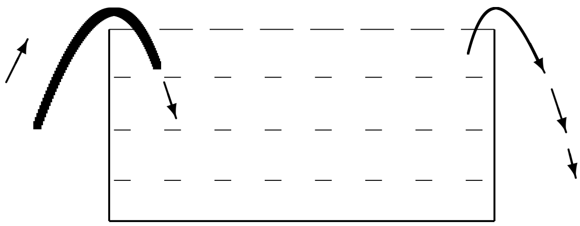


Salt-water mixture pumped into a full tank.

contains $\frac{y(t)}{10}$ kg of salt). Then, the salt flows out at the rate $2\frac{y(t)}{10} = 0.2y(t)$ kg/min. The difference of these two rates gives $y'(t)$, so that

$$y' = 0.6 - 0.2y.$$

This is a linear differential equation. Initially, there was no salt in the tank, so that $y(0) = 0$ is our initial condition. Solving this equation together with the initial condition, we have $y(t) = 3 - 3e^{-0.2t}$. After 4 minutes, we have $y(4) = 3 - 3e^{-0.8} \approx 1.65$ kg of salt in the tank.

Now suppose a patient has alcohol poisoning and doctors are pumping in water to flush his stomach out. One can compute similarly the weight of poison left in the stomach at time t. (An example is included in the Problems.)

1.5 Exact Equations

This section covers *exact equations*. While this class of equations is rather special, it often occurs in applications.

Let us begin by recalling partial derivatives. If a function $f(x) = x^2 + a$ depends on a parameter a, then $f'(x) = 2x$. If $g(x) = x^2 + y^3$, with a parameter y, we have $\frac{dg}{dx} = 2x$. Another way to denote this derivative is $g_x = 2x$. We can also regard g as a function of two variables, $g = g(x, y) = x^2 + y^3$. Then the *partial derivative* with respect to x is computed by regarding y to be a parameter, $g_x = 2x$. Alternative notation: $\frac{\partial g}{\partial x} = 2x$. Similarly, a partial derivative with respect to y is $g_y = \frac{\partial g}{\partial y} = 3y^2$. The derivative g_y gives us the rate of change in y, when x is kept fixed.

The equation (here $y = y(x)$)

$$y^2 + 2xyy' = 0$$

can be easily solved if we rewrite it in the equivalent form

$$\frac{d}{dx}\left(xy^2\right) = 0.$$

Then $xy^2 = c$, and the solution is

$$y(x) = \pm\frac{c}{\sqrt{x}}.$$

We wish to play the same game for general equations of the form

$$M(x, y) + N(x, y)y'(x) = 0. \tag{1.5.1}$$

Here the functions $M(x,y)$ and $N(x,y)$ are given. In the above example, $M = y^2$ and $N = 2xy$.

Definition. The equation (1.5.1) is called *exact* if there is a function $\psi(x,y)$, with continuous derivatives up to second order, so that we can rewrite (1.5.1) in the form

$$\frac{d}{dx}\psi(x,y) = 0. \tag{1.5.2}$$

The solution of the exact equation is (c is an arbitrary constant)

$$\psi(x,y) = c. \tag{1.5.3}$$

There are two natural questions: what conditions on $M(x,y)$ and $N(x,y)$ will force the equation (1.5.1) to be exact, and if the equation (1.5.1) is exact, how does one find $\psi(x,y)$?

Theorem 1.5.1. *Assume that the functions $M(x,y)$, $N(x,y)$, $M_y(x,y)$, and $N_x(x,y)$ are continuous in some disc $D : (x-x_0)^2+(y-y_0)^2 < r^2$, around some point (x_0,y_0). Then the equation* (1.5.1) *is exact in D if and only if the following partial derivatives are equal:*

$$M_y(x,y) = N_x(x,y), \qquad \text{for all points } (x,y) \text{ in } D. \tag{1.5.4}$$

This theorem makes two claims: if the equation is exact, then the partials are equal, and conversely, if the partials are equal, then the equation is exact.

Proof. (1) Assume that the equation (1.5.1) is exact, so that it can be written in the form (1.5.2). Performing the differentiation in (1.5.2), using the chain rule, gives

$$\psi_x + \psi_y y' = 0.$$

But this equation is the same as (1.5.1), so that

$$\psi_x = M,$$
$$\psi_y = N.$$

Taking the second partials,

$$\psi_{xy} = M_y,$$
$$\psi_{yx} = N_x.$$

We know from calculus that $\psi_{xy} = \psi_{yx}$; therefore, $M_y = N_x$.

(2) Assume that $M_y = N_x$. We will show that the equation (1.5.1) is then exact by producing $\psi(x,y)$. We have just seen that $\psi(x,y)$ must satisfy

$$\begin{aligned}\psi_x &= M(x,y),\\ \psi_y &= N(x,y).\end{aligned} \tag{1.5.5}$$

Take the antiderivative in x of the first equation

$$\psi(x,y) = \int_{x_0}^{x} M(t,y)\,dt + h(y), \tag{1.5.6}$$

where $h(y)$ is an arbitrary function of y and x_0 is an arbitrary number. To determine $h(y)$, substitute the last formula into the second line of (1.5.5):

$$\psi_y(x,y) = \int_{x_0}^{x} M_y(t,y)\,dt + h'(y) = N(x,y),$$

or

$$h'(y) = N(x,y) - \int_{x_0}^{x} M_y(t,y)\,dt \equiv p(x,y). \tag{1.5.7}$$

Observe that we denoted by $p(x,y)$ the right side of the last equation. It turns out that $p(x,y)$ does not really depend on x! Indeed, taking the partial derivative in x,

$$\frac{\partial}{\partial x} p(x,y) = N_x(x,y) - M_y(x,y) = 0,$$

because it was given to us that $M_y(x,y) = N_x(x,y)$, so that $p(x,y)$ is a function of y only, or $p(y)$. The equation (1.5.7) takes the form

$$h'(y) = p(y).$$

We determine $h(y)$ by integration and use it in (1.5.6) to get $\psi(x,y)$.

◇

Recall that the equation in differentials

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an alternative form of (1.5.1), so that it is exact if and only if $M_y = N_x$, for all x and y.

Example 1.5.1. Consider

$$e^x \sin y + y^3 - (3x - e^x \cos y)\frac{dy}{dx} = 0.$$

Here $M(x,y) = e^x \sin y + y^3$, $N(x,y) = -3x + e^x \cos y$. Compute

$$M_y = e^x \cos y + 3y^2,$$
$$N_x = e^x \cos y - 3.$$

The partials are not the same, this equation is not exact, and our theory does not apply.

