To the Students

‘Mathematicians,’ [Uncle Petros] continued, ‘find the same enjoyment in their studies that chess players find in chess. In fact, the psychological make-up of the true mathematician is closer to that of the poet or the musical composer, in other words of someone concerned with the creation of Beauty and the search for Harmony and Perfection. He is the polar opposite of the practical man, the engineer, the politician or the . . . ’ — he paused for a moment seeking something even more abhorred in his scale of values — ‘. . . indeed, the businessman.’ — Apostolos Doxiadis [3]

To Those Beginning the Journey into Proof Writing

As undergraduates in the 1970s, we, the authors of this text, learned how to construct proofs essentially by osmosis. We learned much of the material contained in this text in much the same way. That is, we learned the art of proof mostly by imitating the experts. We listened to and watched their presentations. We read their textbooks. For us, the process of learning how to prove theorems began in the calculus sequence and continued through linear algebra and beyond.

This method was similar to learning to sketch or paint a beautiful landscape by imitating the techniques and vision of a master. While such modeling has its place in the development of learning to construct a proof, it is merely one of the first steps toward the creation of your own masterpiece. This text is designed to offer you the tools you will need to create proofs that are both logical and valid and, equally important, to encourage you to create from your own perspective. We hope to bridge the gap between observation and imitation of the work of one who is well-versed in proof writing, and striking out on your own, tools in hand, on your own path that takes you to your masterpiece.

You may have already had some experience observing presentations of well-constructed proofs in a geometry or calculus class. Perhaps you were asked to recreate these proofs as part or your examinations. For those majoring in mathematics, awaiting you in upper-level courses, such as modern and linear algebra, and real and complex analysis, is the expectation of your ability to prove fundamental results and use them to prove further results. Regardless of your field of study, proofs offer valuable insight into theoretical structure and therefore the ability to construct a proof gives the proof-writer the power to see below the surface and observe the underpinnings of an idea. We often wonder how our own mathematical development would have been altered had we taken a bridge course before we dove into such upper level courses in our mathematical training. We hope that you—today’s students of logic and proof-writing from a broad spectrum of disciplines—find this type of textbook useful.

The book has a dual purpose. The primary goal is to guide you in the process of writing proofs. We use, as the medium for your proof constructions, a collection of what we hope are interesting pieces of mathematical content: our secondary goal is for you to get actively involved in order to assimilate this content. As important as having intriguing topics to work with as you develop your proof-writing skills is the process of transforming your thought processes from passive to active and from computational to creative. This is a fundamental step in the development of a foundation from which you can learn any body of knowledge that has an axiomatic structure.

How to Use This Text

The text has five parts. Part I contains introductory material presented in three chapters. Chapter 1 provides some historical background concerning the invention and development of mathematics. In Chapter 2, we define a mathematical statement and use basic symbolic logic to systematize the principle of valid reasoning. Questions we will be concerned with are: How can mathematical statements be combined? When is truth preserved when statements are combined in specific ways characteristic of mathematical thinking? When are two statements equivalent? Chapter 3 presents approaches and techniques commonly used to prove mathematical statements. The techniques will serve as a bridge from low-level cognitive processes, characteristic of computational skills and procedural understanding, to higher-level cognitive processes, focused on conceptual understanding and creative constructions.

The medium you will use to hone your proof-writing skills, the mathematical content, begins in Part II. Chapter 3 deals with some basic proof methods and techniques. This theoretical machinery is used in the study of set theory in Chapter 4. Then it is extended to study functions from one set to another in Chapter 5, relations on a set in Chapter 6, and cardinality of a set in Chapter 7.

Part II—particularly Chapters 4, 5, and 6—builds a foundation for writing many kinds of proofs. Parts III and IV are more difficult, at least in some places. They provide preparation for abstract algebra and advanced calculus (or analysis) courses.

Part III leads you through the construction and examination of various number systems. Chapter 8 offers important properties and relevant proofs related to the development of number systems. Chapters 9 through 14 deal with the sets of natural (or counting or whole) numbers, integers, rational numbers (or fractions), real numbers, and complex numbers.

In physics, you may have functions that describe the position, velocity, and acceleration of a moving object. These quantities may be considered instantaneously in time or on average in a time period. In Part IV we present a structure that allows us to study the properties of such functions and their generalizations. If you have studied calculus, you may know something about this, but we will be more general and precise.

Part V provides hints for the exercises found in the chapters. This is valuable resource for you. Use it wisely. Ultimately our goal is to give you the tools, opportunity, and encouragement to create proofs of your own.

Do the Exercises!

Within each chapter, definitions are presented with examples and comments. Perhaps it is possible to be able to reproduce some mathematical proofs merely by watching, listening, and reading. However, we believe that you will be better off learning actively, that is, by supplying many of the proofs yourself. The text takes small steps forward; proofs are provided at larger leaps.

As you progress through the book, begin each exercise by trying to find a logical path to the conclusion. Only after you have spent time trying to find a plan for your proof should you look at the hint. If you begin by looking at the hint, you will deprive yourself of the joy and excitement that comes with finding your own way; you may discover a different and more interesting path than the one we suggest in the hint. If all else fails, you can always ask the experts.

Do not overlook the resource of your fellow students. Listening to their ideas about how to proceed with a proof and seeing what they take from your ideas is invaluable. It is difficult to do mathematics without discussing it with someone, even a non-expert. It is proper to challenge the ideas your fellow students tell you about their proofs. Free discourse among students and colleagues is the key to developing your own creative style. Do not be satisfied with finding a single path to a result and do not back off your idea until you find that there is a missing link in your argument.

Finally, enjoy yourself creating your own masterpieces!

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For the Professors

To Those Leading the Development of Proof Writing for Students in a Broad Range of Disciplines

The authors of this book were trained in the process of proof writing using the 1970s method of listen, observe, read, and imitate. We were introduced to the art of proof early in our calculus sequences and we developed our skills by studying the techniques used by our professors and presented in our textbooks.

Undergraduate mathematics curricula in the U.S. have undergone some changes since then. The most visible change may be the end product of Calculus Reform. With its integration of numerical, analytical, and geometrical aspects of the subject and the integration of computing technology, some topics have been removed or minimized in the curricula. We saw many proofs in our calculus courses, even if we were not always held responsible for them on the examinations. This is not always the case today. In fact, at many institutions a variety of calculus sequences exists and textbooks used in some do not give any proofs.

Calculus Reform is not the only force driving changes in the curricula. The growing number of high school graduates who continue their educations at post-secondary institutions has put pressure on the calculus curricula by requiring increasingly extensive reviews of algebra and trigonometry. Also, changes in major often leads a future major in mathematics to take a calculus course designed for other majors.

As a result of these fundamental changes in calculus courses, students have been enrolling in course such as linear algebra with weaker backgrounds in construction and understanding proofs. Recently, many institutions have inserted courses into their curricula to address this problem. These courses go by many names. The choice of mathematical content is not universal, but the prime objective is universal. That objective is to prepare students to deal with proof in their later courses. For example, at the University of South Florida, the faculty named their course *Bridge to Abstract Mathematics*. Amazingly, the title was more contentious that the course itself!

The usual content for "Bridge" courses is set theory and related topics. Courses in set theory were common part of the mathematics major earlier in the twentieth century. Between the 1950s and the 1970s most of them were squeezed out of the requirements of the mathematics major by courses in linear algebra and abstract algebra, and a relaxation of requirements in favor of elective courses.

The text has two purposes: to present mathematical content and to guide students on how to create proofs. We hope the content is interesting to your students, but the process of transforming the thought processes from a passive computational orientation to an active creative orientation is more important.

The text has five parts. Part I contains introductory material presented in three chapters. Chapter 1 provides some historical background concerning the invention and development of different kinds of numbers, and gives a brief introduction of the axiomatic method and its scientific and pedagogical merits.

Chapter 2 introduces symbolic logic, which takes a formal approach in building theory with a goal to systemize and codify the principles of valid reasoning. The term "formal" means that we are not concerned with the content and meaning of statements, but exclusively with their form.

Chapter 3 illustrates specific techniques and methods of mathematical proof that are common to mathematical practice and deserve an explicit treatment.

The mathematical content begins in Part II, consisting of Chapters 4 through 7. Chapter 4 presents the elements of set theory, which has been crucial for the modern development of mathematics. It fulfills three important functions. (See [10].) First, set theoretic language permeates a large part of mathematics, supplying its diverse areas with common modes of reasoning. Second, it plays the foundational role of supplying the subject matter of mathematics. Third, it is an important tool used in the study of the infinite.

Chapter 5 is concerned with the fundamental concept of function. Here we formal prove and ask for proofs of some basic facts about functions that are necessary in many areas of mathematics.

Chapter 6 formalizes the concept that two mathematical objects are related according to a specific common characteristic. We explain how a relation on a set gives rise to the generalized notion of order. The important concept of an equivalence relation is given elaborate treatment. This notion is used in the construction of number systems in Part III.

In Chapter 7 we give an elementary treatment of cardinality that extends the notion of number to infinite sets. The relevant theory was primarily produced by the genius of Georg Cantor and contains some landmark results, some of which are included here.

Beginning with Chapter 8 in Part III, where the terminological equipment and some relevant proofs concerning the primary and secondary properties of number systems are laid out, we are is concerned with the construction of number systems. We construct the natural numbers, the integers, the rational numbers, the real numbers, Cantor's real numbers, and the complex numbers in Chapters 9 through 14.

It is our opinion that an understanding of such systems at a fundamental level is a necessary prerequisite for a deeper grasp of mathematical analysis. Usually the real number system with its associated properties is taken for granted in undergraduate courses. Here we give two different constructions of it that present real numbers differing in ontological flavor until we prove they are identical up to isomorphism. Pedagogically, the inclusion of the construction of number systems offers an opportunity for practicing proof and mathematical rigor. The theoretical developments in these chapters culminate with a proof of the existence and uniqueness of a complete ordered field.

Part IV is an introduction to time scale calculus, a structure that unifies discrete and continuous analysis. The benefits of a study of this new area of mathematics at this level is the students' sense of familiarity with the concepts discussed that stems from their study of calculus on the real intervals. Some familiar properties of functions are generalized to functions defined on any closed subset of the real line.

Part V provides some hints to the exercises contained with the main narrative of each chapter. The goal is to provide a resource for students to use only after having spent sufficient time trying to formulate a plan of their own. We do not want to deprive students of the chance to offer a different approach than the one we had in mind and we encourage them to create more than one proof for a given statement.

The goal of this text is to prepare students for the standard mathematics curriculum. The content of the text has little overlap with the standard curriculum. Parts of Part II are generally expected as common knowledge. Part III gives students preparation for an abstract algebra course. Part IV prepares students for a rigorous treatment of calculus or vector analysis.

Within each chapter, definitions are presented with examples and comments. The proofs of propositions and theorems are often left as exercises. To a certain extent one can learn how to prove theorems merely by watching, listening, and reading proofs produced by others. However, we are convinced that active personal involvement and initiative with writing proofs has a profound impact on the development of students ability to create their own arguments. Such development is encouraged by the format of the text, taking small steps forward, with some proofs provided where they involve a larger leap forward. We encourage students to discuss their ideas with their peers and to question ideas presented by everyone involved in these mathematical discussions.

The chapters on the constructions of the natural numbers and the real numbers, Chapters 9 and 12, respectively, are considerably more difficult that the other chapters of Part III. They can be skipped as long as their properties are understood for use in the other chapters. Enjoy and let us know what you think (`oberstevorth@gmail.com`).