

CHAPTER 2

Equations of value and yield rates

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2.1 INTRODUCTION

A financial transaction may be as simple as making a single deposit C , and the quantity being studied may be $A_C(t)$, the accumulation t time periods in the future. However, commonly there is a series of deposits and withdrawals before one focuses on the accumulation. For instance, a young couple may be saving to make a down payment on a home. At the end of each month the couple makes a deposit into a savings account using any available money. Occasionally, the couple has a month with many expenses and a withdrawal must be made from the savings account. Thus there is a contribution at the end of each month and while most of the contributions are positive, sometimes one is negative or conceivably zero. The couple might have a target date of two years from the opening of the savings account for the purchase of the house, and the accumulation at that time (the balance of the account) determines the amount available for a down payment. Alternatively, the couple might have a predictable contribution schedule, and the question is at what point will the balance reach \$20,000, the minimum the couple wants to have saved before

becoming homeowners. To lay down the mathematical foundations for the study of such questions, in Sections (2.2) and (2.3) we introduce **equations of value**. *These are essential to the discussion in the remainder of the text.*

A problem in Interest Theory may often be solved using one or more equations of value. Often you must solve for unknowns. You have already seen problems where there is an unknown rate of interest, and in Section (2.4) we define the **yield rate(s)** to be the unknown rate(s) of interest. Questions of existence and uniqueness are discussed. Reinvestments have yield rate ramifications, which are first studied in Section (2.5). Sometimes an exact solution to a yield rate problem is difficult or even impossible to obtain. A method for finding an approximate solution is introduced in Section (2.6). Finally, a different sort of “yield rate,” the so-called **time-weighted yield rate**, is discussed in Section (2.7). This new “yield rate” is a measure of how well a fund is managed.

2.2 EQUATIONS OF VALUE FOR INVESTMENTS INVOLVING A SINGLE DEPOSIT MADE UNDER COMPOUND INTEREST

The simplest of interest theory problems involves a single investment C for an interval of time of length T under an amount function $A_C(t)$. If we have compound interest at an annual effective rate i and our basic units of time are years, then $A_C(t) = C(1 + i)^t$. The value of the investment at time T is then $A_C(T) = C(1 + i)^T$. If any three of the four basic quantities C , T , i , and $A_C(T)$ are known, we can use the equation

$$(2.2.1) \quad A_C(T) = C(1 + i)^T$$

to solve for the fourth unknown quantity. Equation (2.2.1) is a time T equation of value for the investment under consideration. More generally, a time T equation of value equates two different expressions, each giving the value of some cashflow or sequence of cashflows at time T .

EXAMPLE 2.2.2 Unknown principal

Problem: Aiko’s aunt opens an account for her niece. The account earns compound interest at an annual effective interest rate of 8%. The aunt notifies Aiko of the gift but not of the amount deposited. Five years after the account was opened, Aiko closes the account and receives \$3,673.32. What was the amount C that Aiko’s aunt deposited?

Solution We are given that the final account balance, which was the balance after five years, was \$3,673.32. On the other hand, the compound interest accumulation function (defined for an effective interest rate of 8%) has value $C(1 + .08)^5$ after five years. Therefore, our time 5 equation of value is $\$3,673.32 = C(1 + .08)^5$, and we find $C = \$3,673.32(1 + .08)^{-5} \approx$

\$2,499.999869. Her aunt must have deposited an integral number of cents, so her deposit was for \$2,500. ■

EXAMPLE 2.2.3 Unknown length of investment

Problem: Chen opens an account with \$1,000. The account earns interest at an annual effective rate of 4%. Chen learns that the balance is currently \$1,342. How long did it take to earn the \$342 of interest?

Solution Let T denote the length of time. Then our time T equation of value is $\$1,342 = \$1,000(1.04)^T$. Hence $T = \frac{\ln(1,342/1,000)}{\ln(1.04)} \approx 7.500145074$. Thus the money has been invested for approximately $7\frac{1}{2}$ years. ■

EXAMPLE 2.2.4 Doubling your money / “The rule of seventy-two”

Problem: Antonio plans to open an account. He notes that if the account has an annual effective interest rate of i , then the length of time T required to double his money satisfies the time T equation of value $2 = (1+i)^T$, whence $T = \frac{\ln(2)}{\ln(1+i)}$. If Antonio knows i and has a calculator handy, T may thus be quickly found. However, Antonio would like to make approximate calculations even when no calculator is available. Being mathematically inclined, he notes that $\frac{\ln(2)}{\ln(1+i)} = \left(\frac{\ln(2)}{i}\right)\left(\frac{i}{\ln(1+i)}\right)$, the function $f(i) = \frac{i}{\ln(1+i)}$ grows very slowly for i positive and close to zero, and $\ln(2)f(.08) = \frac{(\ln(2)(.08))}{\ln(1.08)} \approx .720517467$. Use Antonio’s observations to derive a rule for approximating T , then check how well this rule works when $i = 4\%$ and when $i = 9\%$.

Solution

$$\begin{aligned} T &= \frac{\ln(2)}{\ln(1+i)} = \left(\frac{\ln(2)}{i}\right) \left(\frac{i}{\ln(1+i)}\right) \\ &= \left(\frac{\ln(2)}{i}\right) f(i) \approx \left(\frac{\ln(2)}{i}\right) f(.08) = \frac{\ln(2)f(.08)}{i} \approx \frac{.72}{i}. \end{aligned}$$

Thus if i is given as a percent, you can approximate T by dividing 72 by that percentage. For example, if $i = 4\%$, then $T \approx \frac{72}{4} = 18$ and if $i = 9\%$, then $T \approx \frac{72}{9} = 8$. In fact, the formula $T = \frac{\ln(2)}{\ln(1+i)}$ yields that the values are approximately 17.67298769 and 8.043231727, so these are excellent approximations! ■

We note that in Example (2.2.4) we used $f(.08)$ to approximate $f(i)$. This is a standard choice and yields the number 72 which has many divisors. Had we approximated $f(i)$ by 1,¹ we would have had a rule of 69, had we

¹ $f(0)$ is undefined but if you apply l’Hospital’s rule from calculus, you may show that $\lim_{i \rightarrow 0} f(i) = 1$. So approximating $f(i)$ by 1 corresponds to using an interest rate of 0%.

approximated $f(i)$ by $f(.12)$ we would have had a rule of 73, while using $f(.2)$ to approximate $f(i)$ would have given us a rule of 76. The rule of seventy-two has commonly been used.

EXAMPLE 2.2.5 Unknown interest rate

Problem: Adrianna deposits \$400 in an account for her daughter that grows by compound interest. After five years the balance has grown to \$463.71. Assume there is a constant annual effective interest rate i earned by the account, and find that rate.

Solution $\$400(1 + i)^5 = \463.71 . Consequently, $i = (463.71/400)^{\frac{1}{5}} - 1 \approx .030000164 \approx 3\%$. ■

2.3 EQUATIONS OF VALUE FOR INVESTMENTS WITH MULTIPLE CONTRIBUTIONS

In Section (2.2) we considered investments under compound interest with a single contribution of principal. More generally, an interest theory problem involves a sequence of contributions $\{C_{t_k}\}$, the investment amount at time t_k being C_{t_k} . We have a negative contribution C_{t_k} if there is a withdrawal at time t_k . The growth of the contributions is governed by an amount function. The sequence of investments is liquidated at time T , producing a balance of B . If we choose a common date τ at which to value all the contributions, since the total value of the contributions must equal the value of the liquidated amount, we obtain an equation, called the **time τ equation of value** for the investment. If the growth of money is governed by an accumulation function $a(t)$, then by Important Fact (1.7.4), the time τ value of C_{t_k} contributed at time t_k is $C_{t_k} \frac{a(\tau)}{a(t_k)}$ and the time τ value of B at time T is $B \frac{a(\tau)}{a(T)}$. We hence have the time τ equation of value

$$(2.3.1)_\tau \quad \boxed{\sum_k C_{t_k} \frac{a(\tau)}{a(t_k)} = B \frac{a(\tau)}{a(T)} \quad (\text{time } \tau \text{ equation of value});}$$

note that τ is a number, and if you replace τ with another number τ' , you get another equation that we will reference as $(2.3.1)_{\tau'}$.

In particular, since $a(0) = 1$ and the discount function $v(t)$ is defined by $v(t) = \frac{1}{a(t)}$, the time 0 equation of value is

$$(2.3.2) \quad \boxed{\sum_k C_{t_k} v(t_k) = Bv(T) \quad (\text{time 0 equation of value}).}$$

The time T equation of value is

$$(2.3.3) \quad \boxed{\sum_k C_{t_k} \frac{a(T)}{a(t_k)} = B \quad (\text{time } T \text{ equation of value}).}$$

Equation $(2.3.1)_\tau$ and Equation $(2.3.1)_{\tau'}$ are equivalent for all choices of τ and τ' since equation $(2.3.1)_{\tau'}$ may be obtained from $(2.3.1)_\tau$ by multiplying by $\frac{a(\tau')}{a(\tau)}$. Therefore, if you solve for an unknown using a time τ equation of value, the result is independent of the choice of τ .

Before turning to some examples involving multiple contributions, we wish to explain how equation (2.2.1) is an equation of type (2.3.3). With notation as in Section (2.2), let $B = A_C(T)$, the accumulated value of our single contribution at the end of our investment period. Then, as we assumed in Section (2.2) that $a(t) = (1+i)^t$, (2.2.1) may be rewritten as $B = Ca(T)$. Alternatively, since $a(0) = 1$, (2.2.1) may be thought to be of the form $B = C\frac{a(T)}{a(0)}$. This is (2.3.3) for the single contribution time t_1 .

The examples we are about to consider are fairly simple, and you could probably follow them without drawing **timelines**. However, as the financial situations you consider become more complicated, you will likely find it helpful to construct timelines. Therefore, to familiarize you with this illustrative technique, we include timelines for each of the examples in this section.

EXAMPLE 2.3.4 Unknown payment

Problem: John borrows \$1,000. The loan is governed by compound interest at an annual effective interest rate of 10%. John repays \$600 at the end of one year and plans to complete repayment of the loan with a payment of P at the end of the second year. Find P .

Solution From John's perspective, the financial arrangement may be represented by the following timeline.