Example 1.5.2. Solve (for $x > 0$)

$$\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2)\ dy = 0.$$

Here $M(x,y) = \frac{y}{x} + 6x$ and $N(x,y) = \ln x - 2$. Compute

$$M_y = \frac{1}{x} = N_x,$$

and so the equation is exact. To find $\psi(x,y)$, we observe that the equations (1.5.5) take the form

$$\psi_x = \frac{y}{x} + 6x,$$
$$\psi_y = \ln x - 2.$$

Take the antiderivative in x of the first equation,

$$\psi(x,y) = y \ln x + 3x^2 + h(y),$$

where $h(y)$ is an arbitrary function of y. Substitute this $\psi(x,y)$ into the second equation,

$$\psi_y = \ln x + h'(y) = \ln x - 2,$$

which gives

$$h'(y) = -2.$$

Integrating, $h(y) = -2y$, and so $\psi(x, y) = y \ln x + 3x^2 - 2y$, giving us the solution

$$y \ln x + 3x^2 - 2y = c.$$

We can solve this relation for y, $y(x) = \frac{c-3x^2}{\ln x - 2}$. Observe that when solving for $h(y)$, we chose the integration constant to be zero, because at the next step we set $\psi(x, y)$ equal to c, an arbitrary constant.

Example 1.5.3. Find the constant b, for which the equation

$$\left(2x^3 e^{2xy} + x^4 y e^{2xy} + x\right) dx + b x^5 e^{2xy}\, dy = 0$$

is exact, and then solve the equation with that b.

Here $M(x, y) = 2x^3 e^{2xy} + x^4 y e^{2xy} + x$, and $N(x, y) = b x^5 e^{2xy}$. Setting equal the partials M_y and N_x, we have

$$5x^4 e^{2xy} + 2x^5 y e^{2xy} = 5b x^4 e^{2xy} + 2b x^5 y e^{2xy}.$$

One needs $b = 1$ for this equation to be exact. When $b = 1$, the equation becomes

$$\left(2x^3 e^{2xy} + x^4 y e^{2xy} + x\right) dx + x^5 e^{2xy}\, dy = 0,$$

and we already know that it is exact. We look for $\psi(x, y)$ by using (1.5.5), as in Example 1.5.2

$$\psi_x = 2x^3 e^{2xy} + x^4 y e^{2xy} + x,$$
$$\psi_y = x^5 e^{2xy}.$$

It is easier to begin this time with the second equation. Taking the antiderivative in y, in the second equation,

$$\psi(x, y) = \frac{1}{2} x^4 e^{2xy} + h(x),$$

where $h(x)$ is an arbitrary function of x. Substituting $\psi(x, y)$ into the first equation gives

$$\psi_x = 2x^3 e^{2xy} + x^4 y e^{2xy} + h'(x) = 2x^3 e^{2xy} + x^4 y e^{2xy} + x.$$

This tells us that $h'(x) = x$, $h(x) = \frac{1}{2}x^2$, and then $\psi(x, y) = \frac{1}{2} x^4 e^{2xy} + \frac{1}{2} x^2$.

Answer. $\frac{1}{2} x^4 e^{2xy} + \frac{1}{2} x^2 = c$, or $y = \frac{1}{2x} \ln\left(\frac{2c - x^2}{x^4}\right)$.

Exact equations are connected with conservative vector fields. Recall that a vector field $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle M(x, y), N(x, y)\rangle$ is called *conservative* if there is a function $\psi(x, y)$, called the *potential*, such that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \nabla\psi(x, y)$. Recalling that the gradient $\nabla\psi(x, y) = \langle \psi_x, \psi_y \rangle$, we have $\psi_x = M$ and $\psi_y = N$, the same relations that we had for exact equations.

1.6 Existence and Uniqueness of Solution

We consider a general initial value problem

$$y' = f(x, y),$$
$$y(x_0) = y_0,$$

with a given function $f(x, y)$ and given numbers x_0 and y_0. Let us ask two basic questions: is there a solution to this problem, and if there is, is the solution unique?

Theorem 1.6.1. *Assume that the functions $f(x, y)$ and $f_y(x, y)$ are continuous in some neighborhood of the initial point (x_0, y_0). Then there exists a solution, and there is only one solution. The solution $y = y(x)$ is defined on some interval (x_1, x_2) that includes x_0.*

One sees that the conditions of this theorem are not too restrictive, so that the theorem tends to apply, providing us with the existence and uniqueness of the solution. But not always!

Example 1.6.1. Solve

$$y' = \sqrt{y},$$
$$y(0) = 0.$$

The function $f(x, y) = \sqrt{y}$ is continuous (for $y \geq 0$), but its partial derivative in y, $f_y(x, y) = \frac{1}{2\sqrt{y}}$, is not even defined at the initial point $(0, 0)$. The theorem does not apply. One checks that the function $y = \frac{x^2}{4}$ solves our initial value problem (for $x \geq 0$). But here is another solution: $y(x) = 0$. (Having two different solutions to the same initial value problem is like having two prima donnas in the same theater.)

Observe that the theorem guarantees existence of solution only on some interval (it is not "happily ever after").

Example 1.6.2. Solve for $y = y(t)$

$$y' = y^2,$$
$$y(0) = 1.$$

Here $f(t, y) = y^2$ and $f_y(t, y) = 2y$ are continuous functions. The theorem applies. By separation of variables, we determine the solution $y(t) = \frac{1}{1-t}$. As time t approaches 1, this solution disappears, by going to infinity. This phenomenon is sometimes called the *blow-up in finite time*.

1.7 Numerical Solution by Euler's Method

We have learned a number of techniques for solving differential equations; however, the sad truth is that most equations cannot be solved (by a formula). Even a simple looking equation like

$$y' = x + y^3 \tag{1.7.1}$$

is totally out of reach. Fortunately, if you need a specific solution, say the one satisfying the initial condition

$$y(0) = 1, \tag{1.7.2}$$

it can be easily approximated using the method developed in this section (by Theorem 1.6.1, such a solution exists, and it is unique because $f(x, y) = x + y^3$ and $f_y(x, y) = 3y^2$ are continuous functions).

