Preface: Four Ways to Use This Book

Topology is an exciting **subject** to learn. Topological ideas surround us in daily life and in mathematical musings, but we can't fully enjoy their wonders until we learn topology. Is a shoelace knotted? If point p and point q are part of a connected set, is there a path between them? Can we count to infinity and beyond? Can a surface have only one side or one edge? Topology is compelling because it can give us a new perspective and surprising answers to such questions.

Topology is an exciting subject to **learn**. Topology is not just an exciting subject to know—it is an exciting subject to *discover*.

One of the reasons that we, the authors, are writing this book comes from a personal source. For each of us, a topology class was the first setting in which we learned how to prove theorems on our own. Topology was the arena in which our personal relationship with mathematics was transformed. Proving theorems in topology was the experience that shifted our self-image from being purely consumers of mathematics to becoming *producers* of mathematics.

The presentation in this book invites readers, too, to feel the joy of discovering insights for themselves. They will learn not only some fascinating mathematics but also how to create insights and concepts by intent. We will preview a few of the delightful enticements of topology in Chapter 1, but in this preface we describe our vision for how this book might be used by instructors and independent learners.

(1) A textbook for introductory topology: Potential road maps

This book can be used as a textbook in an introductory topology class that is taught in an inquiry-based learning format. In this format, the instructor selects which exercises and theorems the students are to work on for the next sessions of the class. Then in class the students spend most of the time either in group work on unresolved theorems or in making and listening to presentations by students. The standing assignment for the students is to prove selected theorems and do selected exercises on their own, write up their own proofs and solutions, engage in group work during the classroom session, make presentations, and respond to presentations made by other students.

Sources about Inquiry-Based Learning. There are many variations on how inquiry-oriented classes can be conducted. Our Instructors' Resource, which can be accessed at

Francis Su's website (`http://www.math.hmc.edu/~su/`), describes some such methods and also includes recommendations for other sources of information and training about inquiry-based learning. The Instructors' Resource also includes sample syllabi that describe in detail the daily running of the course.

Important note to instructors: It is not possible to do every theorem in any section during an introductory class, so only selected theorems and exercises should be assigned. Every chapter contains many additional concepts, examples, and theorems that an instructor may include.

Core General Topology: Here we list a collection of theorems and exercises that would provide students with the core ideas, examples, and theorems of point-set topology. In general, theorems are more important than exercises, so in any given section, skipping exercises is not likely to affect later theorems. It is still good to encourage students to read over skipped exercises or theorems within each section. In each chapter, reading the introduction and conclusion section is recommended.

- Cardinality—Section 1.1; Section 1.2 except Theorem 1.14 and Exercise 1.15; Theorem 1.16.
- Topological Space Fundamentals—Section 2.1; Section 2.2; Section 2.3.
- Bases, Subspaces, Products—Section 3.1; in Section 3.2, familiarize yourself with the definition of a subbasis; in Section 3.3, become acquainted with the lexicographically ordered square; Section 3.4; in Section 3.5, just do the theorems.
- Separation Properties—Section 4.1 through Theorem 4.9, and the table in Exercise 4.13; in Section 4.2, Exercise 4.18 can be skipped; in Section 4.3, only Theorems 4.19, 4.20, 4.23.
- Countable Features—Section 5.1 through Theorem 5.5; Section 5.2.
- Compactness—in Section 6.1, just focus on theorems; Section 6.2 will be a review if you've had analysis; in Section 6.3, up through the Heine-Borel Theorem.
- Continuity and Homeomorphisms—Section 7.1; in Section 7.2 skip problems using the countably compact and Lindelöf properties; Section 7.3; in Section 7.4, only Theorems 7.32, 7.35, 7.36 and Exercise 7.34; in Section 7.5, through Theorem 7.47.
- Connectedness—in Section 8.1, skip theorems about infinite products.
- Metric Spaces—in Section 9.1, focus on theorems, skip exercises and theorems involving properties you did not cover earlier; in Section 9.2 do the first theorem.

There are many possible variations for an introductory topology course. We sometimes include either the Classification of 2-Manifolds or the Fundamental Group as a part of the experience.

Classification of 2*-Manifolds*: Here is a path to give students an introduction to the classification of 2-manifolds after they have learned some general topology.

