of whatever we are modeling than we are in highly accurate predictions of the future. Will the amount of a drug in the blood stream level off? Will a population of fish die out? The models give us qualitative answers to these questions.

The ability to formulate qualitative predictions is closely tied to the appearance of simple numerical and visual patterns in our models. In fact, all of our models have been developed by assuming simple numerical patterns hold in our data, at least approxmately. This is an important feature of our nonchaotic models. We will see that the sequences in chaotic models do not have simple numerical patterns.

There is another important feature of the models we have examined: moderately changing the values of the parameters does not significantly change the qualitative behavior. So, for a geometric growth model, if we have an incorrect value of r, we can still be pretty sure that the long-term behavior will be accelerating growth to ever larger values (for $r > 1$) or a decrease to 0 (for $r < 1$). Similarly, for certain mixed models and logistic growth models, we have found conditions under which the number sequence levels off. For these models, errors in the parameter values need not invalidate the qualitative prediction of leveling off. As long as the errors are within an acceptable range, we know that the number sequence will level off at a predictable value.

Conclusions that remain valid over a range of parameter values are sometimes described as *robust*. For a mixed model, the qualitative conclusion that the model will level off is robust because it doesn't change when the parameter values are slightly modified. This increases our confidence that the same qualitative conclusion will hold for the actual problem context. Even though the mixed model assumptions are only approximately true and the values of the parameters are only approximately correct, because leveling off will occur in a range of model formulations, we expect that it will also occur in the real context. Thus, robustness is an important aspect of using models to make qualitative predictions.

Two Important Aspects of Chaos. The preceding remarks lead us to two key aspects of chaotic models. First, they do not have simple numerical or visual patterns that can be used to predict future behavior. A chaotic model can shift among many kinds of behavior, seemingly at random. This often includes wild unexpected swings from one extreme to another. The term *chaos* reflects an apparent absence of order or pattern.

Second, it is a characteristic of chaos that the results are *not* robust: even miniscule changes in the parameters can lead to vastly different future behaviors. To emphasize this point, we compare two fictitious examples, first one that is not chaotic, then one that is chaotic.

A Nonchaotic Example. Imagine that you are a doctor interested in the way medicine is absorbed by the body. Your favorite applied mathematician describes a model