In general, we shall deal with the problem

$$y' = f(x, y),$$
$$y(x_0) = y_0.$$

Here the function $f(x, y)$ is given (in the example above we had $f(x, y) = x + y^3$), and the initial condition prescribes that the solution is equal to a given number y_0 at a given point x_0. Fix a step size h, and let $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh$. We will approximate $y(x_n)$, the value of the solution at x_n. We call this approximation y_n. To go from the point (x_n, y_n) to the point (x_{n+1}, y_{n+1}) on the graph of the solution $y(x)$, we use the tangent line approximation:

$$y_{n+1} \approx y_n + y'(x_n)(x_{n+1} - x_n) = y_n + y'(x_n)h = y_n + f(x_n, y_n)h.$$

(We expressed $y'(x_n) = f(x_n, y_n)$ from the differential equation. Because of the approximation errors, the point (x_n, y_n) is not exactly lying on the solution curve $y = y(x)$, but we pretend that it does.) The resulting formula is easy to implement; it is just one computational loop, starting with the initial point (x_0, y_0).

One continues the computations until the points x_n go as far as needed. Decreasing the step size h will improve the accuracy. Smaller h's will require more steps, but with the power of modern computers, that is not a problem, particularly for simple examples, like the problem (1.7.1), (1.7.2), which is discussed next. In that example $x_0 = 0$, $y_0 = 1$. If we choose $h = 0.05$, then $x_1 = 0.05$, and

$$y_1 = y_0 + f(x_0, y_0)h = 1 + (0 + 1^3)0.05 = 1.05.$$

Continuing, we have $x_2 = 0.1$ and

$$y_2 = y_1 + f(x_1, y_1)h = 1.05 + (0.05 + 1.05^3)0.05 \approx 1.11.$$

Next, $x_3 = 0.15$, and

$$y_3 = y_2 + f(x_2, y_2)h = 1.11 + (0.1 + 1.11^3)0.05 \approx 1.18.$$

These computations imply that $y(0.05) \approx 1.05$, $y(0.1) \approx 1.11$, and $y(1.15) \approx 1.18$. If you need to approximate the solution on the interval $(0, 0.4)$, you have to make five more steps. Of course, it is better to program a computer. A computer computation reveals that this solution tends to infinity (blows up) at $x \approx 0.47$. Figure 1.3 presents the solution curve, computed by *Mathematica*, as well as the three points we computed by Euler's method.

Euler's method is using the tangent line approximation, or the first two terms of the Taylor series approximation. One can use more terms of the Taylor series and develop more sophisticated methods (which is done in books on numerical methods and implemented in software packages, like *Mathematica*). But here is a question: if it is so easy to compute numerical approximation of solutions, why bother learning analytical solutions? The reason is that we seek not just to solve a differential equation, but to understand it. What happens if the initial condition changes? The equation may include some parameters. What happens if they change? What happens to solutions in the long term?

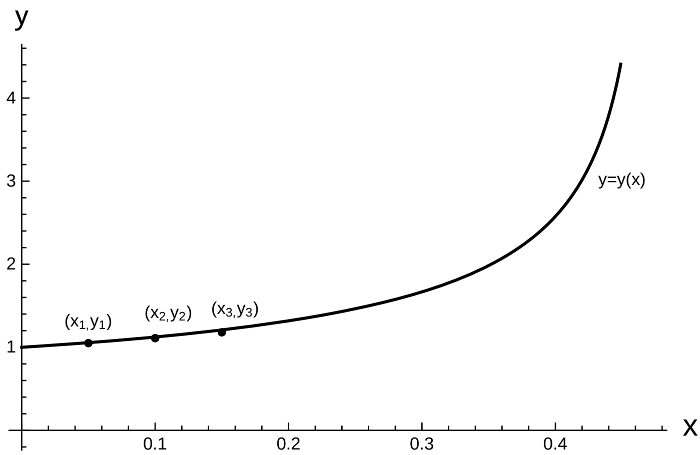


Figure 1.3. The numerical solution of $y' = x + y^3$, $y(0) = 1$.

1.7.1 Problems.

I. Determine if the equation is homogeneous, and if it is, solve it.

1. $\frac{dy}{dx} = \frac{y+2x}{x}$, with $x > 0$.

   Answer. $y = x\,(2\ln x + c)$.

2. $(x + y)\,dx - x\,dy = 0$.

   Answer. $y = x\,(\ln|x| + c)$.

3. $\frac{dy}{dx} = \frac{x^2-xy+y^2}{x^2}$.

   Answer. $y = x\left(1 - \frac{1}{\ln|x|+c}\right)$, and $y = x$.

4. $\frac{dy}{dx} = \frac{y^2+2x}{y}$.

5. $y' = \frac{y^2}{x^2} + \frac{y}{x}$, $y(1) = 1$.

   Answer. $y = \frac{x}{1-\ln x}$.

6. $y' = \frac{y^2}{x^2} + \frac{y}{x}$, $y(-1) = 1$.

   Answer. $y = -\frac{x}{1+\ln|x|}$.

7. $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$, $y(1) = 2$.

   Answer. $y = \frac{2x^2}{3-2x}$.

8. $xy' - y = x \tan \frac{y}{x}$.

   Answer. $\sin \frac{y}{x} = cx$.

9. $xy' = \frac{x^2}{x+y} + y$.

   Answer. $y = -x \pm x\sqrt{2 \ln |x| + c}$.

10. $y' = \frac{x^2+y^2}{xy}$, $y(1) = -2$.

    Answer. $y = -x\sqrt{2 \ln x + 4}$.

11. $y' = \frac{y+x^{-1/2}y^{3/2}}{\sqrt{xy}}$, with $x > 0$, $y > 0$.

    Answer. $2\sqrt{\frac{y}{x}} = \ln x + c$.

12. $x^3 y' = y^2 (y - xy')$.

    Answer. $\ln |y| + \frac{1}{2}\left(\frac{y}{x}\right)^2 = c$, and $y = 0$.

13.* A function $f(x, y)$ is called *quasi-homogeneous* if for any constant α

$$f(\alpha x, \alpha^p y) = \alpha^{p-1} f(x, y),$$

with some constant p.

(i) Letting $\alpha = \frac{1}{x}$ and $v = \frac{y}{x^p}$, verify that

$$f(x, y) = x^{p-1} g(v),$$

where $g(v)$ is some function of one variable.

(ii) Consider a *quasi-homogeneous equation*

$$y' = f(x, y),$$

where $f(x, y)$ is a quasi-homogeneous function. Show that a change of variables $v = \frac{y}{x^p}$ produces a separable equation.

