CHAPTER 1

The growth of money

(1.3) Accumulation and amount functions

(1) In order to determine K, use the property $A_K(0) = K$. We have $A_K(0) = \frac{1{,}000}{100-0} = 10$, so $K = 10$. Therefore, $a(20) = \frac{A_K(20)}{K} = \frac{1{,}000/80}{10} = 1.25$.

(3) Firstly, observe that $a(0) = 1$ forces $\beta = 1$. Secondly, $.02 = i_1 = \frac{a(1)-a(0)}{a(0)} = a(1) - 1$ so $1.02 = \alpha(1^2) + .01(1) + 1 = \alpha + 1.01$, and $\alpha = .01$. Therefore, $a(t) = .01t^2 + .01t + 1$, and $i_4 = \frac{a(4)-a(3)}{a(3)} = \frac{1.2-1.12}{1.12} \approx .071428571 \approx 7.14285\%$.

(5) The amount of interest earned from time 0 to n is the sum of the amount of interest earned in the firtst n time periods, namely

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1);$$

this equality may be obtained by noting that

$$\begin{aligned}
1 + 2 + \cdots + n &= \frac{1}{2}[(1 + 2 + \cdots + n) + (1 + 2 + \cdots + (n-1) + n)] \\
&= \frac{1}{2}[(1 + 2 + \cdots + n) + (n + (n-1) + \cdots + 2 + 1)] \\
&= \frac{1}{2}[(1+n) + (2 + (n-1)) + \cdots + ((n-1) + 2) + (n+1)] \\
&= \frac{1}{2}[(n+1) + (n+1) + \cdots + (n+1) + (n+1)] = \frac{1}{2}n(n+1).
\end{aligned}$$

(7) Note that $A_K(n-1) = 3(n-1)^2 + 2(n-1) + 800 = [3n^2 - 6n + 3] + [2n-2] + 800 = 3n^2 + (-6+2)n + 800 = 3n^2 - 4n + 801$. Therefore,

$$\begin{aligned}
i_n &= \frac{A_K(n) - A_K(n-1)}{A_K(n-1)} = \frac{[3n^2 + 2n + 800] - [3n^2 - 4n + 801]}{3n^2 - 4n + 801} \\
&= \frac{6n - 1}{3n^2 - 4n + 801}.
\end{aligned}$$

If you know calculus, you may thus establish that $\{i_n\}$ is decreasing for $n \geq 17$ by showing that the real-values function $f(x) = \frac{6x-1}{3x^2-4x+801}$ has derivative $f'(x) < 0$ for $x \geq 17$. But,

$$\begin{aligned}
f'(x) &= \frac{6(3x^2 - 4x + 801) - (6x-1)(6x-4)}{(3x^2 - 4x + 801)^2} = \frac{(18x^2 - 24x + 4{,}806) - (36x^2 - 30x + 4)}{(3x^2 - 4x + 801)^2} \\
&= \frac{-18x^2 + 6x + 4{,}802}{(3x^2 - 4x + 801)^2}
\end{aligned}$$

Therefore, $f'(x)$ is negative whenever $-18x^2 + 6x + 4{,}802 < 0$. The quadratic formula may be used to find the roots of $-18x^2 + 6x + 4{,}802 = 0$. One root is negative and the other is $\frac{-6-\sqrt{6^2-4(-18)(4{,}802)}}{-2(18)} \approx 16.5$. For x larger than this positive root, so for $x \geq 17$, $f'(x)$ is negative.

If you do not know calculus, you may solve the problem by showing that for $n \geq 17$, the ratio

$$\frac{i_{n+1}}{i_n} = \frac{6(n+1)-1}{3(n+1)^2 - 4(n+1) + 801} \Big/ \frac{6n-1}{3n^2 - 4n + 801}$$

$$= \frac{(6n+5)(3n^2 - 4n + 801)}{(6n-1)(3n^2 + 2n + 800)}$$

$$= \frac{18n^3 - 9n^2 + 4{,}786n + 4{,}005}{18n^3 + 9n^2 + 4{,}798n - 800}$$

is less than 1. Note that the denominator, $18n^3 + 9n^2 + 4,798n - 800$ is positive for any positive integer n. So, this is equivalent to establishing that for $n \geq 17$, the inequality

$$18n^3 - 9n^2 + 4{,}786n + 4{,}005 < 18n^3 + 9n^2 + 4{,}798n - 800$$

is true. But this inequality is equivalent to $18n^2 + 12n - 4{,}805$ being positive. Since the larger root of the quadratic equation $18n^2 + 12n - 4,805 = 0$ is $\frac{(-12)+\sqrt{12^2-4(18)(-4{,}805)}}{(2)(18)} \approx 16.00850119$, $18n^2 + 12n - 4{,}805$ is indeed positive for $n \geq 17$. So, $\frac{i_{n+1}}{i_n} < 1$ for $n \geq 17$.