PAYMENT:	-\$1,000	\$600	P
TIME:	0	1	2

To determine P consider a time 2 equation of value

$$\$1,000(1.1)^2 = \$600(1.1) + P.$$

Solving for P , we find $P = \$1,000(1.1)^2 - \$600(1.1) = \$550$. We note that P could also have been found by noting that John's original debt was \$1,000. After one year this debt has grown to $\$1,000(1.1) = \$1,100$ but John made a \$600 payment, reducing outstanding debt to \$500. At the end of another year, this debt has grown to $\$500(1.1) = \550 .

We note that the problem may also be done starting with a different (but equivalent) equation of value, for instance a time 0 equation of value; but the solution is simpler if you begin as we did. ■

EXAMPLE 2.3.5 Unknown rate of interest and the quadratic formula

Problem: Caitlin opens a savings account with a deposit of \$5000. She deposits \$3000 a year later and \$2,000 a year after that. The account grows by compound interest at a constant annual effective rate i . Just after Caitlin’s \$2,000 deposit, her balance is \$11,000. Find the effective rate of interest i .

Solution Consider a timeline for Caitlin’s two-year investment.

PAYMENT:	$-\$5,000$	$-\$3,000$	$-\$2,000$
TIME:	0	1	2
BALANCE:			\$11,000

To determine i , note that the time 2 equation of value is

$$\$5,000(1 + i)^2 + \$3,000(1 + i) + \$2,000 = \$11,000.$$

This is equivalent to the equation $5(1 + i)^2 + 3(1 + i) - 9 = 0$. Letting $x = 1 + i$, we have $5x^2 + 3x - 9 = 0$ which is a quadratic equation. Recall that the quadratic formula tells us that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$. Therefore, the quadratic formula and the fact that $x = 1 + i$ is positive tell us that $x = \frac{-3 + \sqrt{189}}{10} \approx 1.074772708$, and $i = x - 1 \approx 7.4772708\%$.

BA II Plus calculator solution Use the **Cash Flow worksheet** to enter the deposit of \$5,000 at time 0 and of \$3,000 at time 1. This, along with advancing to the contribution register C02, is accomplished by pushing

CF 2ND CLR WORK 5 0 0 0
ENTER ↓ 3 0 0 0 ENTER ↓ ↓.

Rather than entering \$2,000 at time 2, we enter the negative of the balance at time 2 just prior to the \$2,000 deposit, namely $-\$9,000 = -(\$11,000 - \$2,000)$. We do so because had we made a withdrawal of \$9,000 at time 2, neither Caitlin nor the savings institution would owe the other party any money. The entering of the \$9,000 imaginary withdrawal is accomplished by continuing with our

Cash Flow worksheet and pushing 9 0 0 0 +/- ENTER. Now push IRR CPT and the display will show IRR = 7.477270849. ■

If we alter the above problem so that we know the balance at time 3 rather than at time 2, then we can still easily write down an equation of value, but

we can not easily solve this equation in closed form for $1 + i$. However, for those equipped with the BA II Plus calculator, the **Cash Flow worksheet** may still be used. Alternatively, a graphing calculator can be used to find a good approximation to $1 + i$. Moreover, as we now demonstrate, we can obtain the rate i to any desired degree of accuracy by the “**guess and check method**” or, for those familiar with Calculus, by **Newton’s method**.

EXAMPLE 2.3.6 Unknown rate of interest and an equation of value that has no easy algebraic solution

Problem: Daphne opens a savings account with a deposit of \$5,000. She deposits \$3,000 a year later and \$2,000 a year after that. The account grows by compound interest at a constant annual effective rate i . Three years after Daphne opened the account, it has a balance of \$11,000. Find the effective rate of interest i .

Timeline Let’s represent Daphne’s three-year investment experience by a timeline.

PAYMENT:	-\$5,000	-\$3,000	-\$2,000	
TIME:	0	1	2	3
BALANCE:				\$11,000

BA II Plus calculator solution Use the **Cash Flow worksheet** to enter the deposits of \$5,000 at time 0, \$3,000 at time 1, and \$2,000 at time 2. This, along with advancing to the register C03, is done by pushing

CF 2ND CLR WORK 5 0 0 0 ENTER ↓ 3 0 0 0
ENTER ↓ ↓ 2 0 0 0 ENTER ↓ ↓.

A withdrawal of \$11,000 at time 3 would result in neither Daphne nor the savings institution owing the other party any money so we continue with our **Cash Flow worksheet** by pushing 1 1 0 0 0 +/- ENTER. Now push

IRR CPT and the display will show IRR = 4.207651054.

Guess and Check Solution To determine i , note that the time 3 equation of value is

$$\$5,000(1 + i)^3 + \$3,000(1 + i)^2 + \$2,000(1 + i) = \$11,000.$$

Comparing this to the scenario in Example (2.3.5), we see that it has taken an extra year for our account balance to grow to \$11,000. Hence, the annual effective interest governing Daphne’s account must be lower than the 7.48% rate we found in (2.3.5). Let $P(i) = \$5,000(1 + i)^3 + \$3,000(1 + i)^2 + \$2,000(1 + i)$.

The polynomial $P(i)$ has only positive coefficients and therefore must be an increasing function of i . We then seek i so that $P(i) = \$11,000$. Note that $P(.03) = \$10,706.335 < \$11,000$, so $.03 < i$, and $P(.04) = \$10,949.12 < \$11,000$. so $.04$ is also lower than the desired rate but fairly close. Further observe that $P(.042) \approx \$10,998.12$. Thus i is just slightly more than 4.2%. In fact $P(.0421) \approx \$11,000.58$. Consequently, i is between 4.2% and 4.21%. Noting that $P(.04205) \approx \$10,999.35$, we see that i is in fact between 4.205% and 4.21%. Hence, to the nearest hundredth of a percent it is 4.21%. Of course this procedure can be continued to get i to any desired degree of accuracy.

Newton’s Method Solution (calculus required) Recall that Newton’s method is an iterative method used to find a solution of an equation of the form $f(x) = 0$. An initial approximation x_0 to the solution is specified and then one forms a sequence of approximations using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The equation we wish to solve here is

$$5,000(1+x)^3 + 3,000(1+x)^2 + 2,000(1+x) = 11,000,$$

so we set

$$f(x) = 5,000(1+x)^3 + 3,000(1+x)^2 + 2,000(1+x) - 11,000.$$

Note that this polynomial has derivative

$$f'(x) = 15,000(1+x)^2 + 6,000(1+x) + 2,000.$$

Just as in our “guess and check” solution, our initial approximation is .03. That is, we set $x_0 = .03$. We then calculate that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \approx .03 - \frac{-293.665}{24,093.5} \approx .042188557.$$

Continuing,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx .042188557 - \frac{2.75000276}{24,545.48617} \approx .04207652,$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx .04207652 - \frac{.00023389}{24,541.31123} \approx .042076511,$$

and

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx .042076511 - \frac{.00000001}{24,541.31087} \approx .042076511.$$

Note that the approximations x_3 and x_4 are equal. Therefore, $i \approx 4.2076511\%$. ■

The “guess and check” method and Newton’s method proceed much better if you make a good initial guess. In general, we may not know as desirable a starting point for our analysis as we did in Example (2.3.6). In that case, just go ahead and guess. Suppose that in Example (2.3.6), we had started by guessing that the effective rate of interest was 8%. In the “guess and check” method, we would then have computed $P(.08)$. Since $P(.08) = \$11,957.76$, we would have known that we needed a considerably lower rate. Noting that $P(0) = \$10,000$, we would next have looked for a rate approximately halfway between 8% and 0% (perhaps slightly closer to 8% since $P(.08)$ is closer to the target \$11,000 than $P(0)$ is). In Newton’s method, an initial guess of $x_0 = .08$ would have led to $x_1 \approx .043129042$ and we would have again quickly obtained the approximation $i \approx 4.2076511\%$. However, a sufficiently poor initial estimate in Newton’s method may lead to a sequence of estimates that is nonconvergent.

Problems involving an unknown time period may also be solved using an equation of value and the “guess and check method” or Newton’s method if needed.

EXAMPLE 2.3.7

Problem: Franklin borrows \$1,000 and the loan is governed by compound interest at an annual effective interest rate of 10%. Franklin agrees to repay the loan by two equally spaced payments of \$525. When should he make these payments?

Algebraic solution The situation may be exhibited by a timeline.

PAYMENT:	$-\$1,000$	$\$525$	$\$525$
TIME:	0	T	$2T$
BALANCE:			$\$0$

We let the time of the loan be $t = 0$. Then, considering a time 0 equation of value, we note that the repayments should take place at times T and $2T$ where

$$\$1,000 = \$525v^T + \$525v^{2T}.$$

Letting $x = v^T$, we can rewrite this equation as $\$1,000 = \$525x + \$525x^2$. Since $x > 0$, the quadratic formula yields

$$x = \frac{-525 + \sqrt{(525)^2 + 2,100,000}}{1,050} \approx .967910728.$$

Then, taking natural logarithms of each side of the equation $x = v^T$, we find $T = \frac{\ln(x)}{\ln(v)} \approx \frac{\ln(.967910728)}{-\ln(1.1)} \approx .342202894$. Consequently, Franklin should make payments at times .342202894 and .684405787.

BA II Plus calculator solution Push **CF** **2ND** **CLR WORK** to open and clear the Cash Flow worksheet. Then push

1 **0** **0** **0** **ENTER** **↓** **5** **2** **5** **+/-** **ENTER** **↓** **2** **ENTER** .

At this point you have entered $CF_0 = 1,000$, $C_01 = -525$, and $F_01 = 2$. Now push **IRR** **CPT** to obtain $IRR = 3.315313209$. Thus the length of time T until the first payment satisfies $(1.1)^T \approx 1.03315313209$. Equivalently, $T \approx \frac{\ln(1.03315313209)}{\ln(1.1)} \approx .342202894$. ■

In the solution of Example (2.3.7), had we not thought to use the quadratic formula, we could have proceeded by setting

$$F(T) = 525v^T + 525v^{2T} = 525(1.1)^{-T} + 525(1.1)^{-2T}.$$

Then $F(T)$ is a decreasing function of T , $F(0) = \$1,050$, and $F(.5) \approx \$977.84$ so T is clearly between 0 and .5. We get better and better estimates of T using the “guess and check” method. Newton’s method can also be used to find T [see Problem (2.3.12)].