(iii) Solve

$$y' = x + \frac{y^2}{x^3}.$$

Hint. Denoting $f(x,y) = x + \frac{y^2}{x^3}$, we have $f(\alpha x, \alpha^2 y) = \alpha f(x,y)$, so that $p = 2$. Letting $v = \frac{y}{x^2}$, or $y = x^2 v$, we get

$$xv' = 1 - 2v + v^2.$$

Answer. $y = x^2\left(1 - \frac{1}{\ln|x|+c}\right)$.

II. Solve the following Bernoulli's equations.

1. $y'(t) = 3y - y^2$.

   Answer. $y = \frac{3}{1+ce^{-3t}}$, and $y = 0$.

2. $y' - \frac{1}{x}y = y^2$, $y(2) = -2$.

   Answer. $y = \frac{2x}{2-x^2}$.

3. $xy' + y + xy^2 = 0$, $y(1) = 2$.

   Answer. $y = \frac{2}{x(1+2\ln x)}$.

4. $y' + y = xy^3$, $y(0) = -1$.

   Answer. $y = -\frac{\sqrt{2}}{\sqrt{2x+e^{2x}+1}}$.

5. $\frac{dy}{dx} = \frac{y^2+2x}{y}$.

   Answer. $y = \pm\sqrt{-1 - 2x + ce^{2x}}$.

6. $y' + x\sqrt[3]{y} = 3y$.

   Answer. $y = \pm\left(\frac{x}{3} + \frac{1}{6} + ce^{2x}\right)^{\frac{3}{2}}$, and $y = 0$.

   Hint. When dividing the equation by $\sqrt[3]{y}$, one needs to check if $y = 0$ is a solution, and indeed it is.

7. $y' + y = -xy^2$.

   Answer. $y = \frac{1}{ce^x - x - 1}$, and $y = 0$.

8. $y' + xy = y^3$, $y(1) = -\frac{1}{e}$.

   Answer. $y = -\frac{1}{\sqrt{-2e^{x^2}\int_1^x e^{-t^2}\,dt + e^{(x^2+1)}}}$.

9. The equation

$$\frac{dy}{dx} = \frac{y^2 + 2x}{y}$$

could not be solved in the preceding problem set because it is not homogeneous. Can you solve it now?

Answer. $y = \pm\sqrt{ce^{2x} - 2x - 1}$.

10. $y' = \frac{x}{y}e^{2x} + y$.

Answer. $y = \pm e^x\sqrt{x^2 + c}$.

11. Solve the *Gompertz population model* (a and b are positive constants)

$$\frac{dx}{dt} = x\,(a - b\ln x)\,, \quad x > 1.$$

Hint. Setting $y = \ln x$, obtain $y' = a - by$.

Answer. $x(t) = e^{a/b}e^{ce^{-bt}}$.

12. Solve

$$x(y' - e^y) + 2 = 0.$$

Hint. Divide the equation by e^y and then set $v = e^{-y}$, obtaining a linear equation for $v = v(x)$.

Answer. $y = -\ln\left(x + cx^2\right)$.

13. $\frac{dy}{dx} = \frac{y}{x + x^2 y}$.

Hint. Consider $\frac{dx}{dy}$, and obtain Bernoulli's equation for $x(y)$.

Answer. $x = \frac{2y}{c - y^2}$.

III. 1. Use parametric integration to solve

$$y'^3 + y' = x.$$

Answer. $x = t^3 + t$, $y = \frac{3}{4}t^4 + \frac{1}{2}t^2 + c$.

2. Use parametric integration to solve

$$y = \ln(1 + y'^2).$$

Answer. $x = 2\tan^{-1} t + c$, $y = \ln(1 + t^2)$. Another solution is $y = 0$.

3. Use parametric integration to solve

$$y' + \sin(y') = x, \quad y(0) = 0.$$

Answer. $x = t + \sin t$, $y = \frac{1}{2}t^2 + t\sin t + \cos t - 1$.

4. Solve the logistic model (for $0 < y < 3$)

$$y'(t) = 3y - y^2, \quad y(0) = 1$$

as a separable equation. What is the carrying capacity? What is $\lim_{t\to\infty} y(t)$?

Answer. $y(t) = \frac{3}{1+2e^{-3t}}$, $\lim_{t\to\infty} y(t) = 3$.

5. A tank is completely filled with 100L of water-salt mixture, which initially contains 10 kg of salt. Water is flowing in at a rate of 5L per minute. The new mixture flows out at the same rate. How much salt remains in the tank after an hour?

Answer. Approximately 0.5 kg.

6. A tank is completely filled with 100L of water-salt mixture, which initially contains 10 kg of salt. A water-salt mixture is flowing in at a rate of 3L per minute, and each liter of it contains 0.1 kg of salt. The new mixture flows out at the same rate. How much salt is contained in the tank after t minutes?

Answer. 10 kg.

7. Water is being pumped into patient's stomach at a rate of 0.5L per minute to flush out 300 grams of alcohol poisoning. The excess fluid is flowing out at the same rate. The stomach holds 3L. The patient can be discharged when the amount of poison drops to 50 grams. How long should this procedure last?

Answer. $t = 6\ln 6 \approx 10.75$ minutes.

8. Temperature in a room is maintained at 70°. If an object at 100° is placed in this room, it cools down to 80° in 5 minutes. A bowl of soup at 190° is placed in this room. The soup is ready to eat at 130°. How many minutes should one wait?

Hint. If $y(t)$ is the temperature after t minutes, it is natural to assume that the speed of cooling is proportional to the difference of temperatures, so that

$$y' = -k(y - 70)$$

for some constant $k > 0$. We are given that $y(5) = 80$, provided that $y(0) = 100$. This allows us to calculate $k = \frac{\ln 3}{5}$. Then assuming that $y(0) = 190$, one calculates t such that $y(t) = 130$.

Answer. $t = 5\frac{\ln 2}{\ln 3} \approx 3.15$ minutes.

9. Find all curves $y = f(x)$ with the following property: if you draw a tangent line at any point $(x_0, f(x_0))$ on this curve and continue the tangent line until it intersects the x-axis, then the point of intersection is $x_0/2$.

   Answer. $y = cx^2$ (a family of parabolas).

10. Find all positive decreasing functions $y = f(x)$ with the following property: in the triangle formed by the vertical line going down from the curve, the x-axis, and the tangent line to this curve, the sum of two sides adjacent to the right angle is a constant, equal to $b > 0$.

    Answer. $y - b \ln y = x + c$.