(1.4) Simple interest

(1) We have $A_{\$1{,}000}(t) = \$1{,}000(1 + .05t)$, so $A_{\$1{,}000}(4) = \$1{,}200$ and $A_{\$1{,}000}(3) = \$1{,}150$. The amount of interest earned in the fourth year is $A_{\$1{,}000}(4) - A_{\$1{,}000}(3) = \$1{,}200 - \$1{,}150 = \$50$.

Alternatively, and more simply, with simple interest and a single investment of capital, the amount of interest is the same each year; it is the product of the amount invested and the annual rate of simple interest: In this case, the amount of interest earned each year is $\$1{,}000 \times .05 = \$50$. The balance at the end of the fourth year is $A_{\$1{,}000}(4) = \$1{,}000 + 4(\$50) = \$1{,}200$; the original \$1,000 had \$50 interest added for each of four years.

(3) The number of months elapsed is twelve times the number of years elapsed, since we view a month as $\frac{1}{12}$-th of a year. With simple interest , the amount of interest is given by the product Krt of the amount invested, the rate of simple interest and the time. Therefore, in order to to compensate for the time being multiplied by a factor of $\frac{1}{12}$, we must multiply the rate of simple interest by 12. Therefore, the annual rate of simple interest is $12 \times .5\% = 6\%$.

(5) We are given that $\$1{,}320 = \$1{,}200(1 + rT)$ where r is the rate of simple interest for this problem. So, $rT = \frac{\$1{,}320}{\$1{,}200} - 1 = .1$. We are asked to calculate $\$500[1 + r(2T)]$. Since $rT = 0.1$, it is equal to $\$500(1.2) = \$600$.

(7) Albert Einstein was born on March 14, 1879, and died on April 18, 1955. We divide the interval between March 14, 1879 and April 18, 1955 into three subintervals, namely interval 1 from March 14, 1879 to December 31, 1949, interval 2 from December 31, 1949 to January 1, 1950, and interval 3 from January 1, 1950 to April 18, 1955. Note that the length of interval 2 was just one day.

Let interval 4 designate the interval from from March 14, 1979 to December 31, 2049; this interval is the interval precisely one hundred years after interval 1, and interval 1 has one fewer day than interval 4 since 1900 was *not* a leap year (since 1900 is divisible by 100 but not by 400) while 2000 was a leap year. We introduced interval 4 because its length may be calculated using the **Date worksheet**.

The number of days that Einstein lived was

$$\begin{aligned} &(\#\text{ days in interval 4 - 1}) + (\#\text{days in interval 2 }) + (\#\text{ days in interval 3}) \\ &= (\#\text{ days in interval 3}) + (\#\text{ days in interval 4}), \end{aligned}$$

a sum of interval lengths that may each be calculated using the **Date worksheet.**

To calculate the number of days in interval 3, first key

2ND DATE 1 • 0 1 5 0 ENTER ↓ 4 • 1 8 5 5 ENTER ;

this will result in "DT1 = 1 – 01 – 1950" and "DT2 = 4 – 18 – 1955". Next key ↓ ↓. If "ACT" is then displayed, find the number of days in interval 3 by keying ↑ CPT, while if the display shows "360", press 2ND SET ↑ CPT. This should result in the display "DBD = 1,933", and there are 1,933 days in interval 3.

Similarly, using "DT1 = 3 – 14 – 79", "DT2 = 12 – 31 – 49", and "ACT", we find that interval 4 comprises 25,860 days. So, Einstein lived for 1,933 + 25,860 = 27,793 days.

(1.5) Compound interest

(1) We are given $A_K(t) = 2,200(1.04)^t$, and we wish to solve $A_K(T) = 8,000$. So, we need to find T such that $(1.04)^T = \frac{8{,}000}{2{,}200}$. To accomplish this, we take natural logarithms of each side of the equation, finding $\ln(1.04)^T = \ln\left(\frac{8{,}000}{2{,}200}\right)$. So, $T\ln(1.04) = \ln\left(\frac{8{,}000}{2{,}200}\right)$, and $T = \frac{\ln\left(\frac{8{,}000}{2{,}200}\right)}{\ln(1.04)} \approx 32.91587729 \approx 32.91588$.

(3) We are given that $(1+i)^9 = 2$. Therefore, $i = 2^{\frac{1}{9}} - 1 \approx .080059739 \approx 8.00597\%$.

(5) If we denote the annual interest rate by i, then we are given that $2 = (1+i)^\alpha$, $3 = (1+i)^\beta$, $10 = (1+i)^\gamma$, and $12 = 5(1+i)^n$. It follows that $(1+i)^n = \frac{12}{5} = \frac{24}{10} = \frac{2^3 3}{10} = \frac{(1+i)^{3\alpha}(1+i)^\beta}{(1+i)^\gamma} = (1+i)^{3\alpha+\beta-\gamma}$. The function $f(x) = (1+i)^x$ is one-to-one (since it is an increasing function), so it follows that $n = 3\alpha + \beta - \gamma$. We therefore have $n = a\alpha + b\beta + c\gamma$ with $a = 3$, $b = 1$, and $c = -1$.