We end this section by considering a method, the so-called **method of equated time** for finding approximate solutions to the following unknown time problem. The method of equated time is sometimes useful for finding an initial value in Newton’s method, the “guess and check method,” or another iterative method.

PROBLEM 2.3.8

An investor makes a sequence of contributions to an account governed by compound interest. There is a contribution of amount C_{t_k} at time t_k , $k = 1, 2, \dots, n$. C_{t_k} is positive if we have a deposit and negative if we have a withdrawal. Find T so that a single payment of $C = \sum_{k=1}^n C_{t_k}$ at time T has the same value at $t = 0$ as the sequence of n contributions of our original scenario.

An exact solution to Problem (2.3.8) may be obtained using the time 0 equation of value

$$Cv^T = \sum_{k=1}^n C_{t_k} v^{t_k}.$$

This equation is equivalent to

$$v^T = \frac{\sum_{k=1}^n C_{t_k} v^{t_k}}{C}$$

and hence to

$$T \ln v = \ln\left(\frac{\sum_{k=1}^n C_{t_k} v^{t_k}}{C}\right).$$

Therefore,

$$(2.3.9) \quad T = \frac{\ln\left(\frac{\sum_{k=1}^n C_{t_k} v^{t_k}}{C}\right)}{\ln v}.$$

An *approximate* solution to Problem (2.3.8) is given by the simpler expression

$$(2.3.10) \quad \bar{T} = \frac{\sum_{k=1}^n C_{t_k} t_k}{C} = \sum_{k=1}^n \left(\frac{C_{t_k}}{C}\right) t_k.$$

This weighted average of payment times is called the **method of equated time approximation** to the solution to Problem (2.3.8). If all the contributions are positive, it can be shown [see Problems (2.3.7) and (2.3.8)] that

$$(2.3.11) \quad \bar{T} \geq T.$$

EXAMPLE 2.3.12

Problem: A loan is negotiated with the lender agreeing to accept \$12,000 after five years and again after 10 years, then \$30,000 after fifteen years in full repayment of the loan. The borrower wishes to renegotiate the loan so that these three payments are replaced by a single payment of \$54,000. It is agreed that the new payment should have the same present value as the old sequence of payments if the present values are calculated using compound interest at an annual effective interest rate of 4.5%. When is the exact time T this payment should be made, and what is the method of equated time approximation \bar{T} ?

Solution This time there are two timelines to consider, namely one for the originally negotiated repayment plan and another for the proposed payment plan. The second of these includes an unknown time T , and each timeline includes an unknown payment of $-L$ at time 0, L being the loan amount.

ORIGINAL

PAYMENT:	$-L$	\$12,000	\$12,000	\$30,000
TIME:	0	5	10	15
BALANCE:				\$0

RENEGOTIATED

PAYMENT:	$-L$	\$54,000
TIME:		T
BALANCE:		\$0

A time 0 equation of value giving T is

$$\begin{aligned} \$12,000(1.045)^{-5} + \$12,000(1.045)^{-10} + \$30,000(1.045)^{-15} \\ = \$54,000(1.045)^{-T}. \end{aligned}$$

Equivalently,

$$T = \frac{\ln \left[\left(12,000(1.045)^{-5} + 12,000(1.045)^{-10} + 30,000(1.045)^{-15} \right) / 54,000 \right]}{-\ln(1.045)}.$$

Calculating we find $T \approx 11.28621406$.

Next we wish to find \bar{T} . By (2.3.10),

$$\bar{T} = \left(\frac{12,000}{54,000} \right) 5 + \left(\frac{12,000}{54,000} \right) 10 + \left(\frac{30,000}{54,000} \right) 15 = \frac{630,000}{54,000} = 11 \frac{2}{3}.$$

The values T and \bar{T} are close. Moreover, as promised by (2.3.11), T does not exceed \bar{T} . ■

2.4 INVESTMENT RETURN

In Examples (2.2.5), (2.3.5), and (2.3.6), we determined an unknown interest rate. The rates we found are known as **yield rates** for the investments. More generally, consider the time τ equation of value (under compound interest at the rate i)

$$(2.4.1) \quad \sum_k C_{t_k} (1+i)^{\tau-t_k} = B(1+i)^{\tau-T}.$$

A rate of interest which satisfies (2.4.1) is called an (annual) **yield rate** or **internal rate of return** for the investment giving rise to (2.4.1). Think of a yield rate as an interest rate on savings and loans that would result in the contributions C_{t_k} accumulating to B at time T [as shown by setting $\tau = T$ in Equation (2.4.1)].

Yield rates are a measure of how attractive a particular financial transaction may be. A lender wishes to have a high yield rate while a borrower searches for a low yield rate. However, as we will consider in Section (2.5) and have already noted in our discussion of Example (1.7.8), there are complications posed for the lender unless repayments may be invested at a rate equal to the original interest rate. Furthermore, if the signs of the contributions C_{t_k} fluctuate, there may not be a consistent “borrower” and “lender.”

We shall see that Equation (2.4.1) does not always have a solution and when it does, that solution need not be unique. However, in (2.4.7) we discuss hypotheses under which uniqueness is guaranteed, and these hypotheses are frequently satisfied in real life applications.

EXAMPLE 2.4.2 A unique yield rate

Problem: Gautam invested \$1,000 on March 1, 1998 and \$600 on March 1, 2000. In return he received \$600 on March 1, 1999 and \$1,265 on March 1, 2001. Show that $i = 10\%$ is the unique yield rate.

Solution A March 1, 2001 equation of value describing Gautam's investment is

$$\$1,000(1+i)^3 - \$600(1+i)^2 + \$600(1+i) - \$1,265 = \$0.$$

Let $p(x) = 1,000x^3 - 600x^2 + 600x - 1,265$. Then i is a yield rate if and only if $1+i$ is a real root of $p(x) = 0$. Note that $p(1.1) = 0$. Therefore, $i = .1 = 10\%$ is a yield rate, and $x - 1.1$ divides the polynomial $p(x)$. In fact, $p(x)$ has the factorization $p(x) = (x - 1.1)(1,000x^2 + 500x + 1,150)$. The quadratic formula shows that $1,000x^2 + 500x + 1,150 = 0$ has no real roots, and consequently 1.1 is the only real root of $p(x) = 0$. So, $.1 = 10\%$ is the only yield rate. We note that if a graphing calculator is available, you might use it to quickly locate this rate. ■

EXAMPLE 2.4.3 No yield rate

Problem: Ace Manufacturing agrees to pay \$100,000 immediately and again in exactly two years in return for a loan of \$180,000 one year from now (to be used to replace a piece of machinery). Their CEO is asked what yield rate is associated with this transaction, but he is unable to answer the question. Why must this be the case?

Solution A time 2 equation of value describing this financial arrangement is

$$\$100,000(1+i)^2 - \$180,000(1+i) + \$100,000 = \$0.$$

Equivalently, $10(1+i)^2 - 18(1+i) + 10 = 0$. Thus, by the quadratic equation, $1+i = \frac{18 \pm \sqrt{18^2 - 400}}{20}$. Since $18^2 - 400 < 0$, this leaves no real solutions to the yield equation. ■

EXAMPLE 2.4.4 Undefined yield

Problem: Banker Johnson is always on the lookout for opportunities to make money without risking any of his own funds. Johnson is able to borrow \$10,000 for one year at an annual effective interest rate of 4%, then loan out the \$10,000 for one year at an annual effective rate of 6%. What is Johnson's yield rate on this transaction?

Solution Banker Johnson must pay $.04(\$10,000) = \400 of interest for the money he borrows. However, he receives interest of $.06(\$10,000) = \600 on the \$10,000 he loans out. He thus makes $\$200 = \$600 - \$400$ on the transaction

without tying up any of his own money. No finite yield rate describes this situation. It might be tempting to say that the yield rate is infinite. However, we refrain from that because this would not give us a way to distinguish Johnson’s situation from the even more favorable situation in which he is able to loan out the \$10,000 at a rate higher than 6%. ■

EXAMPLE 2.4.5 Multiple yield rates

Problem: Alice and Afshan are friends. Afshan agrees to give Alice \$1,000 today and \$1,550 two years from now if Alice will give her \$2,500 in one year. What is the yield rate for this transaction?

Solution A time two equation of value for the given situation is

$$\$1,000(1 + i)^2 - \$2,500(1 + i) + \$1,550 = 0.$$

This is quadratic in $(1 + i)$ and the quadratic formula tells us that

$$1 + i = \frac{2,500 \pm \sqrt{6,250,000 - 6,200,000}}{2,000} = \frac{2,500 \pm \sqrt{50,000}}{2,000} = \frac{25 \pm \sqrt{5}}{20}.$$

Therefore,

$$i = \frac{5 + \sqrt{5}}{20} \approx .361803399$$

or

$$i = \frac{5 - \sqrt{5}}{20} \approx .138196601.$$

We have two distinct positive interest rates. ■

EXAMPLE 2.4.6 Three distinct yield rates

Problem: Parties A and B agree that A will pay B \$1,000,000 at $t = 0$ and \$3,471,437 at $t = 2$. In return, B will pay A \$3,228,000 at $t = 1$ and \$1,243,757 at $t = 3$. Find the possible yield rates, estimating each to at least the nearest 10,000th of a percent.

Solution For those equipped with a BA II Plus calculator, the rate closest to zero may easily be found. (In general, **if the cashflow worksheet gives you a yield rate, it is the one closest to zero.**) The correct sequence of buttons to produce the approximate yield rate of 4.184306654% is

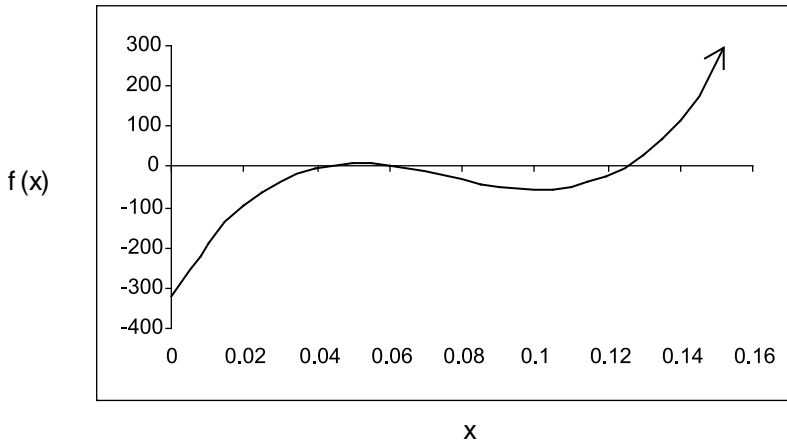
CF	2ND	CLR WORK	1	0	0	0	0	0	0
ENTER	↓	3	2	2	8	0	0	0	+/-
ENTER	↓	↓	3	4	7	1	4	3	7

ENTER	↓	↓	1	2	4	3	7	5	7	+/-
ENTER	IRR	CPT								

For those not using the BA-II Plus, this first yield may be found using either Newton's method or "guess and check," each of which might be facilitated by a good graph of the function

$$f(x) = 1,000,000(1+x)^3 - 3,228,000(1+x)^2 + 3,471,437(1+x) - 1,243,757,$$

perhaps obtained on a graphing calculator.