11. Find all positive decreasing functions $y = f(x)$ with the following property: for the tangent line at $(x_0, f(x_0))$, the length of the segment between the point $(x_0, f(x_0))$ and the y-axis is equal to 1, for all $0 < x_0 \le 1$.

    Answer. $y = -\sqrt{1-x^2} - \ln x + \ln\left[1 + \sqrt{1-x^2}\right] + c$. This historic curve (first studied by Huygens in 1692) is called the *tractrix*.

12. Find all curves $y = f(x)$ such that the point of intersection of the tangent line at $(x_0, f(x_0))$ with the x-axis is equidistant from the origin and the point $(x_0, f(x_0))$, at any x_0.

    Answer. $x^2 + y^2 = cy$, a family of circles. (*Hint.* The differential equation $y' = \frac{2xy}{x^2-y^2}$ is homogeneous.)

13. Solve Riccati's equation

    $$y' + 2e^x y - y^2 = e^x + e^{2x}.$$

    Answer. $y = e^x$, and $y = e^x - \frac{1}{x+c}$.

14. Solve Riccati's equation

    $$y' + (2e^x + 2)\, y - e^x y^2 = e^x + 2.$$

    Answer. $y = 1$, and $y = 1 + \frac{1}{e^x + ce^{2x}}$.

15.* (From the Putnam competition, 2009) Show that any solution of

    $$y' = \frac{x^2 - y^2}{x^2(y^2+1)}$$

    satisfies $\lim_{x\to\infty} y(x) = \infty$.

    Hint. Using "partial fractions", rewrite this equation as

    $$y' = \frac{1 + 1/x^2}{y^2+1} - \frac{1}{x^2}.$$

Then $y'(x) > -\frac{1}{x^2}$, which precludes $y(x)$ from going to $-\infty$. So, either $y(x)$ is bounded or it goes to $+\infty$, as $x \to \infty$ (possibly along some sequence). If $y(x)$ is bounded when x is large, then $y'(x)$ exceeds a positive constant for all large x, and therefore $y(x)$ tends to infinity, a contradiction (observe that $1/x^2$ becomes negligible for large x). Finally, if $y(x)$ failed to tend to infinity as $x \to \infty$ (while going to infinity over a subsequence), it would have infinitely many points of local minimum, at which $y = x$, a contradiction.

16. Solve the *integral equation*

$$y(x) = \int_1^x y(t)\,dt + x + 1.$$

Hint. Differentiate the equation, and also evaluate $y(1)$.

Answer. $y = 3e^{x-1} - 1$.

IV. Determine if the equation is exact, and if it is, solve it.

1. $(2x + 3x^2y)\,dx + (x^3 - 3y^2)\,dy = 0$.

   Answer. $x^2 + x^3y - y^3 = c$.

2. $(x + \sin y)\,dx + (x\cos y - 2y)\,dy = 0$.

   Answer. $\frac{1}{2}x^2 + x\sin y - y^2 = c$.

3. $\frac{x}{x^2+y^4}\,dx + \frac{2y^3}{x^2+y^4}\,dy = 0$.

   Answer. $x^2 + y^4 = c$.

4. Find a simpler solution for the preceding problem.

5. $(6xy - \cos y)\,dx + (3x^2 + x\sin y + 1)\,dy = 0$.

   Answer. $3x^2y - x\cos y + y = c$.

6. $(2x - y)\,dx + (2y - x)\,dy = 0$, $y(1) = 2$.

   Answer. $x^2 + y^2 - xy = 3$.

7. $2x\left(1 + \sqrt{x^2 - y}\right)\,dx - \sqrt{x^2 - y}\,dy = 0$.

   Answer. $x^2 + \frac{2}{3}\left(x^2 - y\right)^{\frac{3}{2}} = c$.

8. $(ye^{xy}\sin 2x + 2e^{xy}\cos 2x + 2x)\,dx + (xe^{xy}\sin 2x - 2)\,dy = 0$, $y(0) = -2$.

   Answer. $e^{xy}\sin 2x + x^2 - 2y = 4$.

9. Find the value of b for which the following equation is exact, and then solve the equation, using that value of b:
$$(ye^{xy} + 2x)\,dx + bxe^{xy}\,dy = 0\,.$$

   Answer. $b = 1$, $y = \frac{1}{x}\ln(c - x^2)$.

10. Verify that the equation
$$(2\sin y + 3x)\,dx + x\cos y\,dy = 0$$
is not exact; however, if one multiplies it by x, the equation becomes exact, and it can be solved.

    Answer. $x^2\sin y + x^3 = c$.

11. Verify that the equation
$$(x - 3y)\,dx + (x + y)\,dy = 0$$
is not exact; however, it can be solved as a homogeneous equation.

    Answer. $\ln|y - x| + \frac{2x}{x-y} = c$.

V. 1. Find three solutions of the initial value problem
$$y' = (y-1)^{1/3}, \quad y(1) = 1.$$
Is it desirable in applications to have three solutions of the same initial value problem? What "went wrong"? (Why doesn't the existence and uniqueness Theorem 1.6.1 apply here?)

   Answer. $y(x) = 1$, and $y(x) = 1 \pm \left(\frac{2}{3}x - \frac{2}{3}\right)^{\frac{3}{2}}$.

2. Find all y_0 for which the following problem has a unique solution:
$$y' = \frac{x}{y^2 - 2x}, \qquad y(2) = y_0.$$

   Hint. Apply the existence and uniqueness Theorem 1.6.1.

   Answer. All y_0 except ± 2.

3. Show that the function $\frac{x|x|}{4}$ solves the problem
$$y' = \sqrt{|y|},$$
$$y(0) = 0,$$
for all x. Can you find another solution?

Hint. Consider separately the cases when $x > 0$, $x < 0$, and $x = 0$.

4. Show that the problem (here $y = y(t)$)

$$\begin{aligned} y' &= y^{2/3}, \\ y(0) &= 0 \end{aligned}$$

has infinitely many solutions.

Hint. Consider $y(t)$ that is equal to zero for $t < a$ and to $\frac{(t-a)^3}{27}$ for $t \geq a$ where $a > 0$ is any constant.

5. (i) Apply Euler's method to

$$y' = x(1+y), \quad y(0) = 1.$$

Take $h = 0.25$, and do four steps, obtaining an approximation for $y(1)$.

(ii) Take $h = 0.2$, and do five steps of Euler's method, obtaining another approximation for $y(1)$.

(iii) Solve the above problem exactly, and determine which one of the two approximations is better.