(7) We need to find i such that $(1+i)^{14} = (1.05)^8(1.006)^{12\times 6}$; the reason we have the exponent 12×6 is that we were given a monthly interest rate during the last six years, and six years consists of 12×6 months. So, $i = [(1.05)^8(1.006)^{72}]^{\frac{1}{14}} - 1 \approx .060398768 \approx 6.03988\%$.

(9) The balance at the end of the first four years was $\$K[1 + (.025)4] = \$1.1K$, and if $t > 4$, the balance t years after the initial deposit of \$K is $\$1.1K(1.05)^{t-4}$. We wish to find t so that this is equal to $\$3K$. Equivalently, we seek t with $1.1(1.05)^{t-4} = 3$. Therefore, $t - 4 = \frac{\ln(3/1.1)}{\ln(1.05)} \approx 20.56361412$ and $t \approx 24.56361412$. Starting in 1963, the balance triples in about 24.56361 years.

(11) (a) Applying the equation $a(s+t) = a(s)a(t)$ with $t = h$, we have $a(s+h) = a(s)a(h)$. Therefore,

$$\begin{aligned} a'(s) &= \lim_{h\to 0}\frac{a(s+h)-a(s)}{h} = \lim_{h\to 0}\frac{a(s)a(h)-a(s)}{h} \\ &= \lim_{h\to 0}\, a(s)\left[\frac{a(h)-1}{h}\right] \\ &= a(s)\lim_{h\to 0}\frac{a(h)-1}{h}. \end{aligned}$$

(b) Using the result from part (a) and the definition of the derivative $a'(0)$, we have

$$a'(s) = a(s)\left(\lim_{h\to 0}\frac{a(h)-1}{h}\right) = a(s)\left(\lim_{h\to 0}\frac{a(0+h)-1}{h}\right) = a(s)a'(0).$$

(c) Observe that $\frac{d}{ds}\ln a(s) = \frac{a'(s)}{a(s)}$ and, [from part (b)] $\frac{a'(s)}{a(s)} = a'(0)$. Therefore,

$$\int_0^t \frac{d}{ds}\ln a(s)ds = \int_0^t \frac{a'(s)}{a(s)}ds = \int_0^t a'(0)ds = a'(0)\int_0^t ds = a'(0)t.$$

(d) It follows from the Fundamental Theorem of Calculus and the result of part (c) that $\ln a(t) - \ln a(0) = a'(0)t$. But $\ln a(0) = \ln 1 = 0$, so $\ln a(t) = \ln a(t) - \ln a(0) = a'(0)t$.

(e) Recall that $a(1) = 1 + i$. Therefore, applying the result of part (d) with $t = 1$, we have $\ln(1+i) = \ln a(1) = a'(0) \times 1 = a'(0)$.

(f) Combining the results of parts (d) and (e) and using an important property of logarithms , we find $\ln a(t) = a'(0)t = [\ln(1+i)]\,t = t\ln(1+i) = \ln(1+i)^t$.

(1.6) Effective discount rates/ Interest in advance

(1) Antonio gets the use of an extra $\$3{,}000 - .08(\$3{,}000) = \$3{,}000 - \$240 = \$2{,}760$.

(3) We are given that $\$1{,}320 = \$1{,}450 - \$1{,}450D = \$1{,}450(1-D)$. Therefore, $D = 1 - \frac{1{,}320}{1{,}450} \approx .089655172 \approx 8.96552\%$. Moreover, $I = (1-D)^{-1} - 1 = \frac{1{,}450}{1{,}320} \approx .098484848 \approx 9.84848\%$.

(5) Note that $1.2 = 1 + i_{[2,4.5]} = (1+i)^{4.5-2}$. So, $1 + i = (1.2)^{\frac{1}{2.5}} = (1.2)^{.4}$. Therefore, $d_{[1,3]} = 1 - (1 + i_{[1,3]})^{-1} = 1 - (1+i)^{-2} = 1 - (1.2)^{-.8} \approx .135718926 \approx 13.57189\%$.

(1.7) Discount functions/ The time value of money

(1) The money is invested at $t = 3$, but the accumulation function gives the growth of 1 invested at $t = 0$. Therefore, we first need to find the amount of money you would need to invest at $t = 0$ in order to have \$3,200 at $t = 8$; you would need $\frac{\$3{,}200}{a(8)} = \frac{\$3{,}200}{1.4}$. But $\frac{\$3{,}200}{1.4}$ deposited at $t = 0$ would grow to $\left(\frac{\$3{,}200}{1.4}\right) a(3) = \left(\frac{\$3{,}200}{1.4}\right)(1.15) \approx \$2{,}628.57$.