The graph is helpful because the yield rates are the roots of this polynomial, and for either Newton's method or the "guess and check" method, an initial approximation is needed. A graph shows that as well as being a root near 4%, there is a root close to 6% and another close to 12%. Here we use the "guess and check" method to find the root close to 6% and calculus-based Newton's method to find the root near 12%.

The root near 6%: We successively calculate $f(.06) = 1.42$, $f(.061) = .25$, $f(.0611) = .127951$, $f(.0612) = .005008$, $f(.06121) = -.007335$, $f(.061205) = -.001163$, $f(.061204) = .000072$, and $f(.0612041) = -.000052$. So, the yield rate is between 6.1204% and 6.12041%.

The root near 12%: Note that

$$f'(x) = 3,000,000(1+x)^2 - 6,456,000(1+x) + 3,471,437$$

and set $x_0 = .12$. Then $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = .12 - \frac{-22.76}{3917} \approx .125810569$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx .125810569 - \frac{4.652858}{5552.278446} \approx .124972561$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx .124972561 - \frac{.104354}{5,303.93506} \approx .124952886$, and $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx .124952886 - \frac{.000054}{5,298.155063} \approx .124952876$. So, the rate is approximately equal to 12.49529%.

Using the BA II Plus calculator for these methods: In the “guess and check” method and also in Newton’s method, one is repeatedly evaluating the polynomial f . It is helpful to store the coefficients of f in successive memories and also to reserve a register for one plus the argument at which the function is evaluated. Likewise, in Newton’s method, use successive memories for the coefficients of the derivative f' . As one calculates f or f' , store the partial results in an available register, remembering that the BA II Plus calculator allows you to add a displayed number to the value stored in register **m** by pushing

STO	+	m
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. ■

EXAMPLE 2.4.7 Three party example

Problem: Brian, Filemon, and Harold are friends. Brian will pay Filemon \$1,000 now. Filemon will pay Brian \$300 and Harold \$800 in exactly one year. Finally, Harold will pay Brian \$900 two years from now. What is Brian’s annual yield for this three-way transaction spanning two years?

Solution Brian’s time 2 equation of value is $\$1,000(1+i)^2 - \$300(1+i) - \$900 = \$100[10(1+i)^2 - 3(1+i) - 9]$. The quadratic equation then tells us that $1+i = \frac{3 \pm \sqrt{9+360}}{20}$. Since Brian invests \$1,000 and receives \$1,200 > \$1,000, he has a positive yield rate, namely $i = \frac{3 + \sqrt{9+360}}{20} - 1 \approx .11046836$. ■

Note that the three-way financial transaction of Example (2.4.7) can be viewed in terms of two-party loans. We could think of it as the following package of loans.

- (1) Brian loans Filemon \$1,000 at $t = 0$ and receives \$1,100 from Filemon at time $t = 1$ in full repayment of the loan.
- (2) Brian loans Harold \$800 at $t = 1$ and receives \$900 from Harold at time $t = 2$ in full repayment of the loan.

Note that the interest rate paid by Filemon is $\frac{\$1,100}{\$1,000} - 1 = .1$ and the interest rate paid by Harold is $\frac{\$900}{\$800} - 1 = .125$. Brian’s yield rate is in between these two rates.

When computing Brian’s yield rate in Example (2.4.7), we were not concerned with the source of the money coming in or the recipient of outgoing money. Rather, we took a “**bottom line approach.**” We were only concerned with the times and amounts of all contributions (positive or negative)

by Brian. This approach is fundamental to the successful determination of yield rates.

When computing the yield rate received by an investor, take a “**bottom line approach.**” Consider only the times and amounts of all contributions (positive or negative) by the investor.

Our next task is to analyze conditions that assure that there is a unique yield rate $i > -1$. We continue with our previous notation and suppose that i is a yield rate. That is, we suppose that we have the equation of value

$$\sum_{k=1}^n C_{t_k} (1+i)^{T-t_k} = B.$$

Define

$$(2.4.8) \quad B_{t_k}(i) = \sum_{q=1}^k C_{t_q} (1+i)^{t_k-t_q}.$$

Then $B_{t_k}(i)$ represents the outstanding balance at time t_k , just after the contribution C_{t_k} , provided that money grows by compound interest at a rate i . An important observation is that

$$B_{t_k}(i) = B_{t_{k-1}}(i)(1+i) + C_{t_k} \quad \text{for } k = 2, 3, \dots, n.$$

We now are ready to explain a set of conditions that forces there to be exactly one yield rate that is also greater than -1 . The proof of Important Fact 2.4.9 is discussed at the end of this section, and may be skipped without hindering your ability to understand the remainder of the book.

IMPORTANT FACT 2.4.9

If the contributions take place at times $t_1 < t_2 < \dots < t_{n-1} < t_n$ and there is a yield rate $i > -1$ such that the first $(n-1)$ consecutive balances $B_{t_1}(i), B_{t_2}(i), \dots, B_{t_{n-1}}(i)$ are all positive or are all negative, then i is the unique yield rate that is greater than -1 .

The condition of (2.4.9) is that one party remains the lender throughout the period of the financial transaction. Among the situations in which this is the case is when all deposits take place before any withdrawals. When this happens, the existence of a unique yield rate i for which $i > -1$ may also

easily be seen using Descartes' rule of signs. Note that in Example (2.4.5), at time 0 Afshan is the lender, but at time 1 Afshan becomes the borrower. Therefore, (2.4.9) does not apply, and it was demonstrated that there are two distinct positive interest rates in Example (2.4.5).

Proof of Important Fact (2.4.9)

Assume that the hypotheses of (2.4.9) are satisfied.

Note that $B_{t_1}(i) = C_{t_1}$, and we will assume that $B_{t_1}(i) > 0$. If this is not the case, interchanging the roles of borrower and lender changes the signs of all the contributions and makes it true, as well as changing the signs of the other balances so $B_{t_k}(i) > 0$ for $k = 1, 2, \dots, n - 1$.

Now suppose that there is another interest rate j with $j > i$ and

$$\sum_{k=1}^n C_{t_k} (1+j)^{T-t_k} = B.$$

Then j is also a yield rate. Just as we did for the rate i , we define

$$B_{t_k}(j) = \sum_{q=1}^k C_{t_q} (1+j)^{t_k-t_q},$$

and observe that

$$B_{t_k}(j) = B_{t_{k-1}}(j)(1+j) + C_{t_k} \quad \text{for } k = 2, 3, \dots, n.$$

We now note that $B_{t_1}(i) = C_{t_1} = B_{t_1}(j)$. But then $B_{t_2}(i) = B_{t_1}(i)(1+i) + C_{t_2}$, $B_{t_2}(j) = B_{t_1}(j)(1+j) + C_{t_2}$, and since $B_{t_1}(i) > 0$, the inequality $j > i$ forces $B_{t_2}(j) > B_{t_2}(i)$. Since $B_{t_3}(i) = B_{t_2}(i)(1+i) + C_{t_3}$, $B_{t_3}(j) = B_{t_2}(j)(1+j) + C_{t_3}$, $j > i$, and $B_{t_2}(i) > 0$, this gives us $B_{t_3}(j) > B_{t_3}(i)$. Continuing inductively, we arrive at $B_{t_n}(j) > B_{t_n}(i)$. However, provided that $1+i > 0$, the inequalities $j > i$ and $B_{t_n}(j) > B_{t_n}(i)$ combine to force

$$B = B_{t_n}(j)(1+j)^{T-t_n} > B_{t_n}(i)(1+i)^{T-t_n} = B.$$

With this contradiction, we see that there is no second yield rate j larger than the yield rate i .

Similarly [see Problem (2.4.7)], we can show that there is no smaller yield rate.

2.5 REINVESTMENT CONSIDERATIONS

In Example (2.4.7), we saw a three-party financial transaction. The yield rate calculated by the lending party, Brian, was not equal to the interest rate paid by either of the other parties. When we have an investor who reinvests money

that is paid to him, that investor may be involved with multiple parties and his yield rate will not in general be equal to the interest rate of his initial borrower. In this section we look at two yield rate calculations, each involving reinvestments. We will look at further important examples in Section (3.10).

EXAMPLE 2.5.1

Problem: Jose loans Martin \$12,000. Martin repays the loan by paying \$5,000 at the end of two years and \$10,000 at the end of four years. The money received at $t = 2$ is immediately reinvested at an annual effective interest rate of 2.4%. Find Martin's rate of interest and Jose's annual yield rate.

Solution Martin's time 4 equation of value is $\$12,000(1+i)^4 - \$5,000(1+i)^2 - \$10,000 = \0 . Let $x = (1+i)^2$. Then Martin's equation of value is equivalent to $12x^2 - 5x - 10 = 0$. From the quadratic formula and the fact that x is non-negative, we deduce that $x = \frac{5 + \sqrt{505}}{24} \approx 1.144675211$ and Martin's yield rate is $i = \sqrt{x} - 1 \approx .069894953$. (Another technique is to use the BA II Plus calculator Cash Flow worksheet with $CFO = 12,000$, $C01 = 0$, $F01 = 1$, $C02 = -5,000$, $F02 = 1$, $C03 = 0$, $F03 = 1$, $C04 = -10,000$, and $F04 = 1$. Pushing $\boxed{\text{IRR}}$ $\boxed{\text{CPT}}$ then gives the rate.)