6. Write a computer program to implement Euler's method for

$$y' = f(x,y), \quad y(x_0) = y_0.$$

It involves a simple loop: $y_{n+1} = y_n + hf(x_0 + nh, y_n)$, $n = 0, 1, 2, \ldots$.

1.8* The Existence and Uniqueness Theorem

In this section, for the initial value problem

$$\begin{aligned} y' &= f(x,y), \\ y(x_0) &= y_0, \end{aligned} \tag{1.8.1}$$

we prove a more general existence and uniqueness theorem than Theorem 1.6.1 stated above.

Define a rectangular box B around the initial point (x_0, y_0) to be the set of points (x, y), satisfying $x_0 - a \leq x \leq x_0 + a$ and $y_0 - b \leq y \leq y_0 + b$, for some positive a and b. It is known from calculus that in case $f(x,y)$ is continuous on B, it is bounded on B, so that for some constant $M > 0$

$$|f(x,y)| \leq M, \quad \text{for all points } (x,y) \text{ in } B. \tag{1.8.2}$$

Theorem 1.8.1. *Assume that the function $f(x,y)$ is continuous on B and for some constant $L > 0$, it satisfies (the Lipschitz condition)*

$$|f(x,y_2) - f(x,y_1)| \leq L|y_2 - y_1|, \tag{1.8.3}$$

for any two points (x,y_1) and (x,y_2) in B. Then the initial value problem (1.8.1) *has a unique solution, which is defined for x on the interval $(x_0 - \frac{b}{M}, x_0 + \frac{b}{M})$, in case $\frac{b}{M} < a$, and on the interval $(x_0 - a, x_0 + a)$ if $\frac{b}{M} \geq a$.*

Proof. Assume, for definiteness, that $\frac{b}{M} < a$, and the other case is similar. We shall prove the existence of solutions first, and let us restrict ourselves to the case $x > x_0$ (the case when $x < x_0$ is similar). Integrating the equation in (1.8.1) over the interval (x_0, x), we convert the initial value problem (1.8.1) into an equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt. \tag{1.8.4}$$

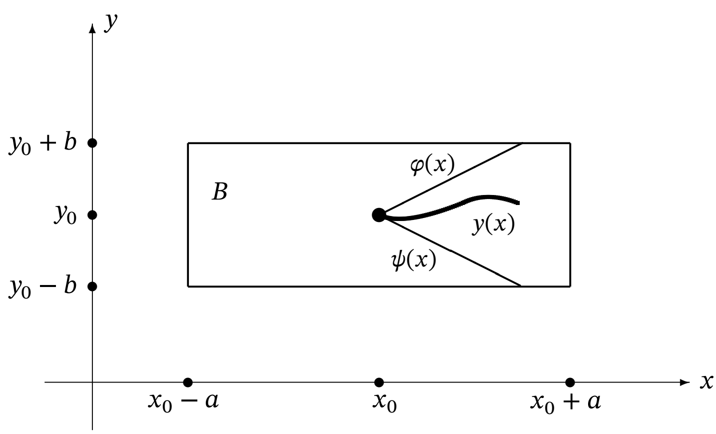
(If $y(x)$ solves (1.8.4), then $y(x_0) = y_0$, and by differentiation $y' = f(x, y)$.) By (1.8.2), obtain

$$-M \le f(t, y(t)) \le M, \tag{1.8.5}$$

and then any solution of (1.8.4) lies between two straight lines

$$y_0 - M(x - x_0) \le y(x) \le y_0 + M(x - x_0).$$

For $x_0 \le x \le x_0 + \frac{b}{M}$ these lines stay in the box B, reaching its upper and lower boundaries at $x = x_0 + \frac{b}{M}$. (In the other case, when $\frac{b}{M} \ge a$, these lines stay in B for all $x_0 \le x \le x_0 + a$.) We denote $\varphi(x) = y_0 + M(x - x_0)$ and call this function a *supersolution*, while $\psi(x) = y_0 - M(x - x_0)$ is called a *subsolution*.



The functions $\varphi(x)$ and $\psi(x)$ exiting the box B.

1. A special case. Let us make an additional assumption that $f(x, y)$ is increasing in y, so that if $y_2 > y_1$, then $f(x, y_2) > f(x, y_1)$, for any two points (x, y_1) and (x, y_2) in B. We shall construct a solution of (1.8.1) as the limit of a sequence of iterates $\psi(x)$, $y_1(x)$,

$y_2(x), \ldots, y_n(x), \ldots$, defined as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt,$$
$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt,$$
$$\vdots$$
$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt.$$

We claim that for all x on the interval $x_0 < x \le x_0 + \frac{b}{M}$, the following inequalities hold:

$$\psi(x) \le y_1(x) \le y_2(x) \le \cdots \le y_n(x) \le \cdots. \tag{1.8.6}$$

Indeed, $f(t, \psi(t)) \ge -M$, by (1.8.5), and then

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt \ge y_0 - M(x - x_0) = \psi(x),$$

giving us the first of the inequalities in (1.8.6). Then

$$y_2(x) - y_1(x) = \int_{x_0}^{x} [f(t, y_1(t)) - f(t, \psi(t))]\ dt \ge 0,$$

using the just established inequality $\psi(x) \le y_1(x)$ and the monotonicity of $f(x, y)$, so that $y_1(x) \le y_2(x)$ and the other inequalities in (1.8.6) are established similarly. Next, we claim that for any x on the interval $x_0 < x \le x_0 + \frac{b}{M}$, all of these iterates lie below the supersolution $\varphi(x)$, so that

$$\psi(x) \le y_1(x) \le y_2(x) \le \cdots \le y_n(x) \le \cdots \le \varphi(x). \tag{1.8.7}$$

Indeed, $f(t, \psi(t)) \le M$, by (1.8.5), giving

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt \le y_0 + M(x - x_0) = \varphi(x),$$

proving the first inequality in (1.8.7) and that the graph of $y_1(x)$ stays in the box B for $x_0 < x \le x_0 + \frac{b}{M}$. Then

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \le y_0 + M(x - x_0) = \varphi(x),$$

and so on, for all $y_n(x)$.

At each x in $(x_0, x_0 + \frac{b}{M})$, the numerical sequence $\{y_n(x)\}$ is nondecreasing, bounded above by the number $\varphi(x)$. Hence, this sequence has a limit which we denote by $y(x)$. The sequence $f(x, y_n(x))$ is also nondecreasing, and it converges to $f(x, y(x))$. By the monotone convergence theorem, we may pass to the limit in the recurrence relation

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt, \tag{1.8.8}$$

concluding that $y(x)$ gives the desired solution of the integral equation (1.8.4).