(3) Each year, the value of the home grows by a factor of 1.028. It cost \$243,000 on June 30, 2018, so its price P on June 30, 2001 satisfied $P(1.028)^{17} = \$243{,}000$. Thus, $P = \$243{,}000(1.028)^{-17} \approx \$151{,}957.91$.

(5) We need to bring the \$5,000 back for ten years, using the appropriate discount factor for each year. Therefore, the present value is $\$5{,}000(1.04)^{-3}(1.05)^{-2}(1.055)^{-5} \approx \$3{,}084.814759 \approx \$3{,}084.81$.

(7) The present value of the first option is $\$6{,}000 + \$5{,}940(1+i)^{-1}$, and the second option has present value $\$12{,}000(1+i)^{-\frac{1}{2}}$. Therefore,

$$\$6{,}000 + \$5{,}940(1+i)^{-1} = \$12{,}000(1+i)^{-\frac{1}{2}}.$$

But this is equivalent to the equation

$$6{,}000(1+i) - \$12{,}000(1+i)^{\frac{1}{2}} + \$5{,}940 = 0.$$

Set $X = (1+i)^{\frac{1}{2}}$. Then,

$$6{,}000X^2 - \$12{,}000X + \$5{,}940 = 0$$

and the quadratic formula gives

$$X = \frac{12{,}000 \pm \sqrt{(12{,}000)^2 - 4(5{,}940)(6{,}000)}}{2(6{,}000)} = \frac{12{,}000 \pm 1{,}200}{12{,}000} = 1 \pm .1.$$

So, $1 + i = X^2$ must either be equal to $1.1^2 = 1.21$ or to $.9.^2 = .81$ Assuming the loan is made at a positive rate of interest, it follows that $i = .21 = 21\%$.

(1.8) Simple discount

(1) Note that \$3,460 to be paid at $t = 9$ has a time $t = 0$ value of $\$3{,}460v(9) = \$3{,}460[1 - 9(.05)] = \$1{,}903$. Bringing this forward to time $t = 4$, we find a value of $\$1{,}903a(4) = \$1{,}903\frac{1}{1-4(.05)} = \$2{,}378.75$.

(3) The specified investment amount \$1,000 is not needed to do the problem. All we need to do is to look at the two accumulation functions, the simple discount accumulation function $a^{\text{s.d.}}(t) = \frac{1}{1-.08t}$ and the simple interest accumulation function $a^{\text{s.i}}(t) = 1 + .12t$ and determine $T > 0$ so that $a^{\text{s.d.}}(T) = a^{\text{s.i.}}(T)$. That is, we need to solve $\frac{1}{1-.08T} = 1 + .12T$. This equation is equivalent to the equation $-.0096T^2 + .04T + 1 = 1$ which has positive solution $T = \frac{.04}{.0096} = \frac{25}{6} \approx 4.16667$. From the investor's perspective, the simple discount investment account looks more and more attractive as time passes while the simple interest account becomes less and less desirable. Thus, the simple discount is preferable if money is kept on deposit for longer than $T \approx 4.16667$.

(5) (a) At the end of three years, the invested \$300 has grown in the simple interest fund to the accumulated amount $\$300[1 + (.06)(3)] = \$354$. After an *additional* T years, during which time the money grows in the 8% simple discount account, the balance is is $\$354(1 - 08T)^{-1}$; this is because we are assuming that the simple discount account has just been opened. Thus, we are trying to determine $T + 3$ where $\$354(1 - 08T)^{-1} = \$650$. This equation has solution $T = \frac{1}{.08}\left(1 - \frac{354}{650}\right) \approx 5.692307692$ and hence our answer is $T + 3 \approx 8.692307692 \approx 8.69231$ years.

(b) We seek i so that $\$300(1+i)^{T+3} = \$650$ where T is as in part (a). Note that $i = \left(\frac{650}{300}\right)^{\frac{1}{T+3}} - 1 \approx .0930271503 \approx 9.30272\%$.

(1.9) Compound discount

(1) The balance at the end of five years is $\$1{,}000(1 - .064)^{-5} \approx \$1,391.9407773 \approx \$1,391.94$.

(3) To bring back money one quarter, you multiply by$(1.068)^{-\frac{1}{4}}$. Therefore, the effective quarterly discount rate is $1 - (1.068)^{-\frac{1}{4}} \approx .016312423 \approx 1.63124\%$.