To calculate Jose's yield rate, we remember the "bottom line" approach of Section (2.4). Jose contributes \$12,000 at $t = 0$. The next time he has any of this \$12,000 available to him is at time $t = 4$ since it is stipulated that the \$5,000 that comes in at $t = 2$ is immediately invested at an annual effective rate of 2.4%. At time $t = 4$ Jose has available the proceeds of this account, namely $\$5,000(1.024)^2 = \$5,242.88$ and the \$10,000 he is paid at time 4. Thus, the equation we must solve to find Jose's annual yield rate i_J is

$$\$12,000(1 + i_J)^4 = \$5,242.88 + \$10,000.$$

It follows that $i_J = \left(\frac{15,242.88}{12,000}\right)^{\frac{1}{4}} - 1 \approx .061625755$.

Jose's yield rate is lower than Martin's interest since Jose reinvested the \$5,000 at an interest rate lower than Martin's rate. ■

EXAMPLE 2.5.2

Problem: Julie pays Chan \$1,000. In return, he pays her \$300 at the end of 1, 2, 3, and 4 years. Each time Julie receives a payment, she reinvests it at 3%. She closes the 3% account at the end of 4 years. Find the annual effective interest rate paid by Chan and the annual yield rate earned by Julie.

Solution Chan's time 0 equation of value is

$$\$1,000 - \$300v - \$300v^2 - \$300v^3 - \$300v^4 = \$0.$$

There is only one change of signs in the sequence of coefficients $-1,000, 300, 300, 300, 300$ so, by Descartes rule of signs, there is a unique interest

rate i which makes this equation true. Chan's rate may be found to be about 7.713847295% using the BA II Plus calculator Cash Flow worksheet with $C_0 = 1,000$, $C_1 = -300$, and $F_0 = 4$. Alternatively, the rate may be estimated using the "guess and check method" or Newton's method.

Julie pays \$1,000 at time 0 and the next time she has any money available is at time 4 when she gets the accumulated balance in her 3% savings account. This amount is $\$300(1.03)^3 + \$300(1.03)^2 + \$300(1.03) + \$300 \approx \$1255.09$. Therefore, her yield rate is i_J where

$$\$1,000(1 + i_J)^4 = \$1,255.09.$$

Consequently, $i_J = \left(\frac{1,255.09}{1,000}\right)^{\frac{1}{4}} - 1 \approx 5.8446028\%$. Julie's rate is lower than Chan's since she reinvested money in a savings account paying interest at 3%, a rate lower than Chan's. ■

2.6 DOLLAR-WEIGHTED YIELD RATES

The yield rates we have defined are solutions to equations of value. Sometimes, these equations may be solved easily by algebraic methods. Other times, the "guess and check" or other iterative method may reasonably be used to arrive at a good approximation(s) to the yield rate(s). Or, perhaps there is a single rate and the timing of the payments is such that the BA II Plus calculator Cash Flow worksheet applies. However, many times these methods are cumbersome, and one might wish for an elementary method to obtain an approximation to the yield rate.

Suppose that we wish to study an investment fund over some finite time interval. We choose new units of time so that in our new units, the interval of our investment has length one. For example, if we wish to study an investment lasting from July 1, 1994 to January 1, 1997, then we let our new unit of time be a period of length two-and-a-half years. Furthermore, we let the starting time of the investment be denoted by time 0 and the ending point be denoted by time 1.

Denote the amount of money in the fund at the beginning ($t = 0$) by A and the amount of money in the fund at the end ($t = 1$) by B . For each time t with $0 < t < 1$,² let C_t denote the contribution to the fund at time t . By assumption (and following real life), there are only finitely many times t at which C_t is nonzero, and we may have a mixture of positive C_t 's (deposits) and negative C_t 's (withdrawals). Define the **net contributions** to the fund by

$$C = \sum_{t \in (0,1)} C_t$$

²That t lies in this interval is denoted $t \in (0, 1)$.

and let I denote the interest earned by the fund from $t = 0$ to $t = 1$. Then,

$$(2.6.1) \quad B = A + C + I.$$

If j denotes the interest rate for the period $[0,1]$ of the investment, then

$$(2.6.2) \quad I = Aj + \sum_{t \in (0,1)} C_t [(1+j)^{1-t} - 1].$$

This last term, giving the interest on the contributions C_t as determined by compound interest, may be hard to calculate if there are a large number of contributions. We therefore *approximate* it by supposing that on each contribution C_t we earn **simple interest** at rate j from the time of deposit until $t = 1$.

$$(2.6.3) \quad C_t [(1+j)^{1-t} - 1] \approx C_t j(1-t) \quad \text{for } t \in (0,1).$$

Combining (2.6.2) and (2.6.3), we obtain the important approximation

$$(2.6.4) \quad I \approx Aj + \sum_{t \in (0,1)} C_t j(1-t).$$

Equivalently, we have the following approximation for j

$$(2.6.5) \quad \boxed{j \approx \frac{I}{A + \sum_{t \in (0,1)} C_t (1-t)}}.$$

The **dollar-weighted yield**, or equivalently, the **dollar-weighted rate of return** is the simple interest approximation of j given by the right-hand side of Equation (2.6.5).

IMPORTANT DEFINITION 2.6.6

The **dollar-weighted yield**, denoted by j_{dw} , is

$$j_{dw} = \frac{I}{A + \sum_{t \in (0,1)} C_t (1-t)}.$$

The denominator in (2.6.6) should be thought of as an average amount of money invested. Approximation (2.6.5) tends to be fairly good if the C_t 's are relatively small compared to A .

There may be times when the j_{dw} is difficult or too time consuming to compute. If we approximate each $t \in (0,1)$ for which $C_t \neq 0$ by a constant k ,

then (2.6.5) gives us the new approximation

$$(2.6.7) \quad j \approx \frac{I}{A + \sum_{t \in (0,1)} C_t(1-k)} = \frac{I}{A + C(1-k)}.$$

Note that approximation (2.6.7) only requires the total contribution C , not all the individual contributions and their times. Recalling (2.6.1), it may be rewritten as

$$(2.6.8) \quad j \approx \frac{I}{A + (B - A - I)(1-k)} = \frac{I}{kA + (1-k)B - (1-k)I}.$$

A good k to use would be the dollar-weighted average $\sum_{t \in (0,1)} \frac{C_t}{C} t$. However, its precise value may take too long to calculate, so perhaps a quick estimate of it will suffice. If one uses $k = \frac{1}{2}$, then one obtains

$$(2.6.9) \quad j \approx \frac{I}{\frac{1}{2}A + (1 - \frac{1}{2})B - (1 - \frac{1}{2})I} = \frac{2I}{A + B - I}.$$

EXAMPLE 2.6.10 One year investment period

Problem: On January 1, 1990, Martin's investment account had a balance of \$10,210. He deposited \$4,000 on March 1, 1990, withdrew \$3,000 on June 1, 1990, and deposited \$1,000 on December 1, 1990. At the end of 1990, Martin's balance was \$12,982. Find Martin's dollar-weighted yield for the year 1990, then find another approximation of the yield for the year 1990 by using Equation (2.6.9).

Solution

DATE	CONTRIBUTION	BALANCE
Jan. 1, 1990	\$0	$A = \$10,210$
Mar. 1, 1990	$C_{\frac{2}{12}} = \$4,000$?
June 1, 1990	$C_{\frac{5}{12}} = -\$3,000$?
Dec. 1, 1990	$C_{\frac{11}{12}} = \$1,000$?
Dec. 31, 1990	\$0	$B = \$12,982$

Since we are considering an investment over a period of exactly one year, our unit of time is a year. With the notation of this section, $A = \$10,210$, $B = \$12,982$, $C_{\frac{2}{12}} = \$4,000$, $C_{\frac{5}{12}} = -\$3,000$, and $C_{\frac{11}{12}} = \$1,000$. Then $C = \$4,000 - \$3,000 + \$1,000 = \$2,000$ and $I = B - A - C = \$12,982 - \$10,210 - \$2,000 = \772 . Then, Martin's dollar-weighted yield is

$$j_{dw} = \frac{\$772}{\$10,210 + \$4,000(1 - \frac{2}{12}) - \$3,000(1 - \frac{5}{12}) + \$1,000(1 - \frac{11}{12})} \approx .065.$$

Note that since the investment period is one year, the annual dollar-weighted yield is 0.065 as well. On the other hand, Equation (2.6.9) gives us another approximation for the annual yield:

$$j \approx \frac{2(\$772)}{\$10,210 + \$12,982 - \$772} \approx .069.$$

■

We note that the (annual) yield j of Example (2.6.10) satisfies the time 1 equation of value

$$\begin{aligned} \$12,982 = & \$10,210(1 + j) + \$4,000(1 + j)^{\frac{10}{12}} - \$3,000(1 + j)^{\frac{7}{12}} \\ & + \$1,000(1 + j)^{\frac{1}{12}}. \end{aligned}$$

We let

$$\begin{aligned} p(j) = & \$10,210(1 + j) + \$4,000(1 + j)^{\frac{10}{12}} - \$3,000(1 + j)^{\frac{7}{12}} \\ & + \$1,000(1 + j)^{\frac{1}{12}}, \end{aligned}$$

the right-hand side of this time one equation of value. Then $p(.064985) \approx \$12,982.00004$, and therefore the yield j is extremely close to 6.4985%. (In this example the BA II Plus calculator Cash Flow worksheet could be used to find the yield rate $(1 + j)^{\frac{1}{12}} - 1$ for a 12-th of a year and then to find j .)

EXAMPLE 2.6.11 Investment period not equal to one year

Problem: On January 1, 1995, Siobhan's investment account had a balance of \$8,412. She deposited \$1,000 on November 1, 1995 and withdrew \$600 on November 1, 1997. Her balance on July 1, 1998 was \$9,620. What was Siobhan's annual dollar-weighted yield for the forty-two month period from January 1, 1995 to July 1, 1998?

Solution

DATE	CONTRIBUTION	BALANCE
Jan. 1, 1995	\$0	\$8,412
Nov. 1, 1995	\$1,000	?
Nov. 1, 1997	-\$600	?
July 1, 1998	\$0	\$9,620

Note that $A = \$8,412$, $B = \$9,620$, $C_{\frac{10}{42}} = \$1,000$, and $C_{\frac{34}{42}} = -\$600$. It follows that $C = \$1,000 - \$600 = \$400$ and $I = \$9,620 - \$8,412 - \$400 = \808 . Therefore, we have a *forty-two month* dollar-weighted yield

$$j_{dw} = \frac{808}{\$8,412 + \$1,000(\frac{32}{42}) - \$600(\frac{8}{42})} \approx .089186973.$$

It follows that the annual (*12 month*) dollar-weighted yield for the *forty-two-month* period is $i_{dw} = (1 + .089186973)^{\frac{12}{42}} - 1 \approx 2.470934465\%$.