(If one starts the recurrence relation (1.8.8) with the supersolution $\phi(x)$, one obtains similarly a decreasing sequence of iterates converging to a solution of (1.8.4).)

2. The general case. Define $g(x,y) = f(x,y)+Ay$. If we choose the constant A large enough, then the new function $g(x,y)$ will be increasing in y, for $(x,y) \in B$. Indeed, using the Lipschitz condition (1.8.3),

$$\begin{aligned} g(x,y_2) - g(x,y_1) &= f(x,y_2) - f(x,y_1) + A\,(y_2 - y_1) \\ &\geq -L\,(y_2 - y_1) + A\,(y_2 - y_1) \\ &= (A-L)\,(y_2 - y_1) \\ &> 0, \end{aligned}$$

for any two points (x,y_1) and (x,y_2) in B, provided that $A > L$ and $y_2 > y_1$. We now consider an equivalent equation (recall that $g(x,y) = f(x,y)+Ay$)

$$y' + Ay = f(x,y) + Ay = g(x,y).$$

Multiplying both sides by the integrating factor e^{Ax}, we put this equation into the form

$$\frac{d}{dx}\left[e^{Ax}y\right] = e^{Ax}g(x,y).$$

Set $z(x) = e^{Ax}y(x)$; then $y(x) = e^{-Ax}z(x)$, and the new unknown function $z(x)$ satisfies

$$\begin{aligned} z' &= e^{Ax}g\left(x, e^{-Ax}z\right), \\ z(x_0) &= e^{Ax_0}y_0. \end{aligned} \tag{1.8.9}$$

The function $e^{Ax}g\left(x,e^{-Ax}z\right)$ is increasing in z. The special case applies, so that the solution $z(x)$ of (1.8.9) exists. Then $y(x) = e^{-Ax}z(x)$ gives the desired solution of (1.8.1).

Finally, we prove the uniqueness of solution. Let $u(x)$ be another solution of (1.8.4) on the interval $(x_0, x_0 + \frac{b}{M})$, so that

$$u(x) = y_0 + \int_{x_0}^{x} f(t,u(t))\,dt.$$

Subtracting this from (1.8.4) gives

$$y(x) - u(x) = \int_{x_0}^{x} \left[f(t,y(t)) - f(t,u(t))\right]\,dt.$$

Assume first that x is in $[x_0, x_0 + \frac{1}{2L}]$. Then using the Lipschitz condition (1.8.3), we estimate

$$\begin{aligned} |y(x) - u(x)| &\leq \int_{x_0}^{x} |f(t,y(t)) - f(t,u(t))|\,dt \\ &\leq L \int_{x_0}^{x} |y(t)) - u(t)|\,dt \\ &\leq L(x-x_0) \max_{[x_0,x_0+\frac{1}{2L}]} |y(x) - u(x)| \\ &\leq \frac{1}{2} \max_{[x_0,x_0+\frac{1}{2L}]} |y(x) - u(x)|. \end{aligned}$$

It follows that

$$\max_{[x_0,x_0+\frac{1}{2L}]} |y(x)-u(x)| \le \frac{1}{2} \max_{[x_0,x_0+\frac{1}{2L}]} |y(x)-u(x)|.$$

But then $\max_{[x_0,x_0+\frac{1}{2L}]} |y(x)-u(x)| = 0$, so that $y(x) = u(x)$ on $[x_0, x_0 + \frac{1}{2L}]$. Let $x_1 = x_0 + \frac{1}{2L}$. We just proved that $y(x) = u(x)$ on $[x_0, x_0 + \frac{1}{2L}]$ and in particular $y(x_1) = u(x_1)$. Repeating (if necessary) the same argument on $[x_1, x_1 + \frac{1}{2L}]$, and so on, we will eventually conclude that $y(x) = u(x)$ on $(x_0, x_0 + \frac{b}{M})$. ◇

Observe that the Lipschitz condition (1.8.3) follows from the easy to check requirement that the function $f_y(x,y)$ is continuous in the box B.

We shall need the following important tool.

Lemma 1.8.1 (Bellman-Gronwall Lemma). *Assume that for $x \ge x_0$, the functions $u(x)$ and $a(x)$ are continuous and satisfy $u(x) \ge 0$, $a(x) \ge 0$. Assume that for some number $K > 0$ we have*

$$u(x) \le K + \int_{x_0}^{x} a(t)u(t)\,dt, \quad \textit{for } x \ge x_0. \tag{1.8.10}$$

Then

$$u(x) \le K e^{\int_{x_0}^{x} a(t)\,dt}, \quad \textit{for } x \ge x_0. \tag{1.8.11}$$

Proof. Divide the inequality (1.8.10) by its right-hand side (which is positive)

$$\frac{a(x)u(x)}{K + \int_{x_0}^{x} a(t)u(t)\,dt} \le a(x).$$

Integrating both sides over (x_0, x) (the numerator of the fraction on the left is equal to the derivative of its denominator) gives

$$\ln\left(K + \int_{x_0}^{x} a(t)u(t)\,dt\right) - \ln K \le \int_{x_0}^{x} a(t)\,dt,$$

which implies that

$$K + \int_{x_0}^{x} a(t)u(t)\,dt \le K e^{\int_{x_0}^{x} a(t)\,dt}.$$

Using the inequality (1.8.10) once more, we get (1.8.11). ◇

In addition to the initial value problem (1.8.1), with $f(x,y)$ satisfying the Lipschitz condition (1.8.3), consider

$$\begin{aligned} z' &= f(x,z), \\ z(x_0) &= z_0. \end{aligned} \tag{1.8.12}$$

If $z_0 = y_0$, then $z(x) = y(x)$ for all $x \in B$, by the Theorem 1.8.1 (observe that the Lipschitz condition (1.8.3) implies the continuity of $f(x,y)$ on B). Now suppose that $z_0 \ne y_0$, but $|z_0 - y_0|$ is small. We claim that $z(x)$ and $y(x)$ will remain close over any bounded interval $(x_0, x_0 + p)$, provided that both solutions exist on that interval and $|z_0 - y_0|$ is small enough. This fact is known as the *continuous dependence of solutions, with respect to the initial condition.*