(5) We are given that in three years, $\$2{,}120 - \$250 = \$1{,}870$ grows to \$2,120. Therefore, $(1+i)^3 = \frac{2{,}120}{1{,}870}$. So, the interest for two years on \$380 is

$$\$380(1+i)^2 - \$380 = \$380\left[\left(\frac{2{,}120}{1{,}870}\right)^{\frac{2}{3}} - 1\right] \approx \$33.15508238 \approx \$33.16.$$

(7) We are given

$$\$320 = \$X[(1+i)^2 - 1] = \$X(i^2 + 2i) = \$Xi(i+2)$$

and

$$\$148 = \$Xd = \$X\left(\frac{i}{1+i}\right).$$

Rewrite this second equation as $\$Xi = \$148(1+i)$, and then substitute this new expression for $\$Xi$ into the first equation, thereby obtaining $\$320 = \$148(1+i)(i+2)$. It follows that $320 = 148i^2 + 444i + 296$, and hence $148i^2 + 444i - 24 = 0$. The quadratic formula then tells us that

$$i = \frac{-444 \pm \sqrt{444^2 - 4(148)(-24)}}{2 \times 148}.$$

So that i is positive, we must take

$$i = \frac{-444 + \sqrt{444^2 - 4(148)(-24)}}{2 \times 148} \approx .053113699 \approx 5.31137\%.$$

Consequently,

$$X = \frac{320}{i(i+2)} \approx \$2{,}934.475103 \approx \$2{,}934.48.$$

(1.10) Nominal rates of interest and discount

(1) The equivalent rates may be found as follows;

$1-d=\left(1-\frac{d^{(4)}}{4}\right)^4$ so $d=1-\left(1-\frac{.08}{4}\right)^4\approx .07763184\approx 7.76318\%.$

$\left(1-\frac{d^{(3)}}{3}\right)^3=\left(1-\frac{d^{(4)}}{4}\right)^4$ so $d^{(3)}=3\left[1-\left(1-\frac{.08}{4}\right)^{\frac{4}{3}}\right]\approx .079732138\approx 7.97321\%$

$1+i=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$ so $i=\left(1-\frac{.08}{4}\right)^{-4}-1\approx .084165785\approx 8.41658\%.$

$\left(1+\frac{i^{(6)}}{6}\right)^6=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$ so $i^{(6)}=6\left[\left(1-\frac{.08}{4}\right)^{-\frac{4}{6}}-1\right]\approx .08135748\approx 8.13575\%.$

(3) Note that $i^{(12)}=12\times .5\%=6\%$. Moreover, $i=(1+.005)^{12}-1\approx .061677812\approx 6.16778\%$. Also, $d=1-(1+i)^{-1}=1-(1+.005)^{-12}\approx .05809466\approx 5.80947\%$.

(5) Since $1=\left(1+\frac{i^{(m)}}{m}\right)\left(1-\frac{d^{(m)}}{m}\right)$, $\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\frac{d^{(m)}}{m}$. Multiplying this equation by by m^2 and substituting in the given values for $i^{(m)}$ and $d^{(m)}$, we find

$$m(.0469936613-.046773854)=(.0469936613)(.046773854).$$

So,

$$m=\frac{(.046993661309)(.046773854)}{.0469936613-.046773854}\approx 10.0000211\approx 10.00002.$$

If we insist that m is an integer, then $m=10$.

(7) (a) Every m years, money grows by a factor $(1+i)^m$ so $1+mi^{(\frac{1}{m})}=(1+i)^m$ and

$$i^{(\frac{1}{m})}=\frac{1}{m}\left[(1+i)^m-1\right].$$

(b) We are given a nominal interest rate of 6% for each year-and-a-half period. So, $(1+i)^{(\frac{3}{2})}=1+\frac{.06}{2/3}=1.09$. Therefore, $i=(1.09)^{\frac{2}{3}}-1\approx .059134217\approx 5.91342\%$.

(c) The discount factor for an m-year period may be expressed as $\left(1+mi^{(\frac{1}{m})}\right)^{-1}$ or as $1-md^{(\frac{1}{m})}$. These must be equal, so

$$d^{(\frac{1}{m})}=\frac{1}{m}\left[1-\left(1+mi^{(\frac{1}{m})}\right)^{-1}\right].$$

The discount factor for m years is also given by the expression $(1-d)^m$, and therefore $1-md^{(\frac{1}{m})}=(1-d)^m$. It follows that

$$d^{(\frac{1}{m})}=\frac{1}{m}\left[1-(1-d)^m\right].$$

(1.11) A friendly competition (Constant force of interest)

(1) We have $\delta=\ln(1+i)=\ln\left[\left(1-\frac{d^{(4)}}{4}\right)^{-4}\right]=-4\ln\left(1-\frac{d^{(4)}}{4}\right)=-4\ln\left(1-\frac{.032}{4}\right)\approx .032128687\approx 3.21287\%$.

(3) We compare the accounts by determining the annual effective interest rates i_A, i_B, and i_C of the three accounts; as the investor, you should choose the account with the highest rate. We calculate $i_B=(1.0044)^{12}-1\approx .054096687\approx 5.40967\%$, and $i_C=e^{.0516}-1\approx .052954476\approx 5.29545\%$. Since we are given $i_A=5.2\%$, it would provide the lowest accumulation. Therefore, you should choose B.