We note that the BA II Plus calculator **Cash Flow worksheet** could be used to find the yield rate in this example; you can first find a monthly yield rate and then calculate the equivalent annual yield rate. Clear the worksheet and enter CFo = 8,412, C01 = 0, F01 = 9, C02=1,000, F02=1, C03 = 0, F03 = 23, C04=-600, F04 = 1, C05 = 0, F05 = 7, C06=-9,620, and F06=1. Then **IRR** **CPT** gives a monthly yield rate of .203699333%. The equivalent annual yield rate is $(1.002036993)^{12} - 1 \approx 2.471964455$, and this is close to the above approximation. ■

The next example is designed so that the **Cash Flow worksheet** (of the BA II Plus calculator or of the BA II Plus Professional calculator) cannot be used to calculate the yield rate. (The reason for this is that one is limited to one initial Cash Flow and then at most twenty-four others, each made at the end of up to 9,999 successive periods.) However, it still might be used to estimate the rate by shifting all the deposits forward one month.

EXAMPLE 2.6.12 Investment period not equal to one year

Problem: On January 1, 1995, Saul's investment account had a balance of \$7,688. He deposited \$100 on February 1, 1995 and then again at the end of every two months (that is on April 1, 1995, June 1, 1995, ..., December 1, 1997). His balance on January 1, 1998 is \$10,830. What was Saul's approximate annual yield for the three-year period, calculated using (2.6.9)?

Solution Note that $A = \$7,688$, $B = \$10,830$, and since there are eighteen \$100 deposits and no withdrawals $C = \$1,800$. Therefore, $I = \$10,830 - \$7,688 - \$1,800 = \$1,342$ and using (2.6.9) we have the approximate *three-year* yield

$$j \approx \frac{2 \times 1,342}{7,688 + 10,830 - 1,342} \approx .156264555.$$

It follows that the approximate annual yield for the three-year period is $(1 + .156264555)^{\frac{1}{3}} - 1 \approx 4.959\%$.

To approximate the yield j by calculating j_{dw} would be quite time-consuming. However, in Chapter 3 we will come back to this problem [see Example (3.2.23)] and see a good approximation to the rate.

Let's take a moment to see why this problem cannot be solved using the BA II Plus calculator. We are given the balance on January 1, 1995 and enter this as our initial cashflow CF_0 . Then, we have a payment one month later (of \$100) which we enter as $C01$. This establishes that the time period being used in the **Cash Flow worksheet** is one month. Thus, we would wish to alternately enter cashflows of \$0 (for the months there were no deposits) and \$100 (for the months when Saul deposited \$100). But then, to span the three-year time period from January 1, 1995 to January 1, 1998 would require registers to enter 36 payments. The problem is that the BA II Plus calculator has 24 cashflow registers, and the BA II Plus Professional has 32 registers, rather than 36 appropriate registers.

Since we can't solve the given problem, we look at the similar problem where the deposits occurred every two months starting with March 1, 1995. The **Cash Flow worksheet** can then be used with $CF_0 = 7,688$, $C01 = 100$, $F01 = 17$, $C02 = -10,830 + 100 = -10,730$, $F02 = 1$. Then

IRR

CPT

 gives a two-month yield of $J \approx .81628332\%$ and this corresponds to an annual yield of $(1 + J)^6 - 1 \approx 4.998742257\%$. Our shift of each of the deposits forward one month (from the original problem) results in this rate being a bit higher than the yield rate for the given problem. ■

We end this section with an example showing the usefulness of approximation (2.6.7).

EXAMPLE 2.6.13 The average date for contributions

Problem: An investment fund had a balance of \$320,000 on January 1, 2002 and a balance of \$374,000 one year later. The amount of interest earned during the year was \$14,000, and the annual yield rate on the fund was 4%. Estimate the (dollar-weighted) average date of contributions to the account.

Solution We are given that $A = \$320,000$, $B = \$374,000$, and $I = \$14,000$. So, $C = B - A - I = \$40,000$, and if k denotes the average (dollar-weighted)

time of the contributions, approximation (2.6.7) gives

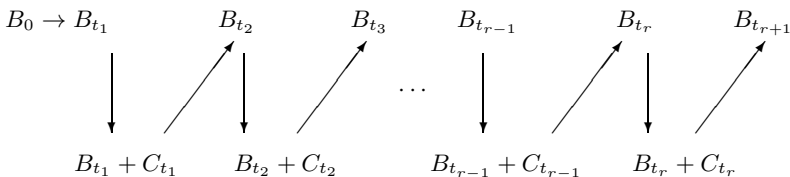
$$.04 \approx \frac{\$14,000}{\$320,000 + \$40,000(1 - k)}.$$

Equivalently, $k \approx \frac{1}{4}$. The average date of contributions to the account is a quarter of the way through the year, that is to say on April 1. ■

2.7 FUND PERFORMANCE

As in Section (2.6), let us consider an investment fund over a given interval of time. However, rather than considering the annual yield which is a measure of how the investment actually grew, we define a new “yield rate,” called the annual **time-weighted yield rate** that quantifies *how well the fund was managed*. The yield rate and time-weighted yield rates are often close, but this is not always the case. For example, if an investor makes large deposits right before the fund does particularly well and makes large withdrawals just before the fund does poorly, the investor’s yield will be considerably higher than the time-weighted yield associated with the fund. This is the case considered in Example (2.7.5).

As in Section (2.6), introduce units so that the given investment period has length one unit. Let B_t denote the account balance at time t , just before any time t contributions. (In the notation of Section (2.6), $B_0 = A$ and $B_1 = B$.) Let C_t denote the contribution at time t , and assume that there are only finitely many t ’s in the interval $(0,1)$ for which $C_t \neq 0$, namely t_1, t_2, \dots, t_r where $0 < t_1 < t_2 < \dots < t_r < 1$. Let $t_{r+1} = 1$. The balance then progresses as is indicated in the following diagram:



The vertical arrows arise due to contributions to the investment fund, and the nonvertical arrows reflect the growth (or shrinkage) of the investment fund over time. In the case of a non-vertical arrow to B_{t_k} , the balance grows by multiplying by $1 + j_k$ where

$$(2.7.1) \quad 1 + j_k = \begin{cases} \frac{B_{t_1}}{B_0} & \text{if } k = 1 \\ \frac{B_{t_k}}{B_{t_{k-1}} + C_{t_{k-1}}} & \text{if } k = 2, 3, \dots, r + 1. \end{cases}$$

Think of j_k as the yield rate for the interval $[t_{k-1}, t_k]$. The **time-weighted yield** j_{tw} for the investment period is defined so that the sum $1 + j_{tw}$ is equal to the product $(1 + j_1)(1 + j_2) \dots (1 + j_{r+1})$; standard mathematical notation for this is $\prod_{k=1}^{r+1} (1 + j_k)$. Thus,

$$(2.7.2) \quad \boxed{j_{tw} = \left[\prod_{k=1}^{r+1} (1 + j_k) \right] - 1.}$$

The motivation for the definition of j_{tw} is as follows. The growth of money over the interval $[0,1]$ is given by its successive growth over the subintervals $[t_{k-1}, t_k]$, and on the subinterval $[t_{k-1}, t_k]$ the growth of money is governed by multiplication by $1 + j_k$.

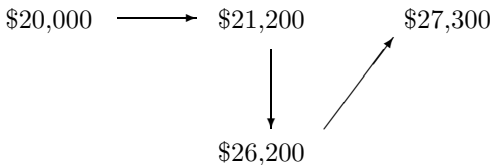
Note that j_{tw} is a “yield rate” for the interval of interest. If the interval lasts T years, the **annual time-weighted yield rate** i_{tw} is

$$(2.7.3) \quad \boxed{i_{tw} = (1 + j_{tw})^{\frac{1}{T}} - 1 = \left[\prod_{k=1}^{r+1} (1 + j_k) \right]^{\frac{1}{T}} - 1.}$$

EXAMPLE 2.7.4 Investment period not equal to one year

Problem: Mohammed had \$20,000 in his investment account on August 15, 1999. On August 15, 2000 his balance was \$21,200 and he deposited an additional \$5,000, giving him a new balance of \$26,200. On August 15, 2001, Mohammed’s account had a balance of \$27,300. Assuming that there are no other contributions to the account, find the annual time-weighted yield and note that it is very close to the annual yield.

Solution The balance grows as follows:



Therefore

$$i_{tw} = \left[\left(\frac{\$21,200}{\$20,000} \right) \left(\frac{\$27,300}{\$26,200} \right) \right]^{\frac{1}{2}} - 1 \approx .050953765.$$

On the other hand, the annual yield i satisfies the equation of value

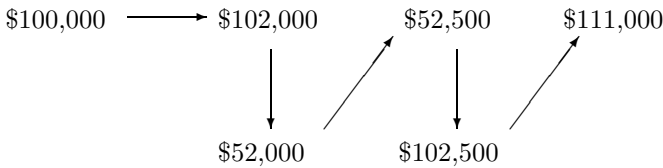
$$\$20,000(1 + i)^2 + \$5,000(1 + i) = \$27,300.$$

This equation may be solved using the quadratic equation, and $i = .05$ is the only positive yield rate. ■

EXAMPLE 2.7.5 Time-weighted yield less than yield

Problem: Astute Mr. Haywood notices that although the “Tomorrow Fund” has an excellent performance history, it performed less well when the price of gasoline experienced a sharp rise. He decides to invest in the fund but, to the extent possible, move his money away from the fund during periods when he anticipates a sharp increase in gasoline prices. On January 1, Mr. Haywood deposits \$100,000 in the fund. On March 1 his balance is \$102,000, and he withdraws \$50,000. On May 1 his balance is \$52,500 and he deposits \$50,000. At the end of the year Mr. Haywood’s fund balance is \$111,000. Find the time-weighted yield for the “Tomorrow Fund” for the year, and show that this is lower than Mr. Haywood’s yield.