We begin the proof of the claim by observing that $z(x)$ satisfies

$$z(x) = z_0 + \int_{x_0}^{x} f(t, z(t))\, dt.$$

From this formula we subtract (1.8.4) and then estimate

$$z(x) - y(x) = z_0 - y_0 + \int_{x_0}^{x} [f(t, z(t)) - f(t, y(t))]\, dt;$$

$$|z(x) - y(x)| \le |z_0 - y_0| + \int_{x_0}^{x} |f(t, z(t)) - f(t, y(t))|\, dt$$

$$\le |z_0 - y_0| + \int_{x_0}^{x} L\, |z(t) - y(t)|\, dt.$$

(We used the triangle inequality for numbers: $|a+b| \le |a| + |b|$, the triangle inequality for integrals: $|\int_{x_0}^{x} g(t)\, dt| \le \int_{x_0}^{x} |g(t)|\, dt$, and the condition (1.8.3).) By the Bellman-Gronwall lemma

$$|z(x) - y(x)| \le |z_0 - y_0| e^{L(x-x_0)} \le |z_0 - y_0| e^{Lp}, \quad \text{for } x \in (x_0, x_0 + p),$$

so that $z(x)$ and $y(x)$ remain close over the interval $(x_0, x_0 + p)$, provided that $|z_0 - y_0|$ is small enough.

1.8.1 Problems.

1. Assume that the function $u(x) \ge 0$ is continuous for $x \ge 1$, and for some number $K > 0$, we have
$$xu(x) \le K + \int_{1}^{x} u(t)\, dt, \quad \text{for } x \ge 1.$$
Show that $u(x) \le K$, for $x \ge 1$.

2. Assume that the functions $a(x) \ge 0$ and $u(x) \ge 0$ are continuous for $x \ge x_0$ and that we have
$$u(x) \le \int_{x_0}^{x} a(t)u(t)\, dt, \quad \text{for } x \ge x_0.$$
Show that $u(x) = 0$, for $x \ge x_0$. Then give an alternative proof of the uniqueness part of Theorem 1.8.1.

   Hint. Let $K \to 0$ in the Bellman-Gronwall lemma.

3. Assume that the functions $a(x) \ge 0$ and $u(x) \ge 0$ are continuous for $x \ge x_0$ and that we have
$$u(x) \le \int_{x_0}^{x} a(t)u^2(t)\, dt, \quad \text{for } x \ge x_0.$$
Show that $u(x) = 0$, for $x \ge x_0$.

   Hint. Observe that $u(x_0) = 0$. When t is close to x_0, $u(t)$ is small. But then $u^2(t) < u(t)$. (Alternatively, one may treat the function $a(t)u(t)$ as known and use the preceding problem.)

4. Show that if a function $x(t)$ satisfies

$$0 \le \frac{dx}{dt} \le x^2 \quad \text{for all } t, \quad \text{and } x(0) = 0,$$

then $x(t) = 0$ for all $t \in (-\infty, \infty)$.

Hint. Show that $x(t) = 0$ for $t > 0$. In case $t < 0$, introduce new variables y and s by setting $x = -y$ and $t = -s$, so that $s > 0$.

5. Assume that the functions $a(x) \ge 0$ and $u(x) \ge 0$ are continuous for $x \ge x_0$ and that we have

$$u(x) \le K + \int_{x_0}^{x} a(t)\,[u(t)]^m\,dt, \quad \text{for } x \ge x_0, \tag{1.8.13}$$

with some constants $K > 0$ and $0 < m < 1$. Show that

$$u(x) \le \left[K^{1-m} + (1-m)\int_{x_0}^{x} a(t)\,dt\right]^{\frac{1}{1-m}}, \quad \text{for } x \ge x_0.$$

This fact is known as *Bihari's inequality*. Show also that the same inequality holds in case $m > 1$, under an additional assumption that

$$K^{1-m} + (1-m)\int_{x_0}^{x} a(t)\,dt > 0, \quad \text{for all } x \ge x_0.$$

Hint. Denote the right-hand side of (1.8.13) by $w(x)$. Then $w(x_0) = K$, and

$$w' = a(x)u^m \le a(x)w^m.$$

Divide by w^m, and integrate over (x_0, x).

6. For the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

or the corresponding integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt,$$

the *Picard iterations* are defined by the recurrence relation

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\,dt, \quad n = 0, 1, 2, \dots,$$

starting with $y_0(x) = y_0$. (Picard's iterations are traditionally used to prove the existence and uniqueness theorem, Theorem 1.8.1.)

(i) Compute the Picard iterations for

$$y' = y, \quad y(0) = 1,$$

and compare them with the exact solution.

(ii) Compute the Picard iterates $y_1(x)$ and $y_2(x)$ for

$$y' = 2xy^2, \quad y(0) = 1,$$

and compare them with the exact solution, for $|x|$ small.

Hint. The exact solution may be written as a series $y(x) = 1+x^2+x^4+x^6+\cdots$.

Answer. $y_0(x) = 1$, $y_1(x) = 1 + x^2$, $y_2(x) = 1 + x^2 + x^4 + \frac{x^6}{3}$. The difference $|y(x) - y_2(x)|$ is very small, for $|x|$ small.

7. Let $y(x)$ be the solution for $x > 0$ of the equation

$$y' = f(x, y), \quad y(0) = y_0.$$

Assume that $|f(x, y)| \le a(x)|y| + b(x)$, with positive functions $a(x)$ and $b(x)$ satisfying $\int_0^\infty a(x)\,dx < \infty$, $\int_0^\infty b(x)\,dx < \infty$. Show that $|y(x)|$ is bounded for all $x > 0$.

Hint. Apply the Bellman-Gronwall lemma to the corresponding integral equation.

8. Assume that for $x \ge x_0$ the continuous functions $y(x)$, $f(x)$, and $g(x)$ are nonnegative and

$$y(x) \le f(x) + \int_{x_0}^{x} g(t)y(t)\,dt, \qquad \text{for } x \ge x_0.$$

Show that

$$y(x) \le f(x) + \int_{x_0}^{x} g(t)f(t)e^{\int_t^x g(u)\,du}\,dt, \qquad \text{for } x \ge x_0.$$

Hint. Denote $I(x) = \int_{x_0}^x g(t)y(t)\,dt$. Since $I'(x) = g(x)y(x) \le g(x)I(x) + g(x)f(x)$, it follows that

$$\frac{d}{dx}\left[e^{-\int_{x_0}^x g(u)\,du} I(x)\right] \le e^{-\int_{x_0}^x g(u)\,du} g(x)f(x).$$

Integration over $[x_0, x]$ gives $I(x) \le \int_{x_0}^x g(t)f(t)e^{\int_t^x g(u)\,du}\,dt$.