(1.12) Force of interest

(1) (a) We begin by finding the accumulation function: $a(t) = e^{\int_0^t \delta_r\,dr} = e^{\int_0^t .05+.006r\,dr} = e^{.05t+.003t^2}$. So, the accumulated value of \$300 deposited at time 0 is $\$300a(3) = e^{.15+.027} = \$300e^{.177} \approx \$358.0893279 \approx$ \$358.09.

(b) If the \$300 deposit is made at time 4, then the deposit three years later is $\$300\frac{a(7)}{a(4)} = \$300e^{\int_4^7 .05+.006r\,dr} = \$300e^{(.05r+.003r^2)|_4^7} = \$300e^{.249} \approx \$384.8226099 \approx \$384.82$.

(3) Note that $a(t) = e^{\int_0^t \delta_r\,dr} = e^{\int_0^t \frac{r^2}{5+r^3}\,dr}$ and $\int_0^t \frac{r^2}{5+r^3}\,dr = \frac{1}{3}\ln|5+r^3||_0^t = \frac{1}{3}\ln|\frac{5+t^3}{5}|$. This gives $a(t) = \left(\frac{5+t^3}{5}\right)^{1/3}$. The present value of \$700 to be paid at time 4 is $\$700v(4) = \$700\left(\frac{69}{5}\right)^{-1/3} \approx \$291.83565 \approx$ \$291.84.

(5) We find $\delta_t = \frac{d}{dt}(\ln a(t)) = \frac{d}{dt}(t\ln(1+.02)) + \frac{d}{dt}\ln(1+.03t) - \frac{d}{dt}\ln(1-.05t) = \ln(1.02) + \frac{.03}{1+.03t} - \frac{-.05}{1-.05t}$. Therefore $\delta_3 = \ln(1.02) + \frac{.03}{1.09} + \frac{.05}{.85} \approx .106149092 \approx 10.61491\%$.

(7) The 10% simple interest account has force of interest function $\delta_t = \frac{.1}{1+.1t}$, which is a decreasing function of T, and the 7% compound interest account has constant force of interest $\delta = \ln(1.07)$. If our goal is to maximize the accumulation at the end of five years or to maximize it at the end of ten years, we wish to always have our money in the account that has a higher force of interest. So, we should move our money when $\frac{.1}{1+.1t} = \ln(1.07)$; that is, at time $t = 10\left(\frac{.1}{\ln 1.07)} - 1\right) \approx 4.780076495 \approx 4.78008$.

(9) We first need to determine the amount functions $A^{(A)}(t)$ and $A^{(B)}(t)$. Note that we have accumulation function

$$a^{(A)}(t) = e^{\int_0^t \delta_r{}^{(A)}\,dr} = e^{\int_0^t \frac{.08}{1+.08r}\,dr} = e^{\ln(1+.08t)} = 1 + .08t,$$

so $A^{(A)}(t) = 600(1+.08t)$. Also,

$$a^{(B)}(t) = e^{\int_0^t \delta_r{}^{(B)}\,dr} = e^{\int_0^t .01r\,dr} = e^{.005t^2},$$

and thus $A^{(B)}(t) = 300e^{.005t^2}$. Therefore,

$$A^{(C)}(t) = A^{(A)}(t) + 2A^{(B)}(t) = 600(1+.08t) + 600e^{.005t^2} = 600 + 48t + 600e^{.005t^2},$$

and

$$\delta_t^{(C)} = \frac{d}{dt}A^{(C)}(t)/A^{(C)}(t) = (48 + 6te^{.005t^2})/(600(1+.08t) + 600e^{.005t^2}).$$

Evaluating at $t = 4$, we obtain $\delta_4^{(C)} = (48 + 24e^{.08})/(792 + 600e^{.08}) \approx .051317832 \approx 5.13178\%$.

(1.13) Note for those who skipped Section (1.11) and (1.12)

(1) Money grows each year by a factor of $e^{.0375}$; so the annual effective rate of interest is $e^{.0375} - 1 \approx .038211997 \approx 3.82120\%$.

(1.14) Quoted rates for Treasury bills

(1) Since the value of the Treasury bill is governed by simple discount on an actual/360 basis,

$$P = \$10{,}000\left[1 - (0.0285)\left(\frac{180}{360}\right)\right] = \$9{,}857.50.$$

Thus, Ayub paid \$9,857.50. Ayub will earn $\$10{,}000 - \$9{,}857.50 = \$142.50$ as interest for this 180-day investment.