Solution The balance grows as follows:



Therefore

$$i_{tw} = j_{tw} = \left(\frac{\$102,000}{\$100,000} \right) \left(\frac{\$52,500}{\$52,000} \right) \left(\frac{\$111,000}{\$102,500} \right) - 1 \approx .115206379.$$

The annual yield i satisfies

$$\$111,000 = \$100,000(1 + i) - \$50,000(1 + i)^{\frac{10}{12}} + \$50,000(1 + i)^{\frac{8}{12}}.$$

Let $q(i) = \$100,000(1 + i) - \$50,000(1 + i)^{\frac{10}{12}} + \$50,000(1 + i)^{\frac{8}{12}}$. Then $q(i_{tw}) \approx \$110,534.53 < 111,000$. Therefore, Mr. Haywood’s yield is higher than i_{tw} . In fact, using the “guess and check” method or Newton’s method, one may determine that Mr. Haywood has a yield rate of approximately .1203. More easily, the BA II Plus calculator **Cash Flow worksheet** may be used to find a monthly yield rate of .951201439 is equivalent to an annual yield rate of 12.03092019%.

The reason that Mr. Haywood's yield exceeds the Tomorrow Fund's time-weighted yield is that he timed his deposits and withdrawals well. Had they been poorly timed, his yield would have been worse than the Fund's time-weighted yield. ■

2.8 PROBLEMS, CHAPTER 2

(2.0) Chapter 2 writing problems

- (1) [following Section (2.3)] Consider the equation

$$\$2,000(1+i)^4 - \$300(1+i)^2 + \$600(1+i) = \$2,925.$$

Describe a financial situation for which this is the associated time 4 equation of value. Give details explaining the sources of any deposits and the reasons for any withdrawals.

- (2) [following Section (2.3)] Consider the equation

$$\sum_{k=1}^{35} \$2,000(1+i)^{35-k} = \sum_{m=1}^{30} \$10,000v^m.$$

Describe a financial situation for which this is the associated time 35 equation of value.

- (3) [following Section (2.7)] Write an advertisement for an investment fund, giving annual yields earned by the fund over each of five years and an annual time-weighted yield for the period.
- (4) Learn about three mutual funds and the portfolio focus of each fund. For each fund, note the annual yield rates over various time periods. Include the period from the date of date of inception to a recent date and a recent five-year period. Comment on similarities and differences in the performance of the funds. Speculate on the causes of any performance discrepancies.

(2.2) Equations of value for investments involving a single deposit made under compound interest

- (1) Mr. Lopez deposits $\$K$ in an account paying 4% annual effective discount. The balance at the end of three years is $\$982$. Find K .
- (2) Marianne deposits $\$2,000$ in a five-year certificate of deposit. At maturity the balance is $\$2,580.64$. Find the annual effective rate of interest governing the account.
- (3) Suzanne remembers that her only deposit into her savings account was a $\$1,800$ deposit. She knows that the account has had a constant nominal

interest rate of 3.2% convertible monthly and that the balance is now \$1,965.35. How long ago did Suzanne make her deposit?

- (4) Use the rule of seventy-two to approximate the length of time it takes money to double at an annual effective interest rate of 5% and then at an annual effective rate of 10%. Then find the exact time it takes for money to double at each of these interest rates.
- (5) Derive a “rule of n ” to approximate the length of time it takes for money to triple. As in the derivation of the “rule of seventy-two,” your rule should be derived to give an especially good estimate when the annual effective interest rate is 8%. After you have stated your rule, compare the approximations it gives for annual effective interest rates of 4% and 12% with the true values at these rates.

(2.3) Equations of value for investments with multiple contributions

- (1) Sidney borrows \$12,000. The loan is governed by compound interest and the annual effective rate of discount is 6%. Sidney repays \$4,000 at the end of one year, X at the end of two years, and \$3,000 at the end of three years in order to exactly pay off the loan. Find X .
- (2) Rafael opens a savings account with a deposit of \$1,500. He deposits \$500 one year later and \$1,000 a year after that. Just after Rafael’s deposit of \$1,000, the balance in his account is \$3,078. Find the annual effective interest rate governing the account.
- (3) Esteban borrows \$20,000, and the loan is governed by compound interest at an annual effective interest rate of 6%. Esteban agrees to repay the loan by making a payment of \$10,000 at the end of T years and a payment of \$12,000 at the end of $2T$ years. Find T .
- (4) Shakari opens a savings account with a deposit of \$3,500. She deposits \$500 six months later and \$800 nine months after opening the account. The balance in Shakari’s account one year after she opened it is \$5,012. Assuming that the account grows by compound interest at a constant annual effective interest rate i , find i .
- (5) A loan is negotiated with the lender agreeing to accept \$8,000 after one year, \$9,000 after two years, and \$20,000 after four years in full repayment of the loan. The loan is renegotiated so that the borrower makes a single payment of \$37,000 at time T and this results in the same total present value of payments when calculated using an annual effective rate of 5%. Estimate T using the method of equated time. Also find T exactly.
- (6) Anne and Frank Smith each borrow \$12,000 from their father. Anne and Mr. Smith have agreed that she will repay her loan in full by paying \$6000 in two years and \$8000 in four years. Frank prefers to make one

lump payment of \$15,000 to fully repay his loan. When should he make that payment so that he and his sister will each have the same effective interest rate?

- (7) Let b_1, b_2, \dots, b_n be positive real numbers. Set $A = (\sum_{k=1}^n b_k)/n$, the arithmetic mean of the numbers, and $G = (\prod_{k=1}^n b_k)^{\frac{1}{n}}$, the geometric mean of the numbers. The objective of this problem is to establish that $A \geq G$ and that $A > G$ whenever b_1, b_2, \dots, b_n are not all equal.
- Write the point-slope equation for the tangent line to $y = \ln x$ at $(A, \ln A)$.
 - Use (a) and concavity to show that $\ln x \leq A^{-1}(x - A) + \ln A$ for all positive x . Moreover, show that equality holds if and only if $x = A$.
 - Use (b) to prove that $\ln G \leq \frac{1}{n} \sum_{k=1}^n A^{-1}(b_k - A) + \ln A$ and that this is a strict inequality unless all the b_k 's are equal.
 - Show that $\frac{1}{n} \sum_{k=1}^n A^{-1}(b_k - A) + \ln A = \ln A$.
 - Conclude that $G \leq A$ and $G < A$ unless all the b_k 's are equal.
- (8) Let C_{t_k} denote the contribution in cents made at distinct times t_k , $k = 1, 2, \dots, n$. Suppose that these are all positive so that we have deposits, but no withdrawals. Then the C_{t_k} 's are positive integers. As in (2.3.9), let

$$T = \ln \left(\frac{\sum_{k=1}^n C_{t_k} v^{t_k}}{C} \right) / \ln v.$$

As in (2.3.10), let $\bar{T} = \sum_{k=1}^n (\frac{C_{t_k}}{C}) t_k$. The objective of this problem is to use the result of Problem (2.3.7) to establish that $\bar{T} \geq T$ and that this is a strict inequality unless $n = 1$.

- Consider C_{t_k} quantities each equal to v^{t_k} , $k = 1, 2, \dots, n$. Then in all we are considering $C = C_{t_1} + C_{t_2} + \dots + C_{t_n}$ quantities. Use the result of Problem [2.3.7(e)] to show that

$$\frac{C_{t_1} v^{t_1} + C_{t_2} v^{t_2} + \dots + C_{t_n} v^{t_n}}{C} \geq v^{\bar{T}}$$

and that this is a strict inequality unless $n = 1$.

- Use (a) to show that the present value of the deposits is at least as large as the present value given by the method of equated time and that it is strictly larger unless $n = 1$.
 - Show that $\bar{T} \geq T$ with strict inequality unless $n = 1$.
- (9) Suppose that you pay \$1,000 at time 0, get \$4,000 at time 1, and pay \$2,000 at time 2. Let $C_0 = \$1,000$, $C_1 = -\$4,000$, and $C_2 = \$2,000$. Set $C = C_0 + C_1 + C_2 = \$1,000 - \$4,000 + \$2,000 = -\$1,000$. Find T such

that getting an inflow of $-C$ at time T has the same present value as the above sequence of financial transactions, assuming that the growth of money is governed by compound interest at $i = 1\%$. Show that T is greater than the weighted average $\bar{T} = \frac{C_0}{C}0 + \frac{C_1}{C}1 + \frac{C_2}{C}2$. [This shows that Inequality (2.3.11) need not hold if you have a negative contribution.]

(10) [recommended for those with a BA II Plus calculator]

Dax borrows \$300,000 and the loan is governed by compound interest at an annual effective interest rate of 4.75%. Dax agrees to repay the loan by ten equally spaced payments, the first four of which are for \$25,000 and the next six of which are for \$40,000. When should he make the first payment?

(11) Find the amount to be paid at the end of eight years that is equivalent to a payment of \$400 now and a payment of \$300 at the end of four years

- (a) if 6% simple interest is earned from the date each payment is made and use a comparison date of right now.
- (b) if 6% simple interest is earned from the date each payment is made and use a comparison date of eight years from now.
- (c) Explain why the fact you get different answers in parts (a) and (b) does not contradict the fact that equations of value at different times are equivalent equations.
- (d) Repeat parts (a) and (b) except replace “simple interest” with “compound interest.”

(12) [calculus needed] Use Newton’s method to solve the problem of Example (2.3.7). More specifically, set $f(t) = 525(1.1)^{-2t} + 525(1.1)^{-t} - 1,000$, make an initial guess T_1 for a root T , and find a sequence of approximations $\{T_1\}$ to T that allow you to obtain T to the nearest hundredth of a percent. Why might $T_1 = .3$ be a reasonable initial guess?

(2.4) Investment return

- (1) Payments of \$3,000 now and \$8,000 eight years from now are equivalent to a payment of \$10,000 four years from now at either rate i or rate j . Find $|i - j|$. Explain why the yield rate is not unique in this case.
- (2) Success, Inc. enters into a financial arrangement in which it agrees to pay \$100,000 today and \$101,000 two years from now in exchange for \$200,000 one year from now. Show that there is no yield rate that can be assigned to this two-year transaction.
- (3) Sigmund, Inc. agrees to pay \$150,000 today and \$40,000 four years from today in return for \$210,000 two years from today. What is the yield rate for this four-year financial arrangement?