- Introduction to Geometric and Algebraic Topology—Chapter 10 is an introduction to geometric topology. It is basically just a short reading assignment.
- 2-Manifolds—Section 11.1; Section 11.2; Section 11.3; Section 11.4 (or Sections 11.6 and 11.7) (these are two different approaches); Section 11.5; Section 11.8; Section 11.9.

Fundamental Group: Here is a path to give students an introduction to the fundamental group after they have learned some general topology.

- The Fundamental Group—in Section 12.1, focus on theorems; Section 12.2; in Section 12.3, focus on theorems; in Section 12.4, the first lemma is a good one to do, then aim to understand the statement of Van Kampen's theorem, and compute examples, if there isn't time to work through the proof.

(2) Topology courses beyond an introductory course

This book contains far more material than could be covered in a single-semester course, and more than could be completely done in a year. So there are several alternatives for a second- or even third-semester course.

One possibility would be to treat point-set topology in the first semester and then do the more geometric and algebraic topology as a second-semester course.

Another possibility is for those who might have an interest in considering some of the more advanced topics in point-set topology. There is plenty of more advanced material in the point-set topology chapters so that an interesting second semester of point-set topology could be offered. Then the geometric and algebraic topology topics could be yet a third semester.

Algebraic and Geometric Topology: Here we list a collection of theorems and exercises that would provide students with the core ideas, examples, and theorems introducing geometric and algebraic topology. Every chapter contains many additional concepts, examples, and theorems that an instructor may include. In each chapter, reading the introduction and conclusion section would be good.

(1) Start with the Classification of 2-Manifolds as described above.

(2) Fundamental Group—Sections 12.1–13.4

(3) Covering Spaces—Sections 13.1–13.4; Section 13.6

(4) Manifolds and Complexes—Sections 14.1–14.4 (Section 14.5 is a fun application)

(5) $\mathbb{Z}_2$-Homology—Sections 15.1–15.4

(6) Applications of $\mathbb{Z}_2$-Homology—Sections 16.1–16.4

(7) Simplicial $\mathbb{Z}$-Homology—Sections 17.1–17.4, 17.7 (Section 17.8 is quite cool)

(8) Singular $\mathbb{Z}$-Homology—Sections 18.1–18.5

(3) Independent study projects

Another use for this book is as a source for many possible independent study projects. Many of the chapters include more advanced theorems than would be treated during a standard course, so many of those theorems or sections could be used as an independent study topic. Individual students or small groups working together could take a topic and work through the theorems and write a booklet describing their work.

For example, the section on metrizability theorems or the section on the Cantor set would be good topics for independent study. The student or group of students could be asked to come to grips with well-ordering, ordinal numbers, transfinite induction, and other concepts involved in the proofs of the basic metrization theorems. These concepts and techniques would form a satisfying, challenging collection of ideas that would be accessible to students during an independent study forum with help from the instructor. Many other such topics are available from this book including topics about issues around product spaces, continua, classification of 2-manifolds, and some algebraic topology, among many others. Some of these possible independent study topics are described in the Instructors' Resource.

(4) Joyful challenges for independent learners

(This category includes those who may have skipped or given short shrift to point-set topology during their mathematical education.) Yet another use of this book is simply for a person who wants to enjoy a rich collection of intriguing mathematical puzzles and challenges. Proving the theorems in this book can be an intrinsically rewarding and satisfying experience. So a person can simply take the whole book as a delightful collection of intellectual treats. Working on the proofs of the theorems can be a truly rewarding enterprise for those who enjoy thinking about challenges for their own sake. And these challenges have the added benefit that, as you master them, you develop a robust understanding of a significant body of mathematics. Perhaps these theorems should be put on a theorem-a-day calendar or should appear in newspapers, where the harder theorems are suggested during the later days of the week.

A word about prerequisites

Little preliminary mathematical knowledge is specifically required to undertake a study of this book; however, realistically speaking, a successful reader will probably need enough mathematical experience to be able to deal with mathematical abstraction. That mathematical maturity will be greatly enhanced while interacting with this book. At our schools, the topology class is generally offered as an upper-division undergraduate course that has the prerequisite of at least one proof-based course in abstract mathematics. Students would do well to have a basic grounding of elementary set theory, such as understanding the concepts of sets, unions, intersections, and DeMorgan's Laws. For the sections on the fundamental group and homology groups, a basic introduction to group theory would be helpful background. The appendix we've provided summarizes the background from group theory that might be useful; for homology theory, a student would really only need the abelian material from the appendix.

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And lastly, we would each like to add that any defects in this book are the other author's fault.