(3) Let P be the purchase price of the U.S. T-bill. Then,

$$P = \$21{,}000\left[1 - \frac{(26)(7)}{360}(0.0305)\right] \approx \$20{,}676.19167.$$

Since money had to change hands for the purchase, we round to the nearest penny which gives $P = \$20{,}676.19$. This gives the purchase price of the Canadian T-bill as $\$35{,}592.52 - \$20{,}676.19 = \$14{,}916.33$. Therefore, the quoted rate of the Canadian T-bill is

$$\frac{\$15{,}000 - \$14{,}916.33}{\$14{,}916.33}\left[\frac{365}{(13)(7)}\right] \approx 0.022498795 \approx 2.24988\%.$$

(5) Let P_u be the price of the U.S. bond, and let P_c be the price of the Canadian bond. Then, since there are $26 \times 7 = 182$ days in 26 weeks, we have

$$P_u = X\left(1 - \frac{182}{360}\,q\right)$$

and

$$P_c = X\left(1 + \frac{182}{365}\,q\right)^{-1}.$$

The expression for P_c implies that $X = P_c\left(1 + \frac{182}{365}\,q\right)$. Using this together with the expression for P_u, we now have

$$P_u = P_c\left(1 + \frac{182}{365}\,q\right)\left(1 - \frac{182}{360}\,q\right).$$

Now, consider the graph of $f(q) = \left(1 + \frac{182}{365}q\right)\left(1 - \frac{182}{360}q\right)$ which is a parabola that opens downward. With or without calculus, you can show that its vertex is $(-\frac{5}{364}, f(-\frac{5}{364}))$. Thus, for all $q > 0$ we have $f(q) < f(0) = 1$. Hence, $P_u < P_c$ and we conclude that the U.S. bond would have the lower price.

(1.15) Inflation

(1) (a) Buying power grows by a factor of $\frac{1.042}{1.03} \approx 1.011650485$. Therefore, the real rate of interest is $\approx 1.16505\%$.

(b) In this case, buying power grows by a factor of $\frac{1.042}{1.046} \approx .99617590 = 1 + (-.003824092)$. So, the real rate of interest is approximately $-.38241\%$.

(3) Buying power grows by a factor of 1.0124 each year while, thanks to interest, money available for purchasing grows by $\left(1 - \frac{.03}{4}\right)^{-4}$. Thus, the real rate of inflation r satisfies the equation $1.0124 = \left(1 - \frac{.03}{4}\right)^{-4}/(1+r)$. Equivalently, $r = \left(1 - \frac{.03}{4}\right)^{-4}(1.0124)^{-1} - 1 \approx .017948488 \approx 1.79485\%$.

(5) During the three-year period, purchasing power changes by a factor $\frac{\left(1+\frac{.024}{12}\right)^{36}}{(1.015)(1.028)(1.034)} \approx .995997571 = 1 - .004002429$. So, purchasing power falls by about $.40024\%$.

(1.16) Choice of quotation base for interest rates

(1) One possible method (out of many) is to calculate the balance at time 1 of \$100 invested at time 0 for each account.

For $\delta = 7.9\%$: $\$100\, e^{0.079} \approx \$108.2204$.
For $d^{(2)} = 8.0\%$: $\$100\left[1 - \frac{0.08}{2}\right]^{-2} \approx \$108.5069$.
For $d^{(4)} = 8.1\%$: $\$100\left[1 - \frac{0.081}{4}\right]^{-4} \approx \$108.5273$.
For $i^{(6)} = 8.1\%$: $\$100\left[1 + \frac{0.081}{6}\right]^{6} \approx \$108.3783$.
For $i^{(12)} = 8.2\%$: $\$100\left[1 + \frac{0.082}{12}\right]^{12} \approx \$108.5153$.
For $i = 8.3\%$: $\$100(1.083) = \$108.30$.

A higher balance reflects faster growth. Therefore, arranging from fastest growth to slowest growth, we have: $d^{(4)} = 8.1\%$, $i^{(12)} = 8.2\%$, $d^{(2)} = 8.0\%$, $i^{(6)} = 8.1\%$, $i = 8.3\%$, and $\delta = 7.9\%$.

(3) We have

$$e^{(0.0704)(5)} = \left(e^{0.054}\right)\left(e^{0.073}\right)\left(e^{0.038}\right)\left(e^{\delta}\right)\left(e^{0.102}\right) = e^{(0.054+0.073+0.038+\delta+0.102)},$$

which gives

$$(0.0704)(5) = 0.054 + 0.073 + 0.038 + \delta + 0.102.$$

Solving for δ, we have $\delta = 0.085 = 8.5\%$.

Chapter 1 review problems

(1) The accumulated value is $\$6{,}208\left(1-\frac{.023}{4}\right)^{-4\times 2}\left(1+\frac{.03}{12}\right)^{12}(1-.042)^{-3}e^{.046\times 2}\approx \$8{,}353.299474 \approx \$8{,}353.30$.

(3) The original annual effective interest rate i_0 corresponds to force of interest δ_0 where $1+i_0 = e^{\delta_0}$. Moreover, $\$1{,}039.98 = \$K(1+i_0)^{-2} = \$Ke^{-2\delta_0}$. On the other hand, we are told that $\$1{,}060.78 = \$Ke^{-2\left(\frac{\delta_0}{2}\right)}$. So, $\$K(1+i_0)^{-1} = \$1{,}060.78$. Therefore, $1+i_0 = \frac{\$K(1+i_0)^{-1}}{\$K(1+i_0)^{-2}} = \frac{1{,}060.78}{1{,}039.98}$, and $K = (1{,}060.78)(1+i_0) = \frac{(1{,}060.78)^2}{1{,}039.98} \approx 1{,}081.996008 \approx 1{,}082$. The interest rate i_0 is equivalent to an annual effective discount rate $d_0 = 1 - \frac{1}{1+i_0} = 1 - \frac{1{,}039.98}{1{,}060.78} = \frac{20.8}{1{,}060.78}$.

If we have a new annual effective discount rate d with $d = \frac{d_0}{2} = \frac{10.4}{1{,}060.78}$, then the present value of $\$K$ is $K(1-d)^2 = \$1{,}082\left(1 - \frac{10.4}{1{,}060.78}\right)^2 \approx \$1{,}060.89$.

(5) The December 1, 2003 value of the first option is $\$6{,}000 + \frac{\$4{,}000}{1.05}$ while the value of the second option on that date is $\$12{,}000(1.05)^{-\frac{N}{12}}$. Setting these two values equal and then dividing by $\$12{,}000$, we find $(1.05)^{-\frac{N}{12}} = .5 + \frac{1}{3(1.05)} \approx .817460317$. Taking natural logarithms, we obtain $-\frac{N}{12}\ln(1.05) \approx \ln(.817460317)$. So, $N \approx -12\ln(.817460317)/\ln(1.05) \approx 49.5721846 \approx 49.57218$.

(7) (a) The accumulation function satisfies $a(0) = 1$, so $c = 1$. To determine the values of the constants a and b, we consider the conditions $i_3 = 50/1{,}088$ and $d_4 = 54/1{,}192$ simultaneously. Note that

$$\frac{50}{1{,}088} = i_3 = \frac{a(3)-a(2)}{a(2)} = \frac{(9a+3b+1)-(4a+2b+1)}{4a+2b+1} = \frac{5a+b}{4a+2b+1}.$$

So,

$$200a + 100b + 50 = 50(4a+2b+1) = 1{,}088(5a+b) = 5{,}440a + 1{,}088b.$$

Equivalently, $a = \frac{50-988b}{5{,}240}$. On the other hand,

$$\frac{54}{1{,}192} = d_4 = \frac{a(4)-a(3)}{a(4)} = \frac{(16a+4b+1)-(9a+3b+1)}{16a+4b+1} = \frac{7a+b}{16a+4b+1}.$$

Therefore,

$$864a + 216b + 54 = 54(16a+4b+1) = 1{,}192(7a+b) = 8{,}344a + 1{,}192b.$$

It follows that $54 = 7480a + 976b = 7480\left(\frac{50-988b}{5{,}240}\right) + 976b$. Thus,

$$282{,}960 = 54(5240) = 7480(50-988b) + 976b(5{,}240) = 374{,}000 - 7{,}390{,}240b + 5{,}114{,}240b,$$

and

$$b = \frac{374{,}000 - 282{,}960}{7{,}390{,}240 - 5{,}114{,}240} = \frac{91{,}040}{2{,}276{,}000} = .04.$$

Moreover, $a = \frac{50-988b}{5{,}240} = \frac{50-988(.04)}{5{,}240} = .002$.

(b) The value at $t = 3$ of \$1,000 to be paid at $t = 8$ is given by the expression $\left(\frac{\$1{,}000}{a(8)}\right)a(3)$. Since

$$a(3) = 9a + 3b + 1 = 9(.002) + 3(.04) + 1 = 1.138$$

and

$$a(8) = (64a + 8b + 1)(1 + .05(8-6)) = [64(.002) + 8(.04) + 1](1.1) = 1.5928,$$

this is equal to $\$714.4650929 \approx \$714.47$.

(9) Note that

$$f(n) = i_{n+i} + 1 = \frac{a(n+1) - a(n)}{a(n)} + \frac{a(n)}{a(n)} = \frac{a(n+1)}{a(n)}.$$

Since

$$a(t) = e^{\int_0^t \delta_r \, dr} \quad \text{for} \quad 2 \leq n \leq 7,$$

we find

$$\begin{aligned} f(n) &= e^{\int_0^{n+1} \delta_r \, dr} \Big/ e^{\int_0^n \delta_r \, dr} \\ &= e^{\int_n^{n+1} \delta_r \, dr} \\ &= e^{\int_n^{n+1} \frac{4}{r-1} \, dr} \\ &= e^{4 \ln (r-1)|_n^{n+1}} \\ &= e^{\ln (n^4) - \ln [(n-1)^4]} \\ &= e^{\ln \left(\frac{n^4}{(n-1)^4} \right)} \\ &= \left(\frac{n}{n-1} \right)^4. \end{aligned}$$