- (4) Firms A, B, C, and D enter into a financial arrangement. Money flush firm A will pay expanding firms B and C each \$1,000,000 today. B will pay D \$2,200,000 three years from today. C will pay B \$800,000 two years from today and D \$350,000 two years from today. Finally, D will pay A \$3,200,000 six years from today. Calculate the yield rate or interest rate, to the nearest hundredth of a percent, that each firm experiences over the period of their involvement (6 years for A, 3 years for B, 2 years for C, and 4 years for D).
- (5) Salsa invests \$8,572.80 at $t = 0$ and \$28,500 at $t = 2$. In return, she receives \$27,074 at $t = 1$ and \$10,000 at $t = 3$. Write down a time 0 equation of value and verify that it is satisfied for $v = .94$, $v = .95$, and $v = .96$. Find the corresponding three yield rates.
- (6) Pedro invests \$100,000 at $t = 0$ and \$60,000 at $t = 2$. In return he gets \$60,000 at $t = 1$ and \$126,500 at $t = 3$. Write down a time 3 equation of value describing Pedro's investment. Explain why there is a unique yield rate and find it.
- (7) Jie purchased two U.S. Treasury bills on the same day. The first one will mature in 13 weeks for \$12,000, with quoted rate of 1.93%. The other one will mature in 26 weeks for \$16,000 and has a quoted rate of 2.35%. What is Jie's annual effective yield for the 26-week period, assuming that there are 365 days in a year?
- (8) Louie borrowed \$2,000 from Phil. Under this agreement, Louie would repay with \$1,200 at $t = 1$ and \$1,700 at $t = 4$ where time is given in years. Louie successfully made the payment in full at $t = 1$, but he faced some financial difficulty and was only able to pay 60% of what he owed at the time of the second payment. What was the annual interest rate for the original loan? What is Phil's annual yield for this four-year period?
- (9) Show that if $-1 < j \leq i$,

$$\sum_{k=1}^n C_{t_k} (1+i)^{T-t_k} = \sum_{k=1}^n C_{t_k} (1+j)^{T-t_k},$$

and $B_{t_1}(i), B_{t_2}(i), \dots, B_{t_{n-1}}(i)$ are all positive [with $B_{t_k}(i)$ as given in (2.4.8)], then $j = i$.

- (10) [recommended for those with a BA II Plus calculator]

On January 1, Ezequiel opens an account at Friendly Bank. His opening deposit is for \$50 and he makes deposits at the end of each quarter for four years, then makes no more deposits. He closes the account exactly seven years after he opens it and receives \$3423.28. Find his annual yield rate for this seven-year period if his quarterly deposits were \$60

in the first year, \$75 in the second year, \$50 in the third year, and were successively \$300, \$450, \$800, and \$240 in the fourth year.

(11) [recommended for those with a BA II Plus calculator]

A loan of \$20,000 is to be repaid by thirty-three end-of-month payments. The first payment is \$400 and then each payment is \$25 more than the previous payment. Find the annual yield rate correct to the nearest hundredth of a percent. [HINT: The **Cash Flow worksheet** only accepts twenty-four payments, or thirty-two if you have a BA II Plus Professional calculator. If you are working with the BA II Plus calculator, suppose that the payments beyond the twenty-fourth, which you do not have registers to accommodate, are made along with the twenty-fourth. Now use the “guess and check” method, obtaining your first estimate by using . This is a challenging problem, especially for those of you with only 24 registers. However, when performing the successive calculations required by the “guess and check” method, you may make judicious use of the **NPV subworksheet** to decrease your work.]

(2.5) Reinvestment considerations

- (1) Angela loans Kathy \$8,000. Kathy repays the loan by paying \$6,000 at the end of one and a half years and \$4,000 at the end of three years. The money received at $t = 1\frac{1}{2}$ is immediately reinvested at an annual effective interest rate of 6%. Find Kathy’s annual effective rate of interest and Angela’s annual yield.
- (2) Kurt loans Randy \$14,000. Randy repays the loan by paying Kurt \$4,000 at the end of one year and \$6,000 at the end of two years and as well as at the end of three years. The money received at $t = 1$ and at $t = 2$ is immediately reinvested at an annual effective interest rate of 3%. Find Randy’s annual effective interest rate and Kurt’s annual yield.
- (3) On January 15, 2000, Enterprise A loans \$6,000 to Enterprise B and \$17,000 to Enterprise C. Enterprise B repays Enterprise A \$7,000 on January 15, 2002 and this money is reinvested at a 5% annual effective rate. Enterprise C repays Enterprise A \$22,500 on January 15, 2004. What is the annual yield received by Enterprise A over the four-year interval. Compare it to the annual effective interest rates paid by Enterprises B and C.

(2.6) Dollar-weighted yield rates

- (1) Sandra invests \$10,832 in the Wise Investment Fund. Three months later her balance has grown to \$11,902 and she deposits \$2,000. Two months later her fund holdings are \$14,308 and she withdraws \$7,000.

Two years after her initial investment, she learns that her holdings are worth \$12,566.

- (a) Write an equation of value involving the (exact) annual yield i over the two-year period.
 - (b) Compute the dollar-weighted annual yield over the investment period, then compute another approximation of the annual yield using (2.6.9).
- (2) On February 1, Arshak's investment account has a balance of \$19,800. He deposited \$1,200 on April 1 and \$2,600 on May 1. He withdrew \$8,400 on July 1. On November 1, Arshak's balance was \$14,820. Find Arshak's dollar-weighted annual yield for this nine-month period.
 - (3) Franklin's investment fund had a balance of \$290,000 on January 1, 1995 and a balance of \$448,000 two years later. The amount of interest earned during the two years was \$34,000, and the annual yield rate on the fund was 5.4%. Estimate the (dollar-weighted) average date of contributions to the account.
 - (4) The investment balance of a firm is \$5,000,000 at the beginning of a two-year period and \$7,000,000 at the end. The firm makes a single contribution during the two-year interval of \$1,200,000. What is the difference between the annual dollar-weighted yield earned by the firm if the contribution is made after 6 months as opposed to it being made after one year?

(2.7) Fund performance

- (1) On January 1, 1988, Antonio invests \$9,400 in an investment fund. On January 1, 1989 his balance is \$10,600 and he deposits \$2,400. On July 1, 1989 his balance is \$14,400 and he withdraws \$1,000. On January 1, 1992 his balance is P . Express his annual time-weighted yield as a function of P .
- (2) Arthur buys \$2,000 worth of stock. Six months later, the value of the stock has risen to \$2,200 and Arthur buys another \$1,000 worth of stock. After another eight months, Arthur's holdings are worth \$2,700 and he sells off \$800 of them. Ten months later, Arthur finds that his stock has a value of \$2,100.
 - (a) Compute the annual time-weighted yield rate of the stock over the two-year period.
 - (b) Compute the annual yield for Arthur over the two-year period.
 - (c) Should the answer in part (a) or part (b) be larger? Why?
- (3) Bright Future Investment Fund has a balance of \$1,205,000 on January 1. On May 1, the balance is \$1,230,000. Immediately after this balance is

noted, \$800,000 is added to the fund. If there are no further contributions to the fund for the year and the time-weighted annual yield for the fund is 16%, what is the fund balance at the end of the year?

Chapter 2 review problems

- (1) Sohail makes an initial investment of \$20,000. In return, he receives \$4,000 at the end of one year and another \$18,000 at the end of three years.
 - (a) Assuming that the investment is made at simple interest at rate r , write down an equation of value for the investment and find r .
 - (b) Assuming that the investment is made at compound interest at effective interest rate i , write down an equation of value for the investment and justify the statement that there is a unique yield rate. Use the “guess and check method” to estimate i to the nearest hundredth of a percent.
 - (c) Starting with the same initial guess for i that you used in (b), check your answer using Newton’s method.
 - (d) (recommended for those with a BA II Plus calculator)
Use the Cash Flow worksheet to find i to the nearest millionth.
- (2) (a) Suppose now that investments are governed by compound interest at an effective interest rate $i \geq 0$. By how much does the sum of the time n value of $\$K$ paid at time 0 and the time n value of $\$K$ paid at time $2n$ exceed $\$2K$? Express your answer as a function $g(n)$ and show that $g(n) > 0$ if $n > 0$.
 - (b) Suppose that investments are governed by the simple interest accumulation function $a(t) = 1 + rt, r \geq 0$. Does the sum of the time n value of $\$K$ paid at time 0 and the time n value of $\$K$ paid at time $2n$ exceed $\$2K$ for all r and n ? Justify your answer.
- (3) Elyse invests \$16,312 at $t = 0$. In return, she gets \$8,000 at $t = 1$ and \$10,000 at $t = 2$. Half of the time 1 payment, she reinvests at an annual effective interest rate of 5%. What is her annual yield rate for the two-year period?
- (4) Sports Manufacturing needs capital for expansion. It borrows \$1,000,000 from Venture Bank for three years at 6% nominal interest convertible quarterly, and \$500,000 for five years from a private investor at a 5% effective discount rate. At the end of two years, Sports Manufacturing makes a \$200,000 three-year loan to its supplier of titanium (for baseball bats) at 7% annual effective interest. What annual internal rate of return should Sports Manufacturing report for the combined cashflows over the five-year period?

- (5) Abiyote invested \$24,500 on January 1, 1994 in the Utopia Fund. On May 1, 1995, his balance was \$28,212 and he withdrew \$10,000. On December 1, 1995, his balance was \$15,892, and he deposited \$8,000. On January 1, 1997 his balance was \$30,309.
- (a) Find the annual time-weighted yield for the Utopia Fund for the three-year period from January 1, 1994 until January 1, 1997.
 - (b) Find an annual dollar-weighted yield received by Abiyote for the three-year period from January 1, 1994 until January 1, 1997.
 - (c) (recommended for those with a BA II Plus calculator) Find the yield received by Abiyote for the three-year period from January 1, 1994 until January 1, 1997, correct to the nearest millionth of a percent.
 - (d) Compare the time-weighted yield experienced by the Utopia Fund and the yield received by Abiyote from his investment in the Utopia Fund. Discuss why the inequality between them is in the direction it is.
- (6) Xiang and Dmitry are friends. They agree that Xiang will pay Dmitry \$800 immediately and another \$200 at the end of three years. In return, Dmitry will pay Xiang \$ K in exactly one year and again at the end of exactly two years.
- (a) Find K if the transaction is based on compound interest at a nominal discount rate of 6% convertible monthly.
 - (b) If $K = 600$, is there a unique positive yield rate for the transaction? Justify